THE ASYMMETRIC ENERGY OF SINGLE-CLOSED-SHELL NUCLEI IN PRESENT-DAY MICROSCOPIC THEORY

A THESIS

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Hagen Michael Kleinert

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THE ASYMMETRIC ENERGY OF SINGLE-CLOSED-SHELL

NUCLEI IN PRESENT-DAY MICROSCOPIC THEORY

Approved:

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DEDICATION

To My Parents

ACKNOWLEDGMENTS

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SUMMARY

The asymmetric energy term of the Weizsaecker mass formula has been derived for single closed shell nuclei. The model of the nucleus used in this work consists of the Nilsson-shell model with two added residual interactions:

(1) a short-range pairing force,

(2) a long-range quadrupole force.

The effect of the first force has been included by the BCS-superconductivity method. The quadrupole force has been treated with the collective "cranking method" introduced by Inglis.

The calculations lead to the asymmetric energy, determined by shell and interaction parameters. The interaction parameters can be expressed in terms of well observable quantities: the even-odd mass difference, and the frequency of the quadrupole radiation. Thus the asymmetric energy turns out to be completely determined by the even-odd mass difference and the quadrupole frequency. The result is checked by two examples and found to be in very good agreement with experiment.

CHAPTER I

INTRODUCTION

The last five years have extended considerably our knowledge of the forces governing the behaviour of nuclei.

A model, first extensively treated by Belyaev[1], has quantitatively explained many nuclear properties[2], such as:

(a) odd-even mass differences,

(b) magnetic dipole moments,

(c) electric quadrupole moments,

(d) electromagnetic transition rates, and

(e) Beta-decay matrix elements.

This model considers the nucleus as a system of protons and neutrons moving in a deformable harmonic oscillator well[3], subjected to certain additional interactions:

(1) The Mayer-Jensen spin-orbit coupling.

(2) A term with l-l coupling, giving the well an effectively more quadratic form.

(3) A pairing force, acting between mutually time reversed particles of equal isotopic spin.

(4) A charge independent quadrupole force.

The level structure of the deformed potential well including spin-orbit and l-l coupling has been calculated by Nilsson[3] and we shall refer to this well as the Nilsson-well.

The Nilsson-well is at present the basis for all microscopic nuclear theories, and it seems that the addition of appropriate small residual interactions will allow a description of all observable nuclear properties.

The advantages of using this basis are that any added residual interaction has rather specific effects and that limitations of the approach can immediately be recognized. For example, pairing and quadrupole force which are so successful in explaining the properties (a) - (e), will certainly not answer questions concerning:

(1) Formation of clusters of other than two nucleons of equal isotopic spin inside the nucleus. (Hence, they fail to explain quantitatively β-decay.)

(2) Multipole vibrations of order higher than 2, which have first been discovered in 1957. Now about 10 nuclei are known to undergo E_3, 8 nuclei to have E_4 transitions[4].

The principal problem connected with (1) is to find the right neutron-proton residual interaction. The attempts made until now[5] have been very unsatisfactory. The fact that also 3-nucleon clusters show a high binding energy indicates the complexity of the additional forces which must be taken into account[6].

The problem we have attacked is to show how the Nilsson-well with pair and quadrupole force gives the asymmetric energy* for those nuclei in which either the neutrons or the protons are in a major closed

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* i.e., the term $\frac{a_s}{A}(A-2Z)^2$ of the Weizsaecker mass formula which determines, together with the Conlomb energy part, the shape of the β-decay valley.

shell. We shall call them, with Kisslinger and Sorensen[2], single closed shell nuclei (S.C.S.). These nuclei do not seem to possess any static equilibrium deformation[7], such that we can treat spherically symmetrical problems. This will simplify our work considerably and allow us to get all results without machine calculations.

For these single closed shell nuclei we shall derive the ground state energy and compare its quadratic dependence on the number of nucleons with that of the asymmetric energy. We do not want to give a theory for the Coulomb term in the mass formula and hence we can treat neutrons and protons symmetrically, neglecting the slight difference of their single particle levels in the Nilsson-well.

In Chapter II, we shall discuss the character of the residual interactions and give reasons for the selection of pairing and quadrupole forces. In Chapter III(A) we shall briefly state the classical Bogoljubov[17] method for obtaining the ground state which results from the pair interaction, and in part (B) we include the quadrupole force and introduce collective coordinates. In Chapter IV we shall calculate the ground state energy and in Chapter V we shall discuss the quadratic terms in the nucleon number.

The reader is assumed to be familiar with the theory of superconductivity; otherwise an understanding of this work is impossible. Chapter III(A) does not attempt to give an introduction into this theory but is a mere statement of the general results.

CHAPTER II

THE CHOICE OF THE RESIDUAL INTERACTIONS

The Nilsson single particle levels are derived for a smooth potential field which is some average of the true potential of one nucleon relative to the others. These levels give a fairly good approximation to the actual ones. For example, the strong static nuclear deformations in the region $N \geq 90$ can be derived by minimizing the sum of the single-particle energies in the Nilsson-well with respect to the deformation[8].

One can expect that the residual interactions which have to be superimposed on the average field will bring about only small changes in the level structure.

Suppose now

$$V = \frac{1}{2} \sum_{i \neq j} v_{ij}$$

is the residual interaction, a sum of two body potentials.

Each v_{ij} can be expanded in a series of Legendre functions. Take v_{12} and denote by ϑ the angle between the position vectors $\bar{r}_1$, $\bar{r}_2$ of the two particles. Then:

$$v_{12} = \sum_{k} f_k(r_1, r_2) \; P_k(\cos\vartheta) \tag{1}$$

The range of the force determines the contributions of the different P_k. Suppose the nucleus has a radius R and the range of v_{12} is r_0, then the range in ϑ is approximately $\frac{r_0}{R}$. Now the Legendre functions

P_k show coherent effects only in the region of width $\frac{1}{k}$ about $\vartheta = 0$. Therefore, in the expansion of V_{12}, those P_k dominate, for which $\frac{1}{k} < \frac{r_0}{R}$. If r_0 is $\ll R_1$ only terms with $k \gg \frac{R}{r_0}$ are important; if the range is of nuclear dimensions, the first terms have to be considered.

It must be the goal of the theory, to select from the superabundance of acting forces those which are essential for explaining the observed phenomena. The shell model fails to predict two strong effects: the high energy required to separate an even nucleon and the emission of quadrupole radiation by collective vibration by many nuclei.* The first effect suggests the introduction of a short range force, which is able to bind pairs of particles together. The second effect is, like all collective phenomena, caused by long range forces and particularly the P_2 term in the expansion of V_{12} will be able to yield the right multipole order of the vibration.

If one now chooses only a force with the angular dependence of P_2 and a short range force as characteristic parts of the residual interaction, one has taken from the expansion in P_k the P_2-part and all those with $k \geq \frac{R}{r_0}$. One has neglected multipole forces with $k \neq 2$. Since this choice of forces has explained quite well low energy level spacings in many nuclei, one can hope that they give also the main contributions to the ground state energy. Calculations of the absolute binding energy of Pb-isotopes by Kisslinger-Sorensen are in good agreement with experiment and therefore support this hope.

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* More than 100 collective states are known.

The problem is now how to build up a Hamiltonian accounting for the selected two forces. We want the Hamiltonian to be simple enough that we can diagonalize it, but it must inherit the characteristic features of the residual interactions such that we can see their effect in the purest way.

This is the old problem of theory, with which already Galilei was confronted when trying to find the law for the free fall. In recognizing the necessity of neglecting nonessential perturbations* he was probably the first theoretician.

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* The *"dissecare naturam"* of Bacon of Verulam.

CHAPTER III

THE GROUND STATE ENERGY

We shall state briefly the Bogoljubov method for finding the ground state if only the pairing force is present and take the quadrupole force into account by collective treatment of the nucleus.

A. The Pairing Force

The interaction Hamiltonian is in second quantized formulation

$$H = \frac{1}{2} \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\, b_\alpha^+ b_\beta^+ b_\gamma b_\delta \tag{1}$$

where b_α^+ and b_α are the creators and annihilators of the nucleons with the quantum numbers

$$\alpha = (n, l, j, m) \tag{2}$$

in the Nilsson-well and $V_{\alpha\beta\gamma\delta}$ is the matrix element of the two body interaction V_{12} between the states $\langle\alpha\beta|$ and $|\gamma\delta\rangle$. The $V_{\alpha\beta\gamma\delta}$ which is used to represent the pairing force is

$$V_{\alpha\beta\gamma\delta} = -\frac{1}{2}\, G\, \delta_{\bar{\alpha}\beta}\, \delta_{\bar{\gamma}\delta} \tag{3}$$

where G the so-called pair interaction constant. $\bar{\alpha}$ denotes the time reversed α , i.e.

$$\bar{\alpha} = (n, l, j, -m) \tag{$\bar{2}$}$$

$V_{\alpha\beta\gamma\delta}$ carries the characteristic features of a short range interaction. This can be seen by investigating the behaviour of the interaction matrix calculated for a δ-function potential; i.e., of

$$\langle \alpha\beta \,|\, \delta(\vec{r}_1 - \vec{r}_2) \,|\, \gamma\delta \rangle$$

For this matrix, one finds that the largest elements lie just in those places where $V_{\alpha\beta\gamma\delta}$ in (3) is different from zero; the interaction $V_{\alpha\beta\gamma\delta}$ gives just an extract of the dominating terms of the δ-force matrix element.*

Let ε_α be the single particle energies in the Nilsson well; then the Hamiltonian including the pair interaction is:

$$H = H_N + H_P \tag{4}$$

with

$$H_N = \sum_\alpha \varepsilon_\alpha b_\alpha^+ b_\alpha \tag{5}$$

and

$$H_P = -\frac{G}{4} \sum_{\alpha,\alpha'} b_\alpha^+ b_{\bar{\alpha}}^+ b_{\bar{\alpha}'} b_{\alpha'} \tag{6}$$

The ground state for such a Hamiltonian has first been given in a good approximation by BCS[9] in the theory of superconductivity.**

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* For detailed discussions, see A. M. Lane, Nuclear Theory, Frontiers in Physics, W. A. Benjamin, Inc., 1964.

** This, of course, is the original reason for the choice (3) of the interaction matrix. In BCS $\alpha, \bar{\alpha}$ denote the quantum numbers

$$\alpha = (\vec{k}, m_s) \quad , \quad \bar{\alpha} = (-\vec{k}, -m_s)$$

Meanwhile, a number of elegant equivalent methods have been developed[10]. Probably the most elegant one is due to Bogoljubov, who transforms the b_ν canonically into "quasi-particle" creators and annihilators by:

$$\alpha_\nu = u_\nu b_\nu - v_\nu b_{\bar{\nu}}^+ \quad \text{with} \quad u_\nu^2 + v_\nu^2 = 1 \tag{7}$$

The α_ν have again fermion character and the state given by BCS as the ground state can be shown to be the vacuum state $|0\rangle$ for the "quasi-particles" with the transformation in (7):

$$v_\nu = s_\nu \sqrt{\tfrac{1}{2}\left(1 - \frac{\varepsilon_\nu - \lambda}{E_\nu}\right)} \tag{8}$$

$$u_\nu = \sqrt{\tfrac{1}{2}\left(1 + \frac{\varepsilon_\nu - \lambda}{E_\nu}\right)} \tag{8a}$$

where

$$s_\nu = \begin{cases} 1 & \text{for} \quad m > 0 \\ -1 & \text{"} \quad m < 0 \end{cases}$$

λ is determined by the number of particles N:

$$N = \sum_\nu v_\nu^2 \tag{9}$$

and

$$E_\nu = \sqrt{(\varepsilon_\nu - \lambda)^2 + \Delta^2} \tag{10}$$

is the energy of the one quasi-particle state $|\alpha_\nu^+\rangle = \alpha_\nu^+|0\rangle$

$$\Delta = \tfrac{1}{2} G \sum_\nu u_\nu v_\nu = \tfrac{1}{4} G \sum_\nu \frac{\Delta}{E_\nu} \tag{11}$$

is the pair correlation energy.

In terms of the old particle operators $|0\rangle$ is as given by BCS:

$$|0\rangle = \prod_{\nu>0} (u_\nu + v_\nu b_\nu^+ b_{\bar{\nu}}^+)|\ \rangle \tag{12}$$*

where $|\ \rangle$ is the vacuum for the b_ν particles. $\nu > 0$ means $m > 0$ in $\nu = (n, \ell, j, m)$

The energy of $|0\rangle$ is

$$E_N = \sum_\nu \varepsilon_\nu v_\nu^2 - \frac{\Delta^2}{G} \tag{13}$$

Since the expectation of the number operator $b_\nu^+ b_\nu$ in the state $|0\rangle$ is

$$\langle 0| b_\nu^+ b_\nu |0\rangle = v_\nu^2 \tag{14}$$

v_ν^2 gives the probability for the state ν to be occupied by a particle.

Hence the single particle levels are filled up to the energy which is therefore the Fermi level of the system. The distribution is a step function in the absence of pairing interaction $(G=0)$. For $G \neq 0$ the distribution smears out around λ with a width of Δ.

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* We neglect throughout self-energy effects which add to ε_ν the term $-G v_\nu^2$, which is $<$ 5 per cent.

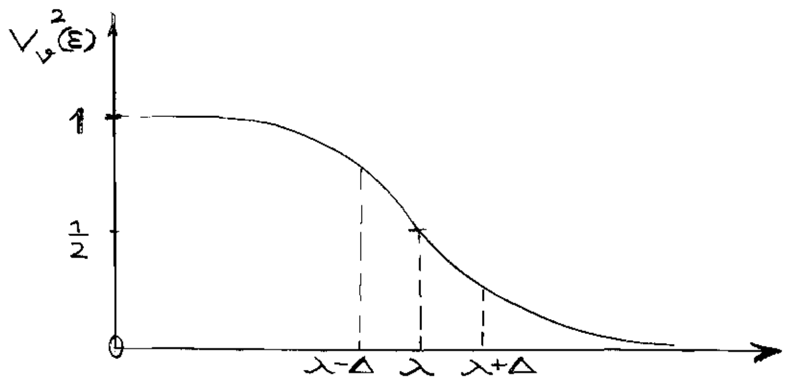

Figure 1. The Occupation Probability of the Single Particle Levels as a Function of the Energy. The Width of the Smeared Out Zone Around the Fermi Level is .

The BCS state can be shown to behave like a system of pairs with Boson-character[11,13], so-called Cooper-pairs[12], which are bound states of two mutually time reversed particles in the $I=0$ state due to the interaction introduced above. Because of this occurrence of bound states the BCS state cannot come out of a perturbation treatment of the interaction H_P. The state and its energy do not depend analytically on the interaction strength G in H_P. From this pair character it is also clear that the BCS state can only yield a good approximation for the ground state of an even number of particles N. In the case of an odd number N + 1, one particle, say ν, cannot find a partner and sits therefore alone in one orbit, the time reversal state being unoccupied. The states ν and $\bar{\nu}$ are unaccessible for the paired particles; they are "blocked." The occupation probability looks like the illustration given in Figure 2.

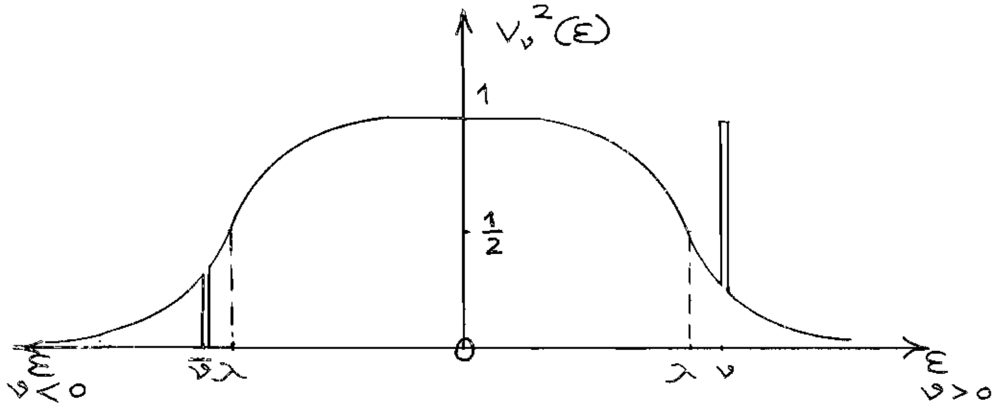

Figure 2. Occupation Probability of the Energy Levels for an Odd Number of Particles. The Right Axis Gives the Particles with $m>0$, the Left One Gives Their Time Reversed with $m<0$.

When calculating λ and Δ this fact: $V_\nu = 1$, $V_{\bar{\nu}} = 0$, must be taken into account. Its being neglected gives a mistake up to 30 per cent in strongly deformed nuclei as was pointed out by Nilsson Prior[14], and Soloviev[15]. As a rule Δ calculated without blocking is 20 per cent larger than Δ with blocking, as can be seen from Reference 14, page 31. Without machine calculations, however, one cannot include this effect and we shall neglect it, realizing that here is a source of large errors.

In this approximation the N + 1 odd-particle ground state is $\alpha_{\nu_1}^{+}|0\rangle$, the energy being

$$E_{N+1} = E_N + \lambda + E_{\nu_1} \tag{15}$$

is a single particle level which lies nearest to the Fermi energy λ.

A form of excitation of the (even) N particle system is the breaking apart of one pair by going from $|0\rangle$ to $\alpha_{\nu_1}^{+}\alpha_{\nu_2}^{+}|0\rangle$. the two quasi-particle state. This requires the energy:

$$E_{\nu_1} + E_{\nu_2} \tag{16}$$

In the weak interaction limit this energy is $\geq 2\Delta$ and appears as energy gap in the excitation spectrum for example in superconductors.

In even-even nuclei, where strong pairing interaction is present, collective excitations have mostly a lower energy than the pair excitation such that the lowest excitation energy is not immediately connected with the E_ν . However, the pairing energy is observable in the nucleus in the purest way: The observed even-odd mass differences defined as

$$P = 2E_{N+1} - E_{N+2} - E_N = 2E_{\nu_1} \qquad *(17)$$

for one nucleon number fixed gives directly the quasi-particle energy.

The (odd) N + 1 particle system can simply be excited by shifting the unpaired particle into another orbit, say from ν_1 to ν_2. The energy varies by an amount of

$$E_{\nu_2} - E_{\nu_1} \approx \frac{\varepsilon_{\nu_2} - \varepsilon_{\nu_1}}{\sqrt{1 + \left(\frac{\Delta}{\bar{\varepsilon} - \lambda}\right)^2}} \quad ; \quad \bar{\varepsilon} = \frac{1}{2}(\varepsilon_{\nu_1} + \varepsilon_{\nu_2}) \tag{18}$$

which is even smaller than the energy difference of the Nilsson levels.

$\varepsilon_{\nu_2} - \varepsilon_{\nu_1}$

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* The right part of the equation follows immediately from (15).

This is the well-known compression effect of the single particle levels in odd-even and odd-odd nuclei.

These are all relations we shall need from the pairing force theory in order to calculate the nuclear ground state energy.

B. The Quadrupole Force

Such drastic simplifications of the interaction Hamiltonian as in the case of the pairing force are impossible. However, for the production of quadrupole vibrations, the angular dependence of the quadrupole force $f_2(r_1, r_2) \cdot P_2(\cos\vartheta)$ (see (1)) is evidently more important than its radial dependence. We therefore have some freedom in the choice of the radial function $f_2(r_1, r_2)$ and take it in such a way, that the interaction matrix is most easily evaluable:

$$f_2(r_1, r_2) \sim r_1^2 \cdot r_2^2$$

With this choice one also will get a nice physical interpretation of the interaction.

The quadrupole force adds to the Hamiltonian a term:

$$H_q = \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | f_2(r_1, r_2) P_2(\cos\vartheta) | \gamma\delta \rangle b_\alpha^+ b_\beta^+ b_\gamma b_\delta \tag{19}$$

where the interaction matrix is:

$$\langle \alpha\beta | f_2 \cdot P_2 | \gamma\delta \rangle = \int \bar{\psi}_\alpha(1) \bar{\psi}_\beta(2) f_2(r_1, r_2) P_2(\cos\vartheta) \psi_\gamma(2) \psi_\delta(1) d1 d2$$

ψ_α being the single particle states with the quantum number α. Now $f_2(r_1, r_2) \sim r_1^2 \cdot r_2^2$ and $P_2(\cos\vartheta)$ can be expressed by spherical harmonics as:

$$P_2(\cos\vartheta) \sim \sum_M Y_{2,M}(1) Y_{2,M}(2)$$

Therefore, r_1 and r_2 terms can be separated and the matrix becomes:

$= -\frac{\chi}{2} \sum_M q_{\alpha\delta}^M q_{\beta\gamma}^M$ with some constant χ. The

$$q_{\alpha\delta}^M = \langle \alpha | r^2 Y_{2,M} | \delta \rangle = \int \bar{\psi}_\alpha(1) r^2 Y_{2,M}(1) \psi_\delta \, d1$$

are the single particle quadrupole moments.

Hence, the Hamiltonian is:

$$H_q = -\frac{\chi}{2} \sum_{M;\alpha\beta\gamma\delta} q_{\alpha\delta}^{M} q_{\beta\gamma}^{M} b_\alpha^+ b_\beta^+ b_\gamma b_\delta \tag{20}$$

If we introduce the quadrupole moment operator of the nucleus

$$D^M = \sum_{\alpha,\beta} q_{\alpha\beta}^{M} b_\alpha^+ b_\beta \tag{21}$$

we can write H_q as

$$H_q = -\frac{\chi}{2} \sum_M D^M \cdot D^M - \frac{\chi}{2} \sum_{M;\alpha\beta\gamma} q_{\alpha\beta}^{M} q_{\beta\gamma}^{M} b_\alpha^+ b_\gamma \tag{22}$$

The second term contains one summation less and is of the order of $\frac{1}{\Omega}$ compared to the first term, if Ω denotes the number of levels under consideration. We neglect this term since it gives a mistake of the same order as the neglection of the "blocking" effect in (15).

Then the Hamiltonian is:

$$H_q = -\frac{\chi}{2} \sum_M D^M \cdot D^M \tag{23}$$

$$= -\frac{\chi}{2} \sum_{M;\alpha,\beta} D^M q_{\alpha\beta}^{M} b_\alpha^+ b_\beta \tag{24}$$

It describes a coupling of all single particle quadrupole moments to the total quadrupole moment of the nucleus. χ is called the quadrupole coupling constant. The total Hamiltonian of the nucleus is now:

$$H = H_N + H_P + H_q \tag{25}$$

The ground state of $H_N + H_q$ is the BCS state given in (12). For the calculation of the effect of H_q on the ground state there exists at present two different methods.

(a) The random phase approximation, which was first used in studying collective effects in electron gases.

(b) The direct collective treatment which was introduced by Inglis as the "cranking method" for finding rotational moments of inertia of nuclei. The theoretical foundation for the second method is rather vague. It has, however, been shown that the more exact random phase approximation yields the same result as the cranking method if the quadrupole frequency is small compared to the quasi-particle energies E_ν[2].

The cranking method has two striking advantages: It gives a physical picture of the collective effect and it allows the calculation of the vibration frequency in a simple way. We therefore shall use this method.

We consider a fixed space direction, say Z, and introduce as a collective parameter: the quadrupole moment of the nucleus in this direction, Q, which is determined by

$$Q = \langle D^\circ \rangle \tag{26}$$

the expectation value of the quadrupole operator

$$D^\circ = \sum_{\alpha,\beta} q^\circ_{\alpha\beta}\, b^+_\alpha b_\beta \qquad *$$

* The superscript 0 will be omitted in the subsequent considerations.

in the given nuclear state. Then the quadrupole vibrations are described by the collective Hamiltonian:

$$H_{Coll} = \frac{C}{2} Q^2 + \frac{B}{2} \dot{Q}^2 \tag{27}$$

where C is the "direction force" and B is the moment of inertia with respect to the collective parameter Q.* If one finds C and B the frequency of the collective vibration is given by

$$\omega = \sqrt{\frac{C}{B}} \tag{28}$$

The zero point energy of this vibration is $\frac{\omega}{2}$. This energy is the contribution of the quadrupole interaction to the ground state energy.

One obtains constants C and B by adiabatic perturbation theory. For this one assumes the orbital frequency of the single particles in the nucleus to be high compared to the macroscopic vibration frequency. Then the single particles do not realize the collective motion and at each moment the intrinsic nuclear state is just determined by the static ground state corresponding to a fixed Q.**

In order to find this state for any prescribed small Q one adds the quadrupole momentum operator D together with a Lagrangian multiplier to the Hamiltonian, forming

$$\bar{H} = H - \mu D \tag{29}$$

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* See S. A. Mozkowski, Handbuch der Physik, 39, 411, (1957).

** The adiabatic condition is not too well satisfied. Both frequencies lie in the same order of magnitude: 10^{21} sec^{-1} or 1 Mev. Only the success of the theory seems to justify this approach.

and looks for the ground state of the new Hamiltonian $\overline{H}$. The multiplier is afterwards determined by the self consistency condition:

$$Q = \langle D \rangle$$

H is composed of $H_N + H_p + H_q$. For $H_N + H_p$ we possess the BCS state as approximate ground state. We suppose $H_q - \mu D$ to be small enough such that we can treat it as a perturbation.

H_q is still a two-particle operator.

$$H_q = -\frac{1}{2} \chi \cdot D \cdot D$$

For our problem we approximate H_q by the one-particle operator:

$$\widetilde{H}_q = \frac{1}{2} \chi Q^2 - \chi Q D \tag{30}$$

The meaning of this approximation will be discussed in a moment. With $\widetilde{H}_q$ the perturbation is

$$H_q - \mu D \approx \frac{1}{2} \chi Q^2 - \tilde{\mu} D \; ; \; \tilde{\mu} = \mu + \chi Q$$

and the second term can be included in $H_N = \sum_\alpha \varepsilon_\alpha b_\alpha^+ b_\alpha$ forming

$$H_N' = \sum_{\alpha, \beta} \widetilde{\varepsilon}_{\alpha\beta} \, b_\alpha^+ b_\beta \tag{31}$$

with

$$\widetilde{\varepsilon}_{\alpha\beta} = \varepsilon_\alpha \delta_{\alpha\beta} - \tilde{\mu} \, q_{\alpha\beta} \tag{32}$$

The ground state of

$$H_N' + H_p + \frac{\chi}{2} Q^2 \tag{33}$$

is readily found, if we succeed to transform the matrix $\widetilde{\varepsilon}_{\alpha\beta}$ to diagonal form, say

$$\widetilde{\varepsilon}_{\alpha'\beta'} = \widetilde{\varepsilon}_{\alpha'}\,\delta_{\alpha'\beta'}$$

Suppose

$$|\alpha'\rangle = \sum_{\alpha} \varphi_{\alpha\alpha'}\,|\alpha\rangle \tag{34}$$

are new single-particle states, in which

$$\varepsilon_{\alpha'\beta'} - \widetilde{\mu}\, q_{\alpha'\beta'} = \widetilde{\varepsilon}_{\alpha'}\,\delta_{\alpha'\beta'} \tag{35}$$

Then the transformation coefficients have to satisfy the equation:

$$\sum_{\alpha\beta} \varphi_{\alpha'\alpha}\,\varepsilon_{\alpha\beta}\,\varphi_{\beta\beta'} - \widetilde{\mu}\,\varphi_{\alpha'\alpha}\,q_{\alpha\beta}\,\varphi_{\beta\beta'} = \widetilde{\varepsilon}_{\alpha'}\,\delta_{\alpha'\beta'} \tag{36}$$

and, since $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha}\,\delta_{\alpha\beta}$ and

$$\sum_{\alpha'} \varphi_{\gamma\alpha'}\,\varphi_{\alpha'\alpha} = \delta_{\gamma\alpha}$$

$$(\varepsilon_{\gamma} - \widetilde{\varepsilon}_{\beta'})\,\varphi_{\gamma\beta'} = \widetilde{\mu} \sum_{\beta} q_{\gamma\beta}\,\varphi_{\beta\beta'} \tag{37}$$

If some states are degenerate, we can combine them linearly such that $q_{\alpha\beta}$ is diagonal.

Then the familiar perturbation solution of this eigenvalue problem is, with the assumption: $\varphi_{\alpha\alpha'} = \delta_{\alpha\alpha'} + \varphi^{(1)}_{\alpha\alpha'}$

$$\varphi^{(1)}_{\alpha\beta'} = \begin{cases} \dfrac{\widetilde{\mu}\, q_{\alpha\beta}}{\varepsilon_{\alpha} - \varepsilon_{\beta}} & \text{for } \varepsilon_{\alpha} \neq \varepsilon_{\beta} \\ 0 & \text{" } \varepsilon_{\alpha} = \varepsilon_{\beta} \end{cases} \tag{38}$$

The energies are in second order:

$$\widetilde{\varepsilon}_{\alpha'} = \varepsilon_\alpha - \widetilde{\mu}\, q_{\alpha\alpha} - \widetilde{\mu}^2 p_\alpha \tag{39}$$

where:

$$p_\alpha = \sum_\beta{}' \frac{|q_{\alpha\beta}|^2}{\varepsilon_\alpha - \varepsilon_\beta} \tag{40}$$

the prime excludes summation over β with $\varepsilon_\beta = \varepsilon_\alpha$.

The quadrupole moment in these new states $|\alpha'\rangle$ is:

$$q_{\alpha'} = \varphi_{\alpha\alpha}\, q_{\alpha\beta}\, \varphi_{\beta\alpha'} = q_{\alpha\alpha} + 2\widetilde{\mu}\, p_\alpha \tag{41}$$

and the single particle energies of $|\alpha'\rangle$:

$$\varepsilon_{\alpha'} = \varepsilon_\alpha + \widetilde{\mu}^2 p_\alpha \tag{42}$$

In terms of $\widetilde{\varepsilon}_{\alpha'}$, H_N' reads:

$$H_N' = \sum_{\alpha'} \widetilde{\varepsilon}_{\alpha'}\, b^+_{\alpha'} b_{\alpha'} \tag{43}$$

where $b^+_{\alpha'}$ is now the creator of a particle in the state $|\alpha'\rangle$ and $b_{\alpha'}$ is its annihilator. The pairing interaction part H_p has the same form in the new states:

$$H_p' = -\frac{G}{4} \sum_{\alpha'\beta'\gamma'\delta'} b^+_{\alpha'} b^+_{\beta'} b_{\gamma'} b_{\delta'} \tag{44}$$

Since the transformation $|\alpha\rangle \rightarrow |\alpha'\rangle$ is unitary.

But the ground state of Hamiltonian $H_N' + H_p'$ is just the BCS state

$$|0'\rangle = \prod_{\alpha'>0} (u_{\alpha'} + v_{\alpha'} b^+_{\alpha'} b^+_{\bar{\alpha}'}) |\ \rangle \tag{45}$$

where now the indices refer to the states $|\alpha'\rangle$. The equations for $v_{\alpha'}$ clearly are: (cf. (12))

$$\begin{aligned} v_{\alpha'} &= s_{\alpha'} \sqrt{\tfrac{1}{2}\left(1 - \frac{\hat{\varepsilon}_{\alpha'} - \lambda}{E_{\alpha'}}\right)} \\ u_{\alpha'} &= \sqrt{\tfrac{1}{2}\left(1 + \frac{\hat{\varepsilon}_{\alpha'} - \lambda}{E_{\alpha'}}\right)} \end{aligned} \tag{46}$$

For odd nuclei always $\alpha^+_{\nu'}|0\rangle$ has to be taken as ground state instead of $|0'\rangle$.

The ground state energy is:

$$\begin{aligned} W = \langle 0'|H|0'\rangle &= \langle 0'|H_N + H_p|0'\rangle + \langle 0'|H_q|0'\rangle \\ &= U - \frac{\chi}{2} Q^2 \end{aligned} \tag{47}$$

where U is the BCS energy of the state $|0'\rangle$ given in IIa (13).

Hence:

$$W = \sum_{\alpha'} \varepsilon_{\alpha'} v_{\alpha'}^2 - \frac{\Delta'^2}{2} - \frac{\chi}{2} Q^2 \tag{48}$$

We now consider the approximation $\hat{H}_q$ taken for H_q.

The state $|0'\rangle$ is the vacuum for the quasi-particles (cf. 7).

$$\alpha^+_{\nu'} = u_{\nu'} b^+_{\nu'} - v_{\nu'} b_{\bar{\nu}'} \tag{49}$$

If one introduces these operators into H_q, we can write

$$H_q = -\frac{1}{2}\chi Q^2 - \chi Q N(D) - \frac{\chi}{2} N(D) N(D) \tag{50}$$

where, as before $Q = \langle 0' | D | 0' \rangle$. N(D) is the usual normal product of the operator D in terms of the quasi-particle operators $\alpha_{\nu'}^{+} \alpha_{\nu'}$. With this, (50) becomes:

$$H_q = \hat{H}_q - \frac{\varkappa}{2} N(D)\, N(D) \tag{51}$$

Hence our approximation $H_q \approx \hat{H}_q$ has neglected the term $-\frac{\varkappa}{2} N(D) N(D)$ which gives an interaction between the quasi-particles $\alpha_{\nu'}^{+}$. But this is a basic approximation in all the BCS theory, where quasi-particles are always considered as independent. The BCS state itself is only determined with this accuracy.*10

The moment of inertia with respect to the collective parameter Q is obtained by the well-known cranking formula of Inglis:

$$B = 2 \sum_{i} \frac{\langle i | \frac{\partial}{\partial Q} | 0 \rangle}{E_i - E_0} \tag{52}$$

where $|0\rangle$ is the nuclear ground state with Q=0 and $|i\rangle$ an orthonormal set containing $|0\rangle$ with the energies E_i.

The effect of a change of Q on the ground state $|0\rangle$ is two-fold as we have seen in the preceding discussion:

(a) The wave functions $|\alpha\rangle$ are changed to $|\alpha'\rangle$ by an amount, say: $K \cdot Q$, where K is the generator of this change.

(b) The single particle energies shift from $\varepsilon_\alpha \longrightarrow \tilde{\varepsilon}_{\alpha'}$.

The second effect can be shown to be the dominating one.[1] Therefore:

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* Clearly this term determines the life times of excited collective states.

$$\frac{\partial}{\partial Q}|0\rangle = \frac{\partial}{\partial Q}\prod_{\nu>0}(u_\nu + v_\nu b_\nu^+ b_{\bar\nu}^+|\rangle \tag{53}$$

$$= \sum_{\nu>0}\left(\frac{\partial u_\nu}{\partial Q} + \frac{\partial v_\nu}{\partial Q} b_\nu^+ b_{\bar\nu}^+\right)\prod_{\substack{\alpha\neq\nu\\ \alpha>0}}(u_\alpha + v_\alpha b_\alpha^+ b_{\bar\alpha}^+)|\rangle$$

The first factor is, because of the condition $u_\nu^2 + v_\nu^2 = 1$

$$\frac{\partial u_\nu}{\partial Q} + \frac{\partial v_\nu}{\partial Q} b_\nu^+ b_{\bar\nu}^+ = -\frac{1}{v_\nu}\frac{\partial u_\nu}{\partial Q}(u_\nu b_\nu^+ b_{\bar\nu}^+ - v_\nu) \tag{54}$$

$$= \frac{1}{v_\nu}\frac{\partial u_\nu}{\partial Q}\alpha_\nu^+ \alpha_{\bar\nu}^+ (u_\nu + v_\nu b_\nu^+ b_{\bar\nu}^+)$$

such that:

$$\frac{\partial}{\partial Q}|0\rangle = \sum_\nu \frac{1}{v_\nu}\frac{\partial u_\nu}{\partial Q}\alpha_\nu^+ \alpha_{\bar\nu}^+|0\rangle \tag{55}$$

where α_ν^+ are the quasi-particle creators $\alpha_\nu^+ = u_\nu b_\nu^+ - v_\nu b_\nu$ as introduced in IIa (7).

The operator $\frac{\partial}{\partial Q}$ can only possess matrix elements from the states $\langle 0|\alpha_\nu \alpha_{\bar\nu}$ to $|0\rangle$ which have the energy $E_\nu + E_{\bar\nu} + E_0$. Such that the moment of inertia:

$$B = \sum_\nu \frac{1}{v_\nu^2}\left(\frac{\partial u_\nu}{\partial Q}\right)^2 \frac{1}{E_\nu} \tag{56}$$

For an odd particle number with the Q=0 ground state $\alpha_{\nu_1}^+|0\rangle$ the summation in (56) runs over all $\nu \neq \nu_1$.

We now have derived all necessary general relations we shall need for this calculation of the ground state energy of the nucleus. The further procedure requires more detailed assumptions concerning the level structure.

CHAPTER IV

CALCULATION OF THE GROUND STATE ENERGY

As we said in the beginning we want to restrict ourselves to spherical, single-closed shell nuclei. Their single-particle levels form fairly well separated highly degenerate groups. These groups we shall call for brevity shells. They do, in general, not coincide with the major closed shells between magic numbers. For example, the levels with numbers between 40 and 50, 70 and 82, 100 and 114 form such a shell in our sense, but only the first one is a major shell.

Then we make the assumption about the pairing force, that only particles inside one shell interact strongly with one another. One has tried to justify this by consideration of the matrix elements $\langle \alpha\bar{\alpha} | \delta(\vec{r}_1 - \vec{r}_2) | \beta\bar{\beta} \rangle$ of the δ-function interaction. However, one finds non-diagonal elements even across a major closed shell quite large. (16)

At any rate, this assumption has led to a hitherto successful theory such that we shall adopt it.

Our model wants to consider only pairing interactions between particles of equal isotopic spin, i.e., p-p or n-n interactions. The neglect of n-p forces is still a great weakness in the present pairing theory. For our single closed shell nuclei, however, their effect might be supposed to be small because of the apparent stability of closed-shell configurations against any kind of perturbation.

With these assumptions the nucleus without quadrupole interaction decomposes into a set of independent shells of nucleons, all but one being completely filled. In order to find the ground state with only the pairing force present, we can therefore restrict ourselves to the consideration of just one shell.

A. The Ground State Without Quadrupole Interaction

Consider one shell of Nilsson levels. They are all degenerate. In this case pairing force theory alone would be very easy. For the determination of the collective oscillator potential C, however, we saw in III(B) that we have to find the ground state for different fixed quadrupole momentum Q, which changes the single particle energies $\mathcal{E}_\alpha$ to $\mathcal{E}_{\alpha'}$. Therefore, the degeneracies, existing for Q=0, are in general removed, only time reversed states maintaining the same energy.

We shall therefore consider immediately the general case that the levels of the shell are split and lie between the energies $\mathcal{E}^{I}$ and $\mathcal{E}^{II}$. We assume the level spacing to be small compared to the correlation energy Δ, since Δ is in the order of the even-odd mass difference, i.e., $\sim$ 1 MeV whereas the levels are arbitrarily close together as long as one keeps Q small enough. Exception is only the case of a completely filled shell, where all $V_\nu=1$, $u_\nu=0$ and hence $\Delta=\frac{G}{2}\sum_\nu u_\nu V_\nu$ vanishes exactly. For only partly filled shells we can approximate all sums over ν in the equations of III by integrals.

Define $\rho(\varepsilon)$ as the density of pair states (i.e., $\nu, \bar{\nu}$ are only counted once). Then Δ is determined by (11):

$$1 = \tfrac{1}{2} G \int_a^b \rho(\varepsilon) \frac{d\varepsilon}{\sqrt{\varepsilon^2+\Delta^2}} \tag{57}$$

with

$$a = \varepsilon' - \lambda \qquad \text{(cf. (9))}$$
$$b = \varepsilon'' - \lambda$$

hence:

$$\Delta = \frac{1}{sh 2\eta}\left[b^2 + a^2 - 2ab\, ch\, 2\eta\right] \tag{58}$$

where

$$\eta = \frac{1}{\bar{\rho} G} \tag{59}$$

and $\bar{\rho}$ is an average level density of the shell defined as

$$\bar{\rho} := \int_a^b \rho(\varepsilon) \frac{d\varepsilon}{\sqrt{\varepsilon^2+\Delta^2}} \cdot \left(\int_a^b \frac{d\varepsilon}{\sqrt{\varepsilon^2+\Delta^2}}\right)^{-1} \tag{60}$$

According to whether $\eta \gg 1$ or < 1 we shall call the interaction weak or strong. Since in our case $\bar{\rho}$ is arbitrarily small for sufficiently small Q we can later go in all results to the limit $\eta < 1$.

The equation for Δ still contains λ , which is determined by III(A)(9) through

$$N = \int_a^b \left(1 - \frac{\varepsilon - \lambda}{\sqrt{\varepsilon^2+\Delta^2}}\right) \rho(\varepsilon)\, d\varepsilon \tag{63}$$

Now we must assume a particular $\rho(\varepsilon)$ in order to solve these equations.

The only ρ which approximates the level density in all shells equally well (or poorly) is

$$g = \text{const} = g_0$$

which is connected with the total number of pairing states Ω of the shell by

$$g = \frac{\Omega}{\varepsilon'' - \varepsilon'} \tag{64}$$

Then

$$\bar{g} = g_0 \tag{65}$$

and

$$\lambda = \frac{\varepsilon'' + \varepsilon'}{2} - \frac{1}{2}(\varepsilon'' - \varepsilon')\,\mathcal{X}_N \,\mathrm{cth}\,\eta \tag{66}$$

where

$$\mathcal{X}_N = 1 - \frac{N}{\Omega} \tag{67}$$

is the parameter showing how empty the shell is

$$\mathcal{X}_N = \begin{cases} 1 & \text{for empty shell} \\ 0 & \text{for half filled shell} \\ -1 & \text{for full shell } (N=2\Omega) \end{cases}$$

We see here a first effect of the pairing force: the stronger the interaction, the stronger is the increase of the Fermi energy with N.

Without any pair interaction $\frac{d\lambda}{dN}$ would be determined by $\frac{1}{2g_0}$, now it is $\frac{\mathrm{cth}\,\eta}{2g_0}$.

For the case of the strong interaction limit, $\eta < 1$, λ is dominated by G:

$$\lambda = \frac{\varepsilon'' + \varepsilon'}{2} - \frac{G}{2} \chi_N \tag{68}$$

Now we put (68) into (58) and get for :

$$\Delta = \frac{\varepsilon'' - \varepsilon'}{2 \mathrm{sh}\, \eta} \Theta_N^{1/2} \tag{69}$$

where

$$\Theta_N = 1 - \chi_N^2 = \frac{2N}{\Omega}\left(1 - \frac{N}{2\Omega}\right) \tag{70}$$

is the statistical factor. For the scattering from the occupied to the empty states. Θ is 0 for full and empty shells and 1 for half filled shells.

With (69) and (13) the ground state energy of the shell becomes:

$$\begin{aligned} U &= \sum_\nu \varepsilon_\nu V_\nu^2 - \frac{\Delta^2}{G} \\ &= \sum_\nu \varepsilon_\nu V_\nu^2 - \frac{\varepsilon'' - \varepsilon'}{4 \mathrm{sh}^2 \eta} \Theta_N \end{aligned} \tag{71}$$

The ground state of a completely filled shell is trivially:

$$U' = \sum_\nu{}' \varepsilon_\nu$$

Since

$$\Delta = 0$$

$\sum_\nu{}'$ denotes the summation over the levels of a closed shell:

The energy of a quasi-particle is

$$\begin{aligned} E_\nu^2 = (\varepsilon_\nu - \lambda)^2 + \Delta^2 &= \frac{1}{4}(\varepsilon'' - \varepsilon')^2 \Theta_N (\mathrm{cth}^2 \eta - 1) \\ &+ \left[\varepsilon_\nu - \frac{\varepsilon'' + \varepsilon'}{2} + \frac{1}{2}(\varepsilon'' - \varepsilon') \chi_N \mathrm{cth}\, \eta\right]^2 \end{aligned} \tag{72}$$

which for weak interaction and ν_1 nearest to the Fermi surface is

$$E_{\nu_1} = \Delta \tag{73}$$

For strong interactions $\eta < 1$, however,

$$E_\nu = \frac{1}{2} G\Omega \tag{74}$$

Hence the excitation of an even nucleus, produced by breaking up a pair, requires the energy

$$E_{\nu_1} + E_{\nu_2} = G\Omega \qquad \text{(cf. (16))} \tag{75}$$

The even-odd mass difference given by , is also

$$P = G\Omega \qquad \text{(cf. (18))} \tag{76}$$

If we add now the energies of all shells together, we obtain as the ground state energy of an even nucleus in the absence of the quadrupole interaction

$$U = \sum_\nu \varepsilon_\nu V_\nu^2 - \frac{\Delta^2}{G} \tag{77}$$

where ν runs in $\sum$ overall states.

$\frac{\Delta^2}{G}$ is given by (69) and reduces in the strong interaction limit $\eta < 1$ to

$$\frac{\Delta^2}{G} = \frac{\Omega^2}{4} G \Theta_N \left(1 - \frac{\eta^2}{3}\right) \tag{78}$$

For an odd nucleus one has to add $E_\nu = \frac{1}{2} G\Omega$ to the energy.

B. Inclusion of the Quadrupole Force

We have to find the collective Hamiltonian (27). The general procedure has been demonstrated in III(B).

We first determine the dependence of the ground state energy on Q and obtain C. Take a spherical Nilsson-well with the single-particle energies ε_α. Let be $|0\rangle$ the BSC state for the corresponding Hamiltonian $H_N + H_p$. The interaction $H_q - \mu D$ has the effect of shifting the single-particle energies in H_N from ε_α to

$$\tilde{\varepsilon}_{\alpha'} = \varepsilon_\alpha - \tilde{\mu} q_{\alpha\alpha} - \tilde{\mu}^2 p_\alpha \quad \text{(cf. (39))}$$

where $q_{\alpha\alpha}$ are the single-particle quadrupole moments of the states $|\alpha\rangle$ in the spherical Nilsson well. By this process, degenerate levels ε_α in one shell are split and spread out over an interval

$$\Delta\varepsilon = \tilde{\mu} q + \tilde{\mu}^2 p \tag{79}$$

where
$$q = \max q_{\alpha\alpha} - \min q_{\alpha\alpha}$$
$$p = \max p_\alpha - \min p_\alpha$$

inside the shell. If we denote by

$$\begin{aligned} \varepsilon' &= \varepsilon - \max \tilde{\mu} q_{\alpha\alpha} - \max \tilde{\mu}^2 p_\alpha \\ \varepsilon'' &= \varepsilon - \min \tilde{\mu} q_{\alpha\alpha} - \min \tilde{\mu}^2 p_\alpha \end{aligned} \tag{80}$$

we come back to the initial condition for the calculation in (A). $\mathcal{G}_0$, Δ and η are now all determined by these ε', ε''.

The ground state energy for a fixed Q is according to III(B)(48)

$$W = \sum_{\alpha'} \varepsilon_{\alpha'} v_{\alpha'}^2 - \frac{\Delta'^2}{G} - \frac{\chi}{2} Q^2$$

Using the relations

$$\varepsilon_{\alpha'} = \varepsilon_\alpha + \tilde{\mu}^2 p_\alpha \qquad \text{(cf. (42))}$$

the first sum becomes:

$$\sum_{\alpha'} \varepsilon_{\alpha'} v_{\alpha'}^2 = \sum_{\alpha} \varepsilon_\alpha v_{\alpha'}^2 + \tilde{\mu}^2 \sum_{\alpha} p_\alpha v_{\alpha'}^2 \tag{81}$$

The left side is constant, since the closed shells contribute $\sum_{\alpha}' \varepsilon_\alpha$ and the unfilled shell gives, because of its degeneracy,

$$\sum_{\alpha}^{\lambda} \varepsilon_\alpha v_{\alpha'}^2 = \varepsilon \sum_{\alpha}^{\lambda} v_{\alpha'}^2 = \varepsilon \cdot N$$

if ε denotes the common energy of all levels and N denotes the number of particles in the unfilled shell. Hence:

$$\sum_{\alpha} \varepsilon_\alpha v_{\alpha'}^2 = \sum_{\alpha}' \varepsilon_\alpha + \varepsilon N \tag{82}$$

We now consider the right side in (81).

We decompose also this sum in a part $\sum'$ over closed shells and part $\sum^{\lambda}$ over the unfilled shell.

$$\tilde{\mu}^2 \sum_{\alpha} p_\alpha = \tilde{\mu}^2 \sum_{\alpha}' p_\alpha + \tilde{\mu}^2 \sum_{\alpha}^{\lambda} p_\alpha v_{\alpha'}^2$$

The Lagrange multiplier in the first part can be eliminated by the quadrupole moment of the closed shells, call it Q_c:

Since

$$Q_c = \sum_{\alpha'}' q_{\alpha'} \tag{83}$$

and

$$q_{\alpha'} = q_{\alpha\alpha} + 2\tilde{\mu}\, p_\alpha \qquad \text{(cf. (41))}$$

and furthermore

$$\sum_\alpha{}' q_{\alpha\alpha} = 0 \tag{84}$$

because of the spherical symmetry of the Nilsson well, we have:

$$Q_c = 2\tilde{\mu} \sum_\alpha{}' p_\alpha \tag{85}$$

The $\sum_\alpha{}' p_\alpha$ is a constant, say P, but then

$$\tilde{\mu}^2 \sum_\alpha{}' p_\alpha = \frac{Q_c^2}{4P} \tag{86}$$

Substituting all this into W we obtain:

$$W = \sum_\alpha{}' \varepsilon_\alpha + \varepsilon \cdot N + \frac{Q_c^2}{4P} - \frac{\Delta'^2}{G} - \frac{\chi}{2} Q^2 - \tilde{\mu}^2 \sum_\alpha^\lambda p_\alpha v_{\alpha'}^2 \tag{87}$$

In this equation we can still neglect the last term compared to χQ^2:

To see this, we decompose

$$Q = Q_c + Q_\lambda \tag{88}$$

where Q_λ denotes the quadrupole moment of the unfilled shell, given by:

$$Q_\lambda = \sum_{\alpha'}^\lambda q_{\alpha'} v_{\alpha'}^2 \tag{89}$$

$$Q_\lambda = \sum_\alpha^\lambda q_{\alpha\alpha} v_{\alpha'}^2 + 2\tilde{\mu} \sum_\alpha^\lambda p_\alpha v_{\alpha'}^2 \tag{90}$$

The right side is a result of the perturbation $\tilde{\mu} q_\alpha$ on the energy levels $\varepsilon_\alpha \equiv \varepsilon$ in the shell. Hence, it is of the order $\frac{\tilde{\mu} q_\alpha}{\varepsilon}$.

Since $\tilde{\mu} = \frac{Q_c}{2P}$, we can neglect this term as long as

$$\frac{Q_c}{2P} \frac{q_\alpha}{\varepsilon} \ll 1 \tag{91}$$

This condition, however, can always be supposed to hold, since we investigate the energy dependence of W only in the neighborhood of Q=0 where also $Q_c=0$. Then

$$Q_\lambda = \sum_\alpha^\lambda q_{\alpha\alpha} v_{\alpha'}^2 \tag{92}$$

The last term in (87) now is negligible against $\chi Q Q_\lambda$ by the same argument, since $\chi Q \approx \tilde{\mu}$. We therefore obtain as ground state energy:

$$W = W_0 + \frac{k}{2} Q_c^2 - \frac{\Delta'^2}{G} - \frac{\chi}{2} Q^2 \tag{93}$$

with:

$$W_0 = \sum_\alpha{}' \varepsilon_\alpha + \varepsilon \cdot N \tag{94}$$

$$k = \frac{1}{2P} = \frac{1}{2\sum_\alpha' p_\alpha} \tag{95}$$

If we substitute $Q_< = Q - Q_>$, W takes the form:

$$W = W_o + \frac{1}{2}(k - \chi)Q^2 - kQQ_> - \frac{k}{2}Q_>^2 - \frac{\Delta'^2}{G} \tag{96}$$

We only have to find $Q_>$ in dependence of Q, then W(Q) is known.

$Q_>$ is determined for small Q, according to equation (92), by

$$Q_> = \sum_{\alpha}^{\lambda} q_{\alpha\alpha} v_{\alpha'}^2$$

In the degenerate shell, where all levels have the same energy ε , we can use the relation $\hat{\varepsilon}_{\alpha'} = \varepsilon - \hat{\mu} q_{\alpha\alpha} - \hat{\mu}^2 p_\alpha$ to write

$$q_{\alpha\alpha} = -\frac{1}{\hat{\mu}}(\hat{\varepsilon}_\alpha - \varepsilon) \tag{97}$$

under condition (91), i.e., for small enough Q. Then:

$$Q_> = -\frac{1}{\hat{\mu}}\sum_{\alpha'}(\hat{\varepsilon}_{\alpha'} - \lambda)v_{\alpha'}^2 + \frac{\varepsilon - \lambda}{\hat{\mu}}N \tag{98}$$

and approximately again the sum by an integral

$$Q_> = -\frac{1}{\hat{\mu}}\int_a^b \left(1 - \frac{\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}}\right)\rho_o\,\varepsilon\,d\varepsilon - \frac{\varepsilon - \lambda}{\hat{\mu}}N \tag{99}$$

which yields for the strong interaction limit:

$$Q_> = \frac{\varepsilon'' - \varepsilon'}{\hat{\mu}}\,\frac{\Omega}{6}\,\Theta_N\,\eta \tag{100}$$

The first factor is a constant which depends only on the shell under consideration: Because of (79) and (80) and the estimate (91) we can write for small Q

$$\frac{\varepsilon'' - \varepsilon'}{\hat{\mu}} = \operatorname{sgn}\hat{\mu}\cdot q \tag{101}$$

If we take $\operatorname{sgn}\tilde{\mu}$ into η, such that

$$\eta = \operatorname{sgn}\tilde{\mu}\ \frac{\varepsilon'' - \varepsilon'}{G \cdot \Omega} \quad , \qquad \text{then} \qquad Q_\lambda \tag{102}$$

is directly proportional to η:

$$Q_\lambda = \alpha \cdot \eta \tag{103}$$

$$\alpha = \frac{q\Omega}{6}\,\Theta_N \tag{104}$$

W therefore depends now only on the quadrupole moments Q, Q_λ.

In order to get their connection we can set in linear approximation

$$Q = f \cdot Q_\lambda \tag{105}$$

in the neighborhood of $Q=0$ (since $Q=0 \rightarrow Q_c=0,\ Q_\lambda=0$). With this W can be expressed as a function only in Q_λ. We choose now f such that:

$$\frac{\partial W}{\partial f} = 0 \tag{106}$$

in a whole neighborhood of $Q_\lambda=0$. This gives:

$$f = \frac{1}{1-\frac{x}{k}} \tag{107}$$

From this and (105) follows

$$Q = f \cdot \alpha \cdot \eta \tag{108}$$

This can be used to express $\frac{\Delta'^2}{G}$, in terms of Q:

For $\eta < 1$ (equation (79))

$$\frac{\Delta^{2}}{G} = \frac{\Omega^2}{4} G \Theta_N \left(1 - \frac{1}{3(f\cdot\alpha)^2} Q^2\right) \tag{109}$$

Finally the energy W(Q) is:

$$W = W_o - \frac{\Omega^2}{4} G \Theta_N + \frac{C}{2} Q^2 \tag{110}$$

$$C = \frac{\Omega^2}{6(f\cdot\alpha)^2} G \Theta_N \left[1 - \frac{\Theta_N}{\Theta_{N_o}}\right] \tag{111}$$

$$\frac{1}{\Theta_{N_o}} = \frac{\chi q^2}{6G(1-\frac{\chi}{k})} \tag{112}$$

C is the direction force. We see that W has its minimum at Q=0 as long as

$$\Theta_N < \Theta_{N_o} \tag{113}$$

Only under this condition the spherical nucleus can undergo quadrupole vibrations.

For $\Theta_N > \Theta_{N_o}$ the nucleus evidently becomes instable. The minimum energy lies then at some $Q \neq 0$ which corresponds to a static deformation. We call Θ_{N_o} the "critical filling." Since single closed shell nuclei show all a stable spherical equilibrium shape, their critical filling Θ_{N_o} must be throughout larger than one.

C. Determination of the Moment of Inertia

According to (56) the moment of inertia is:

$$B = \sum_{\nu} \frac{1}{V_\nu^2} \left(\frac{\partial u}{\partial Q}\right)^2 \frac{1}{E_\nu}$$

or, expressing Q by η

$$B = \frac{1}{(f \cdot \alpha)^2} \sum_{\nu} \frac{1}{V_\nu^2} \left(\frac{\partial u_\nu}{\partial \eta}\right)^2 \frac{1}{E_\nu} \tag{57}$$

This holds also for odd nuclei if one neglects exclusion of one summand ν_1, which gives an error of $\frac{1}{\Omega}$. Now:

$$\frac{1}{V_\nu} \frac{\partial u_\nu}{\partial \eta} = \frac{1}{2u_\nu V_\nu} \frac{\partial u_\nu^2}{\partial \eta} = \frac{1}{2u_\nu V_\nu} \frac{\partial}{\partial \eta} \left\{\frac{1}{2}\left(1 - \frac{\varepsilon_\nu - \lambda}{E_\nu}\right)\right\} \tag{114}$$

Remembering $2u_\nu V_\nu = \frac{\Delta}{E_\nu}$ one obtains:

$$\frac{1}{V_\nu} \frac{\partial u_\nu}{\partial \eta} = \frac{1}{2E_\nu^2}\left[\Delta \frac{\partial}{\partial \eta}(\varepsilon_\nu - \lambda) - (\varepsilon_\nu - \lambda)\frac{\partial \Delta}{\partial \eta}\right] \tag{115}$$

But

$$\frac{\partial \Delta}{\partial \eta} = 0 \qquad \text{at } Q = 0 \qquad \text{(cf. (109))} \tag{116}$$

The expression

$\varepsilon_\nu - \lambda$ is with λ from (68)

$$\varepsilon_\nu - \frac{\varepsilon'' + \varepsilon'}{2} - \frac{G}{2} \chi_N \tag{117}$$

Hence:

$$\frac{\partial}{\partial \eta}(\varepsilon_\nu - \lambda) = \frac{\partial}{\partial \eta}\left(\varepsilon_\nu - \frac{\varepsilon'' + \varepsilon'}{2}\right) \tag{118}$$

Now from (97):

$$\widehat{\varepsilon}_{\nu'} - \frac{\varepsilon' + \varepsilon''}{2} = -\widehat{\mu}\left(q_{\nu\nu} - \frac{\max q_{\nu\nu} + \min q_{\nu\nu}}{2}\right) \qquad (119)$$

$$\varepsilon'' - \varepsilon' = \widehat{\mu} \cdot \operatorname{sgn} \widehat{\mu} \; (\max q_{\nu\nu} - \min q_{\nu\nu})$$

Therefore:

$$\frac{\widehat{\varepsilon}_{\nu'} - \frac{\varepsilon' + \varepsilon''}{2}}{(\varepsilon'' - \varepsilon') \cdot \operatorname{sgn} \widehat{\mu}}$$

is constant in the neighborhood of Q=0. But then, using $\eta = \operatorname{sgn}\widehat{\mu} \frac{\varepsilon'' - \varepsilon'}{G\Omega}$, (118) becomes

$$\frac{\partial}{\partial \eta}\left(\varepsilon_\nu - \frac{\varepsilon' + \varepsilon''}{2}\right) = \frac{\partial}{\partial \eta}\left(\widehat{\varepsilon}_{\nu'} - \frac{\varepsilon' + \varepsilon''}{2}\right)\bigg|_{Q=0} = \frac{\partial}{\partial \eta}\left\{\left.\frac{\widehat{\varepsilon}_{\nu'} - \frac{\varepsilon' + \varepsilon''}{2}}{(\varepsilon'' - \varepsilon')\operatorname{sgn}\widehat{\mu}}\right|_{Q\to 0} \cdot \eta\right\} \qquad (120)$$

and setting this into (115) one has:

$$\frac{1}{V_\nu}\frac{\partial u_\nu}{\partial \eta} = \frac{1}{2E_\nu^2}\Delta \left.\frac{\widehat{\varepsilon}_{\nu'} - \frac{\varepsilon' + \varepsilon''}{2}}{(\varepsilon'' - \varepsilon')\operatorname{sgn}\widehat{\mu}}\right|_{Q\to 0} \Omega G \qquad (121)$$

We remember that for $\eta < 1$

$$E_\nu = \frac{1}{2} G\Omega \qquad \text{(cf. (74))}$$

and

$$\Delta^2 = \frac{\Omega^2 G^2}{4}\Theta_N \qquad \text{(cf. (78))}$$

then

$$B = \frac{1}{(f \cdot \alpha)^2}\frac{2\Theta_N}{\Omega G}\sum_\nu \left(\frac{\widehat{\varepsilon}_{\nu'} - \frac{\varepsilon' + \varepsilon''}{2}}{\varepsilon'' - \varepsilon'}\right)^2_{Q\to 0}$$

and taking the integral

$$B = \frac{1}{(f\cdot\alpha)^2}\frac{2\Theta_N}{\Omega G}\int_{\varepsilon'}^{\varepsilon''}\left(\frac{\varepsilon-\frac{\varepsilon'+\varepsilon''}{2}}{\varepsilon''-\varepsilon'}\right)_{Q\to 0} d\varepsilon$$

$$B = \frac{1}{(f\cdot\alpha)^2}\frac{1}{6G}\Theta_N \tag{122}$$

Together with (111) we obtain then as frequency of the quadrupole vibration:

$$\omega = \sqrt{\frac{C}{B}} = \Omega G\sqrt{1-\frac{\Theta_N}{\Theta_{N_0}}} \tag{123}$$

It is interesting that this frequency is in the absence of quadrupole interaction (i.e. $\frac{1}{\Theta_{N_0}} = 0$) just identical to the energy required for breaking apart a pair (of (74)). We see that in general the collective excitation requires less energy than the creation of two quasi-particles. The zero-point vibration carries an energy $\frac{\omega}{2}$ and since the nucleus is spherical this energy is the same for all five degrees of freedom of quadrupole vibration. Hence, the ground state energy is lifted by

$$\Delta E = \frac{5}{2}\left(\omega - \omega(\chi = 0)\right) \tag{124}$$

by the effect of the quadrupole interaction. For

$$\Theta_N < \Theta_{N_0} \tag{125}$$

this is approximated by

$$\Delta E \approx -\frac{5}{4}\frac{\Theta_N}{\Theta_{N_0}}\Omega G \tag{126}$$

CHAPTER V

THE ASYMMETRIC ENERGY

We now collect the different parts of the total energy calculated in IV. We set in equation (110) Q=0 and obtain the static ground state energy:

$$U = \sum_{\alpha}{}' \varepsilon_\alpha + \varepsilon N - \frac{\Omega^2}{4} G \Theta_N \tag{127}$$

where $\sum_{\alpha}{}' \varepsilon_\alpha$ is the sum of the single particle energies in the closed shells and N the number of nucleons in the outside shell.

The zero point energy of the quadrupole vibration gives according to (126) the contribution

$$\Delta E = -\frac{5}{4} \Omega G \frac{\Theta_N}{\Theta_{N_0}} \tag{126b}$$

Finally, we have to remember that equation (127) for U holds only for even N. For an odd N nucleus we have to add the quasi-particle energy which, from equation (74), we know to be

$$E_\nu = \frac{1}{2} G \Omega$$

Then the total energy of a single closed shell nucleus is in dependence of the number of nucleons in the unfilled shell:

$$E(N) = \sum_{\alpha}{}' \varepsilon_\alpha + \varepsilon N - \frac{\Omega^2}{4} G \Theta_N - \frac{5}{4} \Omega G \frac{\Theta_N}{\Theta_{N_0}} + \begin{cases} E_\nu & \text{for odd } N \\ 0 & \text{for even } N \end{cases} \tag{128}$$

$$\Theta_N = \frac{2N}{\Omega}\left(1 - \frac{N}{2\Omega}\right)$$

being the parameter introduced in (70), Θ_{N_0} is the "critical filling " of the shell where the spherical shape gets unstable (112).

We separate $E(N)$ into terms in N and N^2 and we have:

$$E(N) = \text{const} + \alpha N + \gamma N^2 + \begin{cases} E_\nu & \text{for odd } N \\ 0 & \text{for even } N \end{cases} \tag{129}$$

We are only interested in the coefficient of the quadratic term.

$$\gamma = \frac{G}{4}\left(1 + \frac{5}{\Omega}\frac{1}{\Theta_{N_0}}\right) \tag{130}$$

or

$$\gamma = \frac{G}{4} + \frac{5}{24\Omega}\frac{x}{1 - \frac{x}{k}}\, q^2 \tag{131}$$

This is the quantity we have looked for. It determines the asymmetric energy:

In the Weizsaecker mass formula the quadratic term in N is, for fixed Z:

$$\frac{a_a}{A} \cdot N^2 \tag{132}$$

Therefore γ should give:

$$\gamma^{-1} = \frac{a_a}{A} \tag{133}$$

and, since $a_a = 19$ MeV

$$\gamma^{-1} = \frac{19}{A} \tag{133a}$$

γ depends on the constants of pairing and quadrupole interaction G and χ, respectively, and on the shell parameters

Ω = Number of pair states in the unfilled shell

$$k = \frac{1}{2\sum_{\alpha} p_\alpha^{-1}} \tag{95}$$

$$q = \max q_{\alpha\alpha} - \min q_{\alpha\alpha} \tag{79}$$

taken in the unfilled shell.

G and χ are unknown. For G we cannot take the value $G = \frac{23}{A}$ successfully applied by all former workers[2,14,7,15]. The reason is that our G is an interaction constant acting only in one of the degenerate level groups, while all other people use one G for a total major closed shell.

We possess, however, a good experimental source for G.
According to (76) the even-odd mass difference is given by:

$$P = G\Omega$$

hence,

$$G = \frac{P}{\Omega} \tag{134}$$

P is listed for example by Kisslinger Sorensen[2], p. 883.

In a similar way we can use equation (123) to determine the critical filling parameter Θ_{N_0} by measurements of the quadrupole frequency ω. Experimental ω-values are given in the same reference on p. 866. For each unfilled shell we need just one ω, if $\Theta_N \neq 0$. Then (123) gives:

$$\frac{1}{\Theta_{N_0}} = \frac{1}{\Theta_N}\left(1-\left(\frac{\omega}{6\Omega}\right)^2\right) \tag{135}$$

With these two relations γ becomes:

$$\gamma = \frac{P}{4\Omega}\left\{1+\frac{5}{\Omega}\,\frac{1}{\Theta_N}\left(1-\left(\frac{\omega}{P}\right)^2\right)\right\} \tag{136}$$

Thus γ is determined by only one shell parameter and two observable quantities, the pairing force and the quadrupole frequency. We want to check this equation by evaluating γ and comparing it with γ' from the empirical Weizsaecker formula (133). In view of the approximations made in the course of the calculations, we can expect quantitatively good results only if the unfilled shell shows a high degeneracy. The best example with respect to this property is:

(a) $\quad {}_{50}Sn^{124}_{74}$

For this nucleus the tables of Ref. 2 give:

$\omega = 1.2 \qquad P = 2.8$

The unfilled shell contains six pair levels. These are the levels with the numbers 28, 32, 35, 36, 37, 38 in the Nilsson scheme, given in Fig. 5 of Ref. 3.

Hence Ω =6. The shell contains N=4 neutrons. Then:

$$\frac{1}{\Theta_{N_0}} = \frac{1}{\Theta_N}\left(1-\left(\frac{\omega}{P}\right)^2\right) = 0.8 < 1$$

which tells that the spherical shape of all Sn-isotopes with neutrons in the shell under consideration are stable. Thus:

$$\gamma = 0.14 \tag{137}$$

the empirical formula (133) gives:

$$\gamma' = \frac{19}{124} = 0.15 \tag{137a}$$

which is in very good agreement with (137). Another example is:

(b) $Pb^{200}_{82\ 118}$

for which the tables in (2) give: ω =1.1, P=1.5

Here the five levels with the numbers 61, 62, 63, 70, 71 have to be considered as the outside shell. These levels are not quite degenerate; however, they lie all close together on an energy interval $\Delta\varepsilon$ of less than $\frac{2}{10}$ MeV, which is much smaller than the pairing energy P=1.5 .
This fact allows to consider them as degenerate*. Then: Ω =5, N=4, which gives: $\frac{1}{\Theta_{N_0}}$ =0.48 and thus:

$$\gamma = 0.11 \tag{138}$$

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* The zero-point vibration amplitude of η is $=\overline{\eta} = \sqrt{\frac{\omega}{C(f-\alpha)^2}} = \sqrt{\frac{6}{\Omega\Theta_N}\frac{P}{\omega}}$ i.e., of the order of one and this much larger than the equilibrium value $\eta_0 = \frac{\Delta\varepsilon}{P} \sim \frac{1}{10}$. Hence the vibration can be considered as well around $\eta_0 = 0$ and our calculations apply.

The empirical value is:

$$\gamma' = \frac{19}{200} = 0.95 \tag{138a}$$

Again we obtain a very good agreement. Similarly one can check other single-closed-shell nuclei. It is interesting that the contributions of both residual forces to the value of γ lie in the same order of magnitude. Let:

$$\gamma_P = \frac{P}{4\Omega} \quad , \quad \gamma = \frac{5P}{4\Omega^2} \frac{1}{\Theta_N} \left(1 - \left(\frac{\omega}{P}\right)^2\right) \tag{139}$$

be the individual parts of pairing and quadrupole interactions, respectively. Then their ratio is:

$$\frac{\gamma_P}{\gamma_q} = \frac{\Omega}{5} \Theta_N \tag{140}$$

In our examples this takes the values:

$$\begin{aligned} &(a) \qquad \frac{\gamma_P}{\gamma_q} \approx 1.5 \\ &(b) \qquad \frac{\gamma_P}{\gamma_q} \approx 2.1 \end{aligned} \tag{141}$$

We remark that in equation (131) one cannot conclude $\gamma = 0$ for the absence of residual interactions, since all formulas are derived for the strong interaction limit. The asymmetric energy of the independent particle shell model is determined by the increase of the Fermi level λ with N, which is

$$\frac{d\lambda}{dN} = \frac{1}{2 \times \text{mean level density}} \tag{142}$$

Forming this average over many levels gives

$$\frac{d\lambda}{dN} \approx \frac{25}{A}$$

for nuclei of mass number A.

The asymmetric energy following from this value is

$$\gamma_0 = \frac{1}{2}\frac{d\lambda}{dN} \sim \frac{12}{A} \; *$$

which is somewhat more than half of the total $\frac{a_a}{A} = \frac{19}{A}$.

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* Since $dE = \lambda\, dN + \frac{1}{2}\frac{d\lambda}{dN}(dN)^2$ in a Fermi sea for small changes dN of the particle number N.

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