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Group Dynamics of Elementary Particles

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Abstract

It is proposed that particle interactions can be described by means of simple group operations in non-compact Lie groups. Prescriptions for the structure of these group operations are formulated which are motivated by the study of simple models, in particular of the dynamics of the hydrogen atom. If they are fulfilled we call the structure "group dynamics."

Neglecting at first internal symmetries, a few simple models are investigated in which group dynamics is possible. The group $O(3,1)$ turns out to describe the coupling of pseudoscalar mesons, $O(4,2)$ that of photons to baryons quite well: The pionic decay rates of baryon resonances up to spin 19/2 and the electromagnetic form factors of the nucleons are predicted in good agreement with experiment.

The internal symmetry $SU(3)$ is included in the $O(3,1)$ model of the pionic coupling in the simplest way, by assuming $O(3,1) \times SU(3)$ to be the dynamical group. This gives rise to a minimal symmetry breaking of the amplitudes and relates it to the mass differences in $SU(3)$ multiplets: The amplitude consists of a product of a $SU(3)$ Clebsch Gordan coefficient and a universal function of the velocity of the final baryon. The way the particle masses enter the decay rates is uniquely prescribed in this approach. The agreement with experiment is excellent.

Finally, the connection of this purely algebraic approach with related ideas, like the use of infinite component wave equations, is discussed.

Contents

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I. Prologue

The algebra of observables of a highly symmetric quantum mechanical system can be generated by the Lie algebra of some small, in general non-compact, group [0]. The physical Hilbert space is then a representation space of that group. If all the diagonalized observables, whose eigenvalues label the Hilbert space, are generators of the group and the representation is irreducible, we call this group "*the group of quantum numbers*". External interactions or scattering processes can cause transitions between the states of the system. If there is a group, which not only generates the quantum numbers but also contains all transition operators as group or Lie algebra elements for a certain set of interactions, we call this group "*dynamical group*" with respect to these interactions. Transitions due to a single type of interaction are usually described by the operators of some non compact subgroup of the dynamical group, which we shall call the "*transition group*" of this interaction. For illustration, some physical systems and their groups are listed in Table 1. To be complete, we have added the maximal group of degeneracy connecting levels with fixed energy. We also have given the operator in the algebra which has to be identified with the energy in each system.

Table 1
Some Physical Systems and their Characteristic Lie Groups

System	Maximal Group of Degeneracy	Group of Quantum Numbers	Dynamical Group	Transition Group	Energy Operator
1. N-dimensional harmonic oscillator	$U(N)$	$U(N,1)$	$U(N,1)$	$O(2,1)$	L_{12}
2. Rigid Rotator	$O(3)$	$SL(2,C)$	$SL(2,C)$	$O(2,1)$	L^2
3. Hydrogen Atom	$O(4)$	$O(4,1)$	$O(4,2)$	$O(2,1)$	$-1/(2L_{56}^2)$

Given the dynamical group of a system, the representation space, and identification prescriptions of observables and transition operators with the generators and elements of the group, Schrödinger theory can be completely substituted by group theory. For highly symmetric systems, the dynamical group description simplifies more complicated dynamical calculations considerably, as we shall see in the evaluation of transition form factors of the hydrogen atom. The advantage gained is comparable to the one in potential theory, where the choice of coordinates carrying the geometrical symmetries of the system leads to the simplest calculations.

The strength of Schrödinger theory is that the correspondence principle specified the Hamiltonian and the observables uniquely; for the dynamical group approach operator identifications are at present an extra input. Symmetry properties with respect to certain subgroups, like rotation and parity groups, impose, however, strong restrictions upon the choice of the observables. We shall see that in small groups these restrictions will leave in general little or no freedom for these operator identifications. In particle physics there is no correspondence principle telling us how to construct observable operators on the Hilbert space of asymptotic states. One observes, on the other hand, high symmetries in the spectra of particle quantum numbers and in scattering processes. This suggests strongly that particle dynamics allows in fact for a simple description through a dynamical group. The results obtained until now have been very encouraging.

The problem of finding the dynamical group of particle interactions can be subdivided into several parts:

1. Find the group of degeneracy (or approximate degeneracy) which relates processes only differing by internal quantum numbers. $SU(3)$ seems to be a good candidate. Symmetries of the interactions implied by groups of degeneracy will be called "horizontal symmetries".
2. Find the group of quantum numbers labeling correctly the states of all particles at rest.
3. Find the transition group for certain interactions, for example electromagnetic coupling. Every transition group implies certain symmetries of energy levels at fixed internal quantum numbers. Such symmetries will be called "vertical".
4. Find the dynamical group for all processes. This group clearly has to contain the groups of degeneracy and all transition groups as subgroups.

Much work in particle physics has been concentrated on the first problem. The principal difficulty there is the fact that all horizontal symmetries are broken and one lacked for quite some time a definite prescription of how to incorporate this effect. Current algebras define symmetry breaking in an implicit way and have been quite successful in relating breaking effects to other observable quantities. We shall show that an understanding of the vertical symmetries leads to an alternative, direct and definite way of breaking of horizontal symmetries in transition amplitudes, which is in excellent agreement with experiment.

Our work will first study the simplest possible transition group, $O(3,1)$, for relativistic processes to get some feeling for what effects one can describe (Ch. II). Assuming this group extended by $SU(3)$ to $O(3,1)\times SU(3)$ as a dynamical group for the Pseudoscalar-Baryon coupling, we shall obtain extremely good results for all observed amplitudes of this kind (Ch. III).

It turns out that the electromagnetic interactions cannot be described by this group. For these one needs a larger dynamical group. In order to get some hints of what structure the electromagnetic interaction may have in such a larger group, we study first the exactly soluble case of the H-atom (Ch. IV) and give complete electromagnetic form factors for all transitions. We then propose $O(4,2)$ as the dynamical group of electromagnetic interactions of baryons as long as internal symmetries are neglected and calculate the form factors of the spin $^1/_2$ ground state which are in quite satisfactory agreement with experiment (Ch. V).

The prescriptions of how to obtain dynamical information from a dynamical group are not well defined yet and far from being on the level of a theory. We therefore shall not state them at the beginning following an axiomatic approach, but shall rather formulate them as we go along and learn from the examples we are treating. The soluble case of the H-atom will give the most important hints for these prescriptions. A theory of interactions built up on the representation space of a group, having the same mathematical structure as observed in the H-atom, will tentatively be called *"group dynamics"*.

II. The Minimal Dynamical Group

1. Introduction

Suppose all states of particles at rest contained in the Hilbert space spanned by the states $|\alpha\rangle$, where α denotes collectively all quantum numbers characterizing the particles. The Hilbert space of rest states must be invariant under rotations, hence one can decompose it with respect to its spin contents. Therefore we can assume, that α contains the spin labels j, j_3 among its quantum numbers. If we want to describe any relativistic interactions between the particles, we have to define a representation of the Lorentz group on this Hilbert space. The trivial, kinematical way to do this is to increase the Hilbert space by adding the momentum as an additional quantum number to the states forming $|\alpha, p\rangle$ and by representing the Lorentz group elements Λ separately for every spin j by means of the Wigner [*1*] rotations $W^j(\Lambda, p)$ as

$$U(\Lambda)\,|\alpha, p\rangle = |\alpha' \Lambda p\rangle\, W^j_{\alpha'\alpha}(\Lambda, p). \tag{2.1}$$

Another possibility is to leave the Hilbert space the same and to represent the Lorentz group directly on the rest states $|\alpha\rangle$. The representation is then in general non-unitary as in the case of the Dirac spinor representation.

Each moving state can be characterized by the "rapidity" $\boldsymbol{\zeta}(\equiv \tanh^{-1}\boldsymbol{v}/c)$ of its motion, and we shall denote it by $|\alpha, \zeta\rangle$. If $\boldsymbol{M}$ are the generators of the Lorentz transformations, then

$$|\alpha, \zeta\rangle = e^{i\boldsymbol{M}\vec{\zeta}}|\alpha\rangle \equiv B(\zeta)\,|\alpha\rangle. \tag{2.2}$$

An irreducible representation of the Lorentz group couples in general a tower of spins from $j = j_0$ until $j = j_1 - 1$ together. Its states are denoted by $|jm[j_0, j_1]\rangle$. If $j_1 = i\nu$ is purely imaginary, the tower has no upper end; the multiplet then contains infinitely many particles and the representation is unitary. In that case we shall often use the alternative labels $[j_0, \nu]$ to characterize the representation. The state $|\alpha, \zeta\rangle$ can be written in more detail as:

$$|jm[j_0, j_1]\zeta\rangle = \sum_{j'm'} \langle j'm'[j_0, j_1]|B(\zeta)|jm[j_0, j_1]\rangle \, |j'm'[j_0, j_1]\rangle \tag{2.3}$$

A summary of the properties of the representations of the Lorentz group is given in App. A [*1*].

The states $|jm[j_0, j_1]\rangle$ can in general not yet be identified with particles of spin j, For this they would have to be eigenstates of parity. Since parity reverses the sign of the Casimir operator $LM = -ij_0j_1$ of the Lorentz group, only $[j_0, j_1] = [0, j_1]$ or $[j_0, 0]$ possess states invariant under parity. For the other representations we have to form the linear combinations characterized by the in variant $\eta (= +1$ or $-1)$

$$|jm[j_0, j_1]\eta \pm\rangle = [|jm[j_0, j_1]\rangle \pm \eta |jm[-j_0, j_1]\rangle] \tag{2.4}$$

which have the parity:

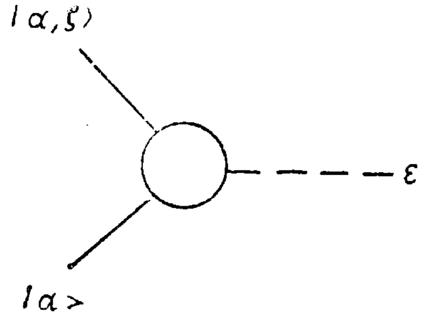
$$\Pi |jm[j_0, j_1]\eta \pm\rangle = \pm (-)^{j-j_0} \eta |jm[j_0, j_1]\eta \pm\rangle. \tag{2.5}$$

As we see, η coincides with the parity of the ground state $|j_0m[j_0, j_1]\eta +\rangle$. These states can then be identified with particles. It is clear that this construction gives the smallest possible Hilbert spaces on which one can construct a Lorentz invariant dynamical theory containing parity (i.e. invariant under the semidirect product $O(3,1) \sim \Pi$).

If the representation contains more than one state, we can interpret the matrix elements

$$\langle j'm'[j_0, j_1]\eta' \pm |B(\zeta)|jm[j_0, j_1]\eta \pm\rangle \tag{2.6}$$

in some sense as transition amplitudes under an acceleration process: The boost from the rest frame to rapidity ζ has been achieved operationally by means of some external interaction ε.



The usual kinematical boost can practically be done only in the passive way by the observer going to another inertial reference frame. Any active acceleration would need an external interaction, which would not leave the particle as it is, but excite it or produce other particles. The boost we have written down in (2.6) is obviously describing just this alternative. The only problem is to find

out which, if any, physical interaction may be associated with the transition amplitudes (2.6). Every representation of the Lorentz group clearly specifies a different interaction of this kind. (See the note added in proof (1), p. 74.)
The characteristic property of this interaction is that any particle with spin j in the tower of states from j_0 to $j_1 - 1$, being accelerated to rapidity ζ, will with the amplitude (2.6) be excited to any other particle in the tower. The interactions must therefore have the quantum numbers of a neutral non-strange meson, while all the levels of the tower differ only by their spin (and parity). Such processes exist indeed in great number. There are towers of baryon resonances, and photons or pions can cause transitions up and down the levels. Neglecting their isospin, we therefore shall identify tentatively the isospin 1/2 baryons N (940), (1525), (1688), (2190), (2650), (3030) with parts of an $O(3,1)$ tower (the "N-tower") and the isispin 3/2 resonances Δ (1236), (1920), (2420), (2850), (3230), as parts of another tower (the "Δ-tower"). The N-tower can obviously by only a representation $[1/2, \nu]$ because it contains the spin 1/2 nucleon. The Δ-tower may be $[1/2, \nu]$ or $[3/2, \nu]$. We shall see that the existence of the vertex $\Delta(1236) \to N(940) + \gamma$ requires it to be $[1/2, \nu]$. It turns out that we can by means of the Lorentz group describe the pionic coupling very well in the region of small momentum transfer of the pions. The electromagnetic form factors will need a mixing of many irreducible representations of $O(3,1)$ by means of the larger dynamical group $O(4,2)$ and will be discussed later (in Ch. VI).

2. Electromagnetic Interaction

For the particles in the representation of $O(3,1)$ extended by parity $|jm[j_0, j_1]\eta\rangle$ we postulate the existence of a vector operator Γ^μ coupling to the electromagnetic field. This problem has been discussed in great detail by GELFAND and YAGLOM [2]. Their results are given in App. B. One obtains the selection rule that Γ^μ can couple $[j_0, j_1]$ only to $[j_0 \pm 1, j_1]$ or $[j_0, j_1 \pm 1]$. From this we see that a single irreducible representation of $O(3,1)$ can contain a vector operator only if $[j_0 j_1] = [1/2, 0]$ or $[0, 1/2]$ (since the representation $[j_0, j_1]$ is equivalent to $[-j_0, -j_1]$), which is just the case where parity can be added to the $O(3,1)$ representation without doubling of the Hilbert space. On our doubled Hilbert space (2.4) of parity eigenstates it is clear that a vector Γ^μ exists if, and only if $[j_0, j_1] = [1/2, j_1]$ with arbitrary j_1. The particles are then all fermions.
We see that the N-tower has always a vector. But so does the Δ-tower since it has to be assigned to $[1/2, j_1]$. Otherwise $\Delta(1236) \to N(940) + \gamma$ would be forbidden which has been seen experimentally. As stated in App. B, the component Γ_0 has on the basis

$$\begin{bmatrix} |jm, 1\rangle \\ |lm, 2\rangle \end{bmatrix} \equiv \begin{bmatrix} |jm[j_0, j_1]\rangle \\ |jm[j_0, -j_1]\rangle \end{bmatrix} \tag{2.7}$$

the form

$$\Gamma_0 = \sigma_1 \gamma^j, \quad \text{with} \quad \gamma^j = (j + 1/2)\gamma \tag{2.8}$$

where γ is an arbitrary real constant, or

$$\Gamma_0 = \sigma_2 \cdot \gamma^j. \tag{2.9}$$

σ_i are the Pauli matrices $(i = 1,2,3)$.

Since parity is defined on the states (217) by

$$\Pi = (-)^{j-j_0}\sigma_1 \tag{2.10}$$

as follows from (2.5), we see that only (2.8) generates a vector, while (2.9) gives an axial vector. If Γ_0 is interpreted as the current operator, we can introduce charge conjugation reversing the sign of the electromagnetic interaction by

$$C \equiv \sigma_3 \tag{2.11}$$

and we see that among the states (2.4), + and − combinations have to be interpreted as particles and antiparticles for $\eta = +1$ (or vice versa for $\eta = -1$)

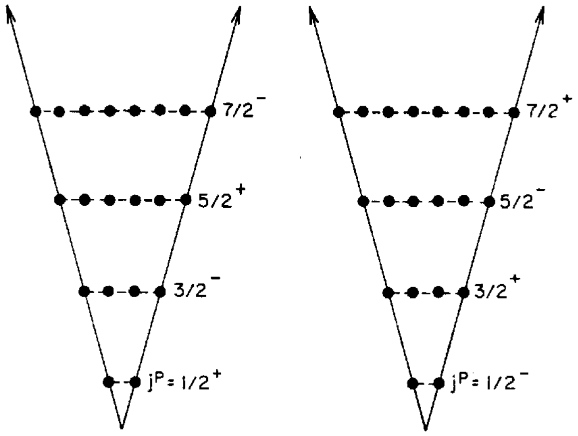


Fig. 1. The states of the doubled representation of $O(3,1)$ extended by parity are shown, assuming the representation to be unitary with $[j_0, j_1] = [1/2, i\nu]$. To these states we may tentatively assign the isospin 1/2 baryons N(940), (1525), (1688), (2190), (2650) and (3030), or the isospin 3/2 baryons Δ(1236), (1920), (2420), (2850), and (3230) together with their antiparticles, if one neglects isospin.

The multiplets are shown in Fig. 1 for $\eta = +1$ and the doubled Hilbert space. Note that the degenerate case $[j_0, j_1] = [1/2, 0]$ gives a fermion representation with no antiparticles.

From (2.10), (2.11) we see that charge conjugation anticommutes with parity, as it should, since fermions and antifermions have opposite parities.

From $\Gamma^0 = \sigma_1 \gamma^j$ we obtain Γ^j through the commutation rule

$$[M_i, \Gamma^0] = i\Gamma^j \quad \text{(See App. B)} \tag{2.12}$$

which gives on the particle-antiparticle states (2.4):

$$\Gamma^0 |jm\,\pm\rangle = \pm (j + 1/2)\, |jm\,\pm\rangle,$$

$$\begin{aligned} i\Gamma^+ |jm\,\pm\rangle = {} & \pm [(j-m)(j-m-1)]^{1/2} C_j |j-1, m+1, \pm\rangle \mp \\ & \mp [(j-m)(j+m+1)]^{1/2} (2j+1) A_j |j, m+1, \mp\rangle \mp \\ & \mp [(j+m+1)(j+m+2)]^{1/2} C_{j+1} |j+1, m+1, \pm\rangle, \end{aligned}$$

$$\begin{aligned} i\Gamma^- |jm\,\pm\rangle = {} & \mp [(j+m)(j+m-1)]^{1/2} C_j |j-1, m-1, \pm\rangle \mp \\ & \mp [(j+m)(j-m+1)]^{1/2} (2j+1) A_j\, |j, m-1, \mp\rangle \mp \\ & \pm [(j-m+1)(j-m+2)]^{1/2} C_{j+1}\, |j+1, m-1, \pm\rangle, \end{aligned}$$

$$i\Gamma^3|jm\pm\rangle = \pm[j^2 - m^2]^{1/2} C_j|j-1, m, \pm\rangle \mp m(2j+1)A_j|jm, \mp\rangle \pm$$
$$\pm[(j+1)^2 - m^2]^{1/2} C_{j+1}|j+1, m, \pm\rangle, \tag{2.13}$$

where

$$C_j = \frac{i}{2j}[j^2 + \nu^2]^{1/2}, \quad A_j = \frac{\nu/2}{j(j+1)} \tag{2.14}$$

and Γ^+, Γ^- denote the usual combinations:

$$\Gamma^+ = \Gamma^2 + i\Gamma^2, \quad \Gamma^- = \Gamma^1 - i\Gamma^2. \tag{2.15}$$

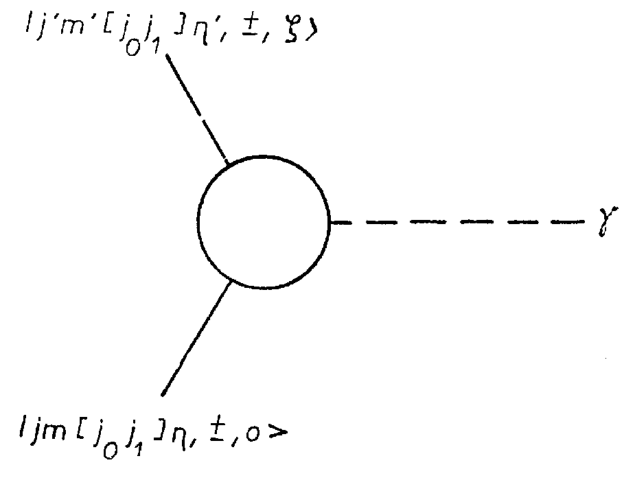
We have assumed $\eta = +1$. For $\eta = -1$ one has to exchange $\Gamma^\mu \to -\Gamma^\mu$. Observe that Γ^μ as a vector operator has the property, that under a Lorentz transformation $\Lambda^\mu{}_\nu$ it transforms as

$$U^{-1}(\Lambda)\,\Gamma^\mu\,U(\Lambda) = \Lambda^\mu{}_\nu\Gamma^\nu \tag{2.16}$$

where $U(\Lambda)$ is the representation of Λ on our Hilbert space (2.4). Note also that our Γ^μ reduces to the Dirac matrices γ^μ in the particular case $[j_0 j_1] = [1/2, 3/2]$ where it becomes

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} = \gamma^\mu \quad \left(\sigma^\mu = (\sigma^0, \sigma), \, \tilde{\sigma}^\mu = (\sigma^0, -\sigma)\right) \tag{2.17}$$

on the $\begin{bmatrix} |1\rangle \\ |2\rangle \end{bmatrix}$ basis (2.7).



Consider now the process in which a member of the multiplet is accelerated from rest to rapidity ζ by an external photon. We require that the electromagnetic current for this process contains only the vector Γ^μ given in (2.15). Other vectors which would in principle be possible are

$$\Gamma_1^\mu = f(q^2)\,(p'^\mu + p^\mu), \quad \Gamma_2^\mu = f'(q^2)\,(p'^\mu - p^\mu) \tag{2.18}$$

or

$$\Gamma_3^\mu = g(q^2)\,L^{\mu\nu}(p' + p)_\nu, \quad \Gamma_4^\mu = g'(q^2)\,L^{\mu\nu}(p' - p)_\nu \tag{2.19}$$

where p', p are the four-momenta of the initial and final particle and q is the momentum transfer $q = p - p'$. Since such vectors may contain the arbitrary

functions of f and g of the momentum transfer and allow too much freedom of the theory, we shall exclude them for the time being from our consideration, focussing only on the purely algebraic vector Γ^μ. We note in advance, however, that the current of the hydrogen atom will in fact contain a term of the form Γ_1^μ with a constant f for its complete description (see. Ch IV). (See the note added in proof (2), p. 74)
For unitary representations, the electromagnetic current can then be written, using the special frame in which the final particle moves in the Z-direction:

$$I^\mu = \langle j' m' [j_0 j_1] \eta', \pm \,|\Gamma^\mu|\, j m [j_0 j_1] \eta \pm, \zeta\rangle \tag{2.20}$$

ζ can be calculated in terms of the invariant momentum transfer through

$$t \equiv q^2 = (p - p')^2 = (m - m')^2 - 2m'm(\operatorname{ch}\zeta - 1) \tag{2.21}$$

The global representation of $O(3,1)$ on the $|jm[j_0 j_1]\rangle$ basis has been given by Ström [3]. For a Lorentz transformation in the Z-direction he finds

$$|jm[j_0 j_1]\zeta\rangle = |j' m [j_0 j_1]\rangle\, B_m^{j'j}(\zeta[j_0 j_1]) \tag{2.22}$$

where for $m = j,\ j' > j$ (See App. A)

$$B_j^{j'j}(\zeta[j_0 j_1]) = N^{j'j}(j_0 j_1)\operatorname{sh}^{j'-j}\zeta\, e^{j_1\zeta} e^{-\zeta(j'+j_0+1)} \times F(j' + 1 - j_1,\ j' + 1 + j_0,\ 2j' + 2,\ 1 - e^{-2\zeta})$$

$$N^{j'j}(j_0 j_1) = 2^{j'-j}\left[\frac{(j' + j_0)!\,(j' - j_0)!\,(j' + j)!}{(j + j_0)!\,(j - j_0)!\,(j' - j)!} \times \right. \\ \left. \times \frac{(2j + 1)!\,[(j + 1)^2 - j_1^2]\cdot\ldots\cdot[j'^2 - j_1^2]}{(2j' + 1)!\,(2j')!}\right]^{1/2}. \tag{2.23}$$

On the states with definite parity the Lorentz transformation is then

$$|jm[j_0 j_1]\eta +\rangle = |j' m [j_0 j_1]\eta \pm\rangle\, B_m^{\pm j'j}(\zeta[j_0 j_1]) \tag{2.24}$$

where:

$$B_m^{\pm j'j}(\zeta[j_0 j_1] \equiv \frac{1}{2}\left(B_m^{j'j}(\zeta[j_0 j_1]) \pm B_m^{j'j}(\zeta[j_0 - j_1])\right). \tag{2.25}$$

In the case $j_1 = 0$, the representation becomes reducible; only B_n^+ exists due to the fact that this representation has no pseudoscalar, the only one, $L \cdot M$, being zero since $L \cdot M = -i j_0 j_1$.
Besides this fermion representation, also the boson representation $[0, 1/2]$ is given by the same B's. One just takes the limit $j_1 \to 0$ before $j_0 \to 1/2$ and gets the same result as for $[1/2, 0]$. Then this case doesn't have to be discussed separately.
To compare the electromagnetic form factor for the spin $1/2^+$ ground state with experimentally observed quantities, we have to separate I^μ into electric and magnetic parts. This can be done by using the conventional current form:

$$I^\mu(t) = \bar{u}(p')\left[\gamma^\mu F_1(t) + i\,\frac{\varkappa}{2M}\,\sigma^{\mu\nu} q_\nu F_2(t)\right] u(p) \tag{2.26}$$

with

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \tag{2.27}$$

and

$$\varkappa = \mu - 1 = \text{anomalous magnetic moment.}$$

In terms of the functions $F_1(t)$, $F_2(t)$ the electric and magnetic form factors are then defined as

$$G_E(t) = F_1(t) + \frac{t}{4M^2} \varkappa F_2(t), \; G_M(t) = F_1(t) + \varkappa F_2(t). \tag{2.28}$$

With them the current I^μ can also be written in the BARNES [4] form:

$$I^\mu(t) = \frac{1}{1 - \frac{t}{4M^2}} \bar{u}(p') \left[G_E(t) \frac{(p'+p)^\mu}{2M} + G_M(t) \frac{i\Gamma'^\mu}{4M^2} \right] u(p) \tag{2.29}$$

where:

$$\Gamma'^\mu = \varepsilon^{\mu\nu\lambda\varkappa} (p'+p)_\nu (p'-p)_\lambda \gamma_\varkappa \gamma_5,$$
$$\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.30}$$

Putting $p' = 0$ and p in $\pm Z$-direction, we obtain between the indicated spin directions (with $p/M = \operatorname{sh}\zeta$)

$$I^0_{\text{up, up}}(t) = \operatorname{ch}\zeta/2 \; G_E(t),$$
$$I^3_{\text{up, up}}(t) = \pm \operatorname{sh}\zeta/2 \; G_E(t),$$
$$I^1_{\text{down, up}}(t) = \pm \operatorname{sh}\zeta/2 \; G_M(t). \tag{2.31}$$

Current conservation implies $(p'-p)_\mu \Gamma^\mu = 0$, hence in our particular frame:

$$I^3/I^0 = \pm \operatorname{th}\zeta/2. \tag{2.32}$$

$\mu \equiv G_M(0)$ is the magnetic moment of the particle in units $e/2Mc$. In order to find the magnetic moment of the ground state we need to calculate I^μ only for small ζ. From (2.20) we find

$$I^0 = \langle {}^1\!/_2\, {}^1\!/_2 + | \Gamma^0 | \, {}^1\!/_2\, {}^1\!/_2 +, \zeta \rangle = B^{+\,1/2\,1/2}_{1/2}(\zeta[{}^1\!/_2, j_1]). \tag{2.33}$$

$$I^3 = \langle {}^1\!/_2\, {}^1\!/_2 + | \Gamma^3 | \, {}^1\!/_2\, {}^1\!/_2 +, \zeta \rangle = \langle {}^1\!/_2\, {}^1\!/_2 + | \Gamma^3 | \, {}^1\!/_2\, {}^1\!/_2 - \rangle \, B^{-\,1/2\,1/2}_{1/2}(\zeta) +$$
$$+ \langle {}^1\!/_2\, {}^1\!/_2 + | \Gamma^3 | \, {}^3\!/_2\, {}^1\!/_2 + \rangle \, B^{+\,3/2\,1/2}_{1/2}(\zeta) =$$
$$= -{}^2\!/_3\, j_1 B^{-\,1/2\,1/2}_{1/2}(\zeta) + \frac{\sqrt{2}}{3} [{}^9\!/_4 - j_1^2]^{1/2} B^{+\,3/2\,1/2}_{1/2}(\zeta)$$

$$I^1 = \langle {}^1\!/_2 - {}^1\!/_2 + | \Gamma^1 | \, {}^1\!/_2\, {}^1\!/_2 +, \zeta \rangle = \langle {}^1\!/_2 - {}^1\!/_2 + | \Gamma^1 | \, {}^1\!/_2\, {}^1\!/_2 - \rangle \, B^{-\,1/2\,1/2}_{1/2}(\zeta) +$$
$$+ \langle {}^1\!/_2 - {}^1\!/_2 + | \Gamma^1 | \, {}^3\!/_2\, {}^1\!/_2 + \rangle \, B^{+\,3/2\,1/2}_{1/2}(\zeta). \tag{2.34}$$

For small ζ, we may use the first term in the Taylor series of the B's (see App. A)

$$B_{1/2}^{-1/2\,1/2} = -\frac{j_1}{3}\,\zeta\,,$$

$$B_{1/2}^{+3/2\,1/2} = \frac{\sqrt{2}}{3}\,[9/4 - j_1^2]^{1/2}\,\zeta \tag{2.35}$$

to get:

$$I^0 = 1\,,$$

$$I^3 = 1/2\,\zeta\,,$$

$$I^1 = \left(\frac{2}{3}\,j_1^2 - \frac{1}{2}\right)\frac{\zeta}{2}\,. \tag{2.36}$$

Comparing with (2.31) we see that the current is conserved to this order in ζ. As we shall show in Section 5, it is even conserved for all ζ. Since the charge is one, we get for the magnetic moment

$$G_M(0) \equiv (2/3\,j_1^2 - 1/2)\,. \tag{2.37}$$

In the case $j_1 = 3/2$, which represents the limit of the Dirac theory, we obtain the magnetic moment $\mu = 1$ of the electron. The case $j_1 = 0$ gives $\mu = -1/2$ and was discussed as early as 1932 by MAJORANA [5] using an infinite component wave equation. We shall come back to this in Section 6.

Since the theory doesn't contain isospin, one probably should identify $\mu/2$ with the isoscalar magnetic moment of the nucleons ($\mu_s = 44$). The agreement is not too good. For $\nu \neq 0$ it is even worse, being $< -1/4$.

Let's now consider the electric form factor of the $1/2^+$ ground state. We have plotted it in Fig. 2. (The $\nu = 0$ fermion electric form factor also coincides with the spin zero form factor of the boson representation $[0, 1/2]$ which is defined by

$$I^\mu = G(t)\,\frac{(p' + p)^\mu}{2M}\,.$$

Except for $\nu = 0$, all form factors oscillate for larger momentum transfers. Like in the case of the magnetic moment, $\nu = 0$ gives also here the most physical result.

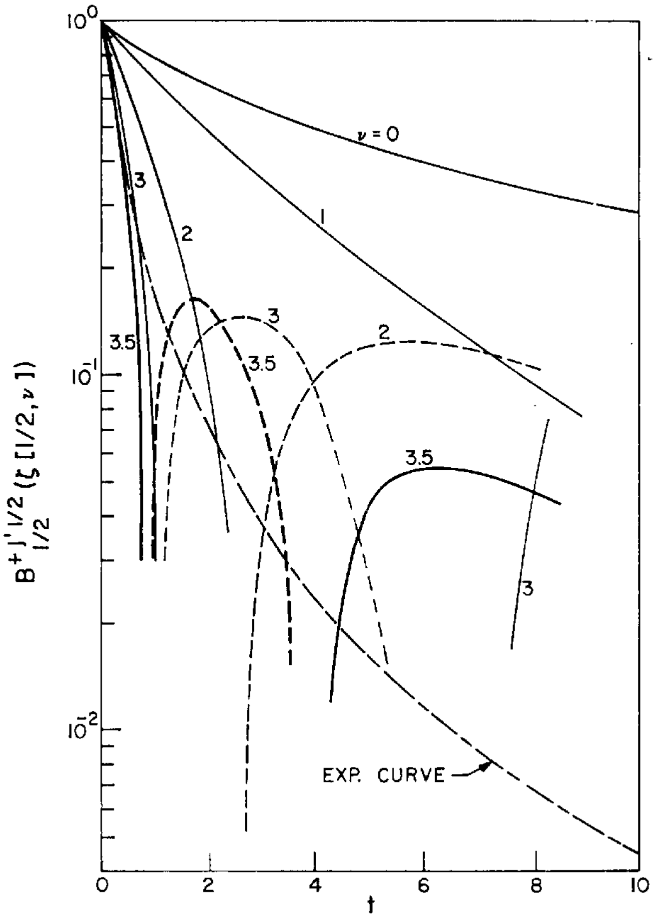


Fig. 2. The electric form factor of the $j^P = 1/2^+$ ground states is plotted as a function of the invariant momentum transfer $t = q^2 = (p - p')^2$, and compared with the nucleon form factor. We see that the $\nu = 0$ form factor falls off far too slowly in t. Higher ν values improve the slope for small t but give oscillations for larger t. The curve with $\nu = 0$ gives also the form factor of the boson representation $j = 0, 1, 2, \ldots$

Observe now, that this case is also the only one in which there really exists a dynamical group of electromagnetic interactions. According to our definition in Ch. I., transition operators like Γ^μ have to be generators of the dynamical group of the system. But Γ^μ close back under commutation to form a Lie algebra in the unitary case only if $\nu = 0$. Then we can define

$$L_{ij},\ L_{i4} = M_i,\ L_{i5} = \Gamma^i,\ L_{45} = \Gamma^0 \tag{2.38}$$

to form the Lie algebra $O(3,2)$. (See App. (B. 12), (B. 13)).
Hence, the minimal dynamical group of electromagnetic interaction in this sense, constructed on an $O(3,1)$ representation space, is really $O(3,2)$. And since this case gives the most physical result, our requirements of Ch. I upon the theory seem to be quite reasonable concerning this model.
Even in the optimal case $\nu = 0$ the agreement of the theoretical form factor with the nucleon form factor is rather bad. We see that a model based on just one irreducible representation of $O(3,1)$ fails to describe the electromagnetic properties of the nucleons. If one tries to obtain an agreement by including terms of the form (2.18) (2.19) into the current, strongly q^2 dependent functions f and g would be necessary, whide cannot be obtained from group theory. We conclude, that in order to improve the results, representation mixing is obviously needed. We may define a "wave function" $\psi^j_{j_0}(\nu)$ such that the physical particle consists of the mixture

$$|jm\rangle = \sum_{j_0 \leq j} \int d\nu\, \psi^j_{j_0}(\nu)\, |jm[j_0\nu]\eta \pm\rangle \tag{2.39}$$

Such a mixture leads then to a nucleon current

$$I^\mu(\zeta) = \sum_{j_0 < j} \int_{\infty}^{\infty} d\nu\, \psi^{*j}_{j_0}(\nu)\, \psi^j_{j_0}(\nu)\, I^\mu(\zeta[j_0\nu]) \tag{2.40}$$

which amounts at large ζ essentially to taking Fourier transforms of $\psi^{*j}_{j_0}(\nu)\,\psi^j_{j_0}(\nu)$ (see the asymptotic forms of $I^\mu(\zeta[j_0,\nu])$ in App. (A. 57)). Since in Schrödinger theory the form factor is the Fourier transform of $\psi^*(x)\,\psi(x)$, we see that $\psi^j_{j_0}(\nu)$ corresponds to a relativistic wave function of the particle, which is defined in an invariant way in ν space, instead of x. This suggests one method of getting better form factors: One may try to write down a Schrödinger equation for a particle moving in some potential well $V(\nu)$ and try to determine $\psi^j_{j_0}(\nu)$ to fit the form factors of the particles with spin j. For example, the wave function

$$\psi^j_{j_0}(\nu) = e^{-a\nu}({}^1/_2 + \nu^2) \tag{2.41}$$

leads for large t to an electric form factor of the $j^P = {}^1/_2{}^+$ state

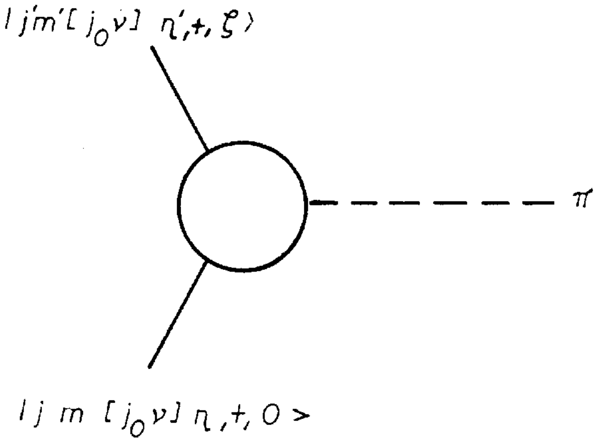
$$G_E = \frac{1}{2}\, e^{-\zeta/2} \left(\frac{4a\zeta}{(a^2+\zeta^2)^2} + \frac{a}{a^2+\zeta^2} \right) \tag{2.42}$$

which decreases faster than the $\nu = 0$ form factor and doesn't oscillate.
Another possibility, and this is the one we shall pursue, is to postulate a larger dynamical group for the electromagnetic interactions. This automatically introduces a larger number of particles into the theory which will be in general mix-

tures of irreducible representations of some Lorentz subgroup of the dynamical group. The next larger group after $O(3,2)$ with a similar structure is $O(4,2)$. It will be discussed in great detail in Chs. IV and VI.

3. Pionic Interactions

We proceed along similar lines as in the case of the electromagnetic interactions. Instead of a vector Γ^μ we now need a pseudoscalar interaction. Such an operator is in the extended Hilbert space $\begin{bmatrix} |1\rangle \\ |2\rangle \end{bmatrix}$ given by $P \equiv C = \sigma_3$. This operator does not



belong yet to the group. We can, however, simply extend $O(3,1) \sim \Pi$ by P as well and may then within $O(3,1) \sim \Pi \sim P$ dynamics use P as a current operator, in consistency with our philosophy. Then, between two particle multiplets with lowest parities η', η we find for the vertex the amplitude:

$$A(\zeta) = \frac{G}{\sqrt{2}} \langle j' m [j_0 \nu] \eta' + | P | j m [j_0 \nu] \eta +, \zeta \rangle$$

$$= \frac{G}{\sqrt{2}} B_m^{\pm j'j}(\zeta [j_0, \nu]), \{\pm \text{ for } \eta \cdot \eta' = \mp 1\} \tag{2.43}$$

G is a coupling constant. The decay rates can be calculated from

$$\Gamma = \Phi \frac{G^2}{2} \sum_m | B_m^{\pm j'j}(\zeta [j_0, \nu]) |^2 \tag{2.44}$$

where Φ is the invariant phase space

$$\Phi = \frac{1}{2j' + 1} \frac{P}{M_i} M_f. \tag{2.45}$$

p the momentum of the decay products and M_i, M_f are the masses of initial and final baryon, respectively.

The amplitude, which is analytic in the external spins j' and j, is interpreted to give, after fixing the coupling constant G and the Casimir operators of the representations, the complete form factors for the decay of any baryon resonance in

the Δ- or N-tower (see p. 6) into any other one and a pion. In order that the decay $\Delta \to N\pi$ is allowed, we have to assume again that $[j_0, \nu]$ are the same for the Δ- and N-towers. The N-tower has necessarily $j_0 = {}^1/_2$ because it contains the nucleons, hence both towers have to be $[{}^1/_2, \nu]$. Notice that the existence of a vector operator Γ^μ on the Hilbert space follows then and does not have to be postulated.
The pionic form factors of baryon resonances have not been measured yet. Only one point on some of them is known due to decay processes. The best measured ones are the processes

$$\Delta \to N + \pi \tag{2.46a}$$

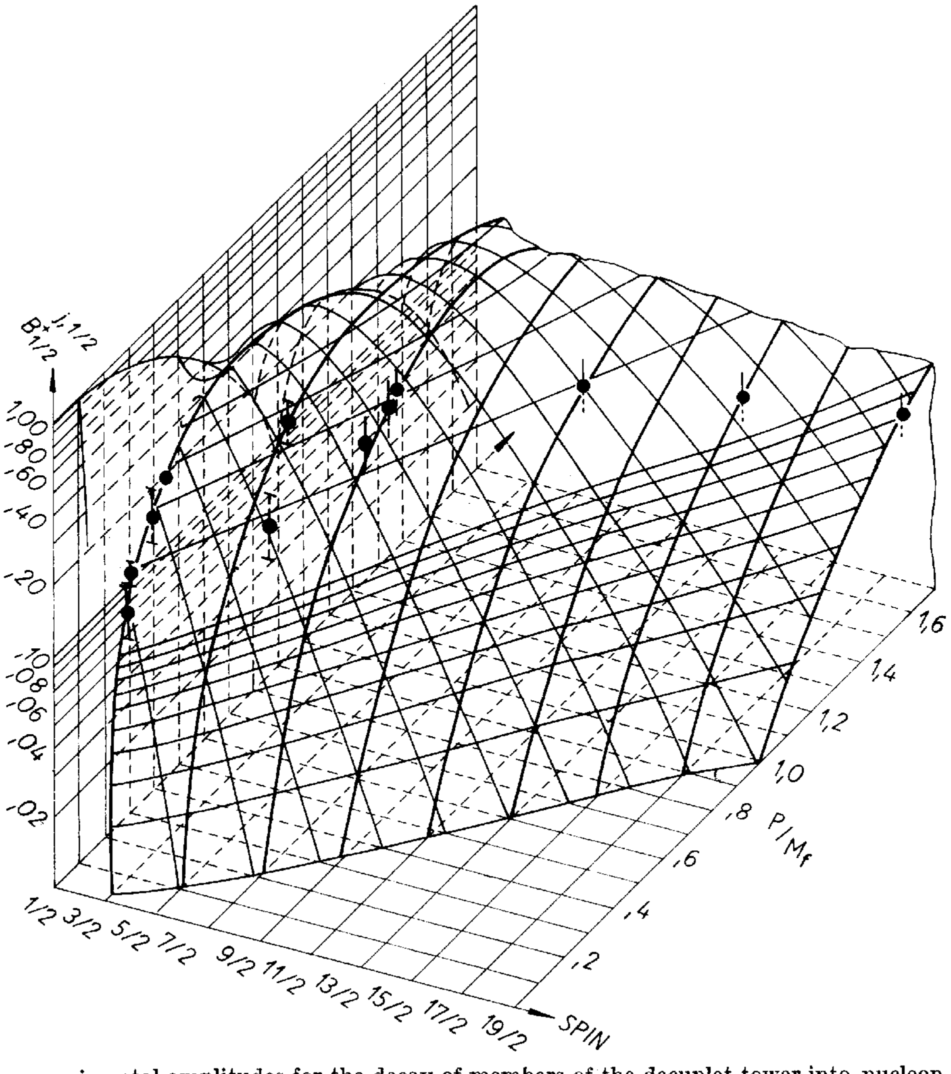


Fig. 3a. The experimental amplitudes for the decay of members of the decuplet tower into nucleon and pseudoscalar meson octet are plotted and compared with the theoretical function $B^{+\,j'\,1/2}_{1/2}[\zeta({}^1/_2, 3.5)]$ as a function of spin of the decaying particles and the variable $p/M_f = \mathrm{sh}^{-1}\zeta$. Couting towards increasing p/M_f, the dots stand for the decays

Spin $3/2^+$: $\Sigma(1385) \to \Sigma\pi$, $\Xi(1530) \to \Xi\pi$
$\Sigma(1385) \to \Lambda\pi$, $\Delta(1236) \to N\pi$
Spin $5/2^-$: $\Sigma(1770) \to \Lambda\pi$, $\Sigma(1770) \to N\bar{K}$
Spin $7/2^+$: $\Sigma(2035) \to \Lambda\pi$, $\Sigma(2035) \to N\bar{K}$
$\Delta(1920) \to N\pi$
Spin $11/2^+$: $\Delta(2420) \to N\pi$
Spin $15/2^+$: $\Delta(2850) \to N\pi$
Spin $19/2^+$: $\Delta(3230) \to N\pi$

The agreement is excellent except for the $\Sigma(1770) \to \Lambda\pi$ which is off by a factor of 2. It may be that this resonance doesn't belong to a decuplet. For the points with open error lines, the experimental errors are not known.

and

$$N^* \to N + \pi. \tag{2.46b}$$

In these particular cases the width is then given by

$$\Gamma = \Phi G^2 \,|\, B_{1/2}^{\pm j'\,1/2}(\zeta[j_0 \nu])\,|^2. \tag{2.47}$$

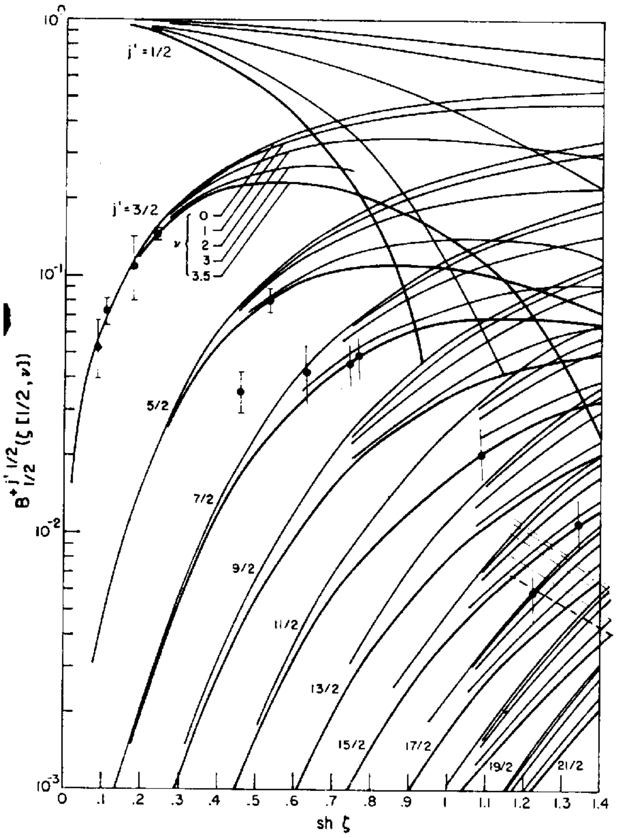


Fig. 3b. The figure shows the same points as on Figure 3a, only that they are now plotted on a two-dimensional graph which contains more than one value of $\nu(0,1,2,3,3.5)$. We see that the value $\nu = 3.5$ gives the best fit. The curves for the highest spin have been reflected on the right boundary to get all points onto one graph. See the note added in proof (3) on page 74.

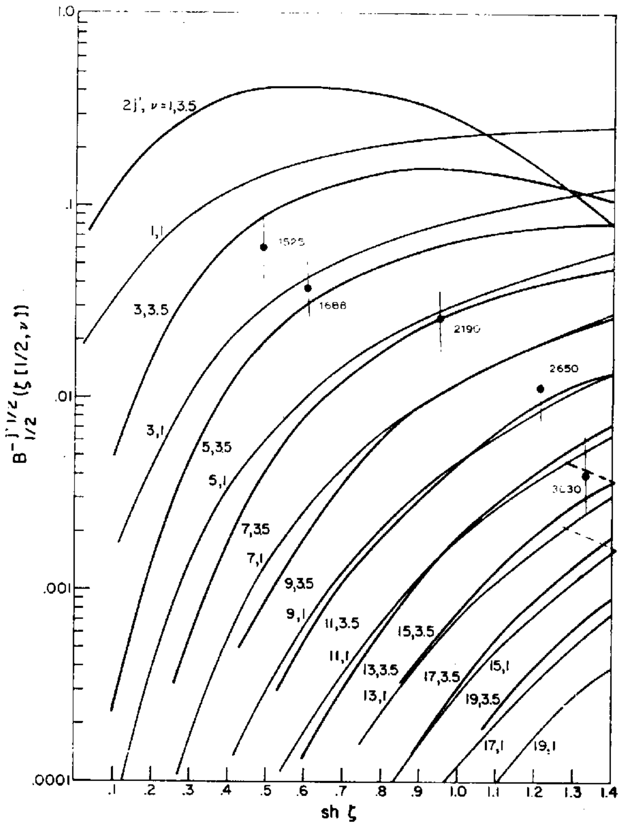


Fig. 4. The experimental amplitudes for the decay of the isospin $^1/_2$ baryons: $N(1525)$, (1688), (2190), (2650), (3030) $\to N\Pi$ are plotted and compared with theoretical curves $B_{1/2}^{-j'\,1/2}[\zeta(^1/_2, \nu)]$, which have been taken from Figure 7. We see that $\nu = 3.5$ gives the best fit. Again the curves for the highest spin have been reflected on the right boundary to get all experimental points onto one graph.

We have plotted on Figs. 3,4 the experimental amplitudes for the decays (2.46a) and (2.46b), and compared them with our theoretical function $B_{1/2}^{\pm j'\,1/2}(\zeta[j_0 \nu])$, respectively. The parameters which fit best are $\nu = 3.5$ and $G = 13.5$ and 19 [*6*]. The agreement is excellent. There are some more points for strange resonances on Fig. 3 which will be discussed in Ch. III.

4. Algebraic Meaning of $O(3,1)$ Dynamical Symmetry

a) Expansion of the Current into $O(3,1)$ Harmonics

Quite naturally the question arises now as to what kind of approximation of a possible exact theory is given by our $O(3,1)$ currents. This point is illuminated best by reducing the exact currents with respect to $O(3,1)$ symmetry, i.e. by performing an $O(3,1)$ "partial wave analysis." Our approximation then proves, as we expect, to be equivalent to assuming dominance of one partial wave in this expansion.

The most general form factor of a particle with mass M and spin s can be written as

$$I^\mu = \langle p' s_3' [M', s] | \Gamma^\mu | p s_3 [M s] \rangle \tag{2.48}$$

where Γ^μ is some vector operator and $|p s_3 [M s]\rangle$ are unitary representations of the Poincare group [1]. The state $|p s_3 [M s]\rangle$ can be expanded into the complete set of unitary $O(3,1)$ representations as:

$$|p s_3 [M, s]\rangle = \sum_{j_0 \leqq s} \int d\nu \, |j m [j_0 \nu]\rangle \langle j m [j_0 \nu] | p s_3 [M s]\rangle \tag{2.49}$$

where $\langle j m [j_0 \nu] | p s_3 [M s]\rangle$ are the "spherical harmonics" of $O(3,1)$. That j_0 has to be smaller than s is seen by going to the rest frame. There $|j m\rangle$ is necessarily $|s, s_3\rangle$ from rotational invariance. Then

$$|0 s_3 [M s]\rangle = \sum_{j_0} \int d\nu \, |s s_3 [j_0 \nu]\rangle \langle s s_3 [j_0 \nu] | 0 s_3 [M s]\rangle \tag{2.50}$$

and only $j_0 \leqq s$ therms contribute.

The matrix element

$$A_{j_0}^{M,s}(\nu) = \langle s s_3 [j_0 \nu] | 0 s_3 [M s]\rangle \tag{2.51}$$

is independent of s_3. By boosting we get

$$|p s_3 [M s]\rangle = \sum_{j_0} \int d\nu \, B(\zeta) |s s_3 [j_0, \nu]\rangle A_{j_0}^{M,s}(\nu) . \tag{2.52}$$

The form factor (2.48) becomes then, putting again $p' = 0$ and p in the Z-direction

$$I^\mu = \sum_{j_0' j_0} \int d\nu \, d\nu' \, A_{j_0'}^{M',s'*}(\nu) A_{j_0}^{M,s}(\nu) \langle s' s_3' [j_0' \nu'] | \Gamma^\mu | s s_3 [j_0 \nu] \zeta\rangle . \tag{2.53}$$

According to the selection rules for the vector operator, the matrix element can only exist if $[j_0' \nu'] = [j_0 \pm 1, \nu]$. For a spin $^1/_2$ particle then only $j_0 = {}^1/_2$ and one ν-integration survive and constructing as in (2.5) particles and antiparticles, we are left with

$$I^\mu = \int d\nu \, |A_{j_0}^{M,s}(\nu)|^2 \langle {}^1/_2 s_3' [{}^1/_2, \nu] \eta + | \Gamma^\mu | {}^1/_2 s_3 [{}^1/_2, \nu] \eta +, \zeta\rangle . \tag{2.54}$$

The right matrix element is, however, according to the Wigner Eckart theorem nothing else but our current calculated in (2.20) times some reduced matrix element $\gamma(\nu)$

$$\langle {}^1/_2 s_3' [{}^1/_2, \nu] \eta + | \Gamma^\mu | {}^1/_2 s_3 [{}^1/_2, \nu] \eta +, \zeta\rangle = \gamma(\nu) \, I^\mu(\zeta, [{}^1/_2, \nu]) . \tag{2.55}$$

The assumption that $O(3,1)$ (or $O(3,2)$) is the dynamical group of electromagnetic interactions obviously now amounts to the dominance of some ν, as we stated in the beginning of this section, in the integral (2.54). The current then reduces to the one calculated in (2.20). We see that we could have started the approach from this side and might have justified the same approximation by the existence of Δ- and N-towers of baryons as $O(3,1)$ multiplets. Otherwise we get what we interpreted as the wave function in (2.39)

$$\psi^{1/2}_{j_0=1/2}(\nu) = A^{M,1/2}_{j_0=1/2}(\nu)\,\sqrt{\gamma(\nu)} \qquad (2.56)$$

of the system. A fixed ν corresponds to the particle being concentrated on a spherical shell of radius ν. This interpretation is supported by the fact that $O(3,1)$ can be considered as the dynamical group of the rigid rotator (see Table 1).

b) The $O(3,1)$ Spherical Harmonics $\langle jm[j_0\nu]\,|\,ps_3[Ms]\rangle$

The state $|ps_3[Ms]\rangle$ can be written as

$$U(\hat{p})\,B(\zeta)\,|0s_3'[Ms]\rangle\,D^{s*}_{s_3s_3'}(\hat{p}) \qquad (2.57)$$

where $B(\zeta)$ bootss by $\zeta = sh^{-1}\,p/M$ into the Z-direction and $U(\hat{p})$ rotates from Z into the $\hat{p}$-*direction*. Then the matrix element $\langle ps_3[Ms]\,|\,jm[j_0\nu]\rangle$ takes the form:

$$\begin{aligned}\langle ps_3[Ms]\,|\,jm[j_0\nu]\rangle &= D^{s}_{s_3\bar{s}_3}(\hat{p})\,\langle 0\bar{s}_3[Ms]\,|\,B^{-1}(\zeta)\,U^{-1}(p)\,|\,jm[j_0\nu]\rangle\\ &= D^{s}_{s_3\bar{s}_3}(\hat{p})\,\langle 0\bar{s}_3[Ms]\,|\,B^{-1}(\zeta)\,|\,j\bar{s}_3[j_0\nu]\rangle\,D^{j*}_{m\bar{s}_3}(\hat{p})\,.\end{aligned} \qquad (2.58)$$

Using the spectral function $A^{Ms}_{j_0}(\nu)$ of (2.51) and the global representation of B from (2.23) we obtain

$$\langle ps_3[Ms]\,|\,jm[j_0\nu]\rangle = D^{s}_{s_3\bar{s}_3}(\hat{p})\,D^{j*}_{m\bar{s}_3}(\hat{p})\,B^{js*}_{\bar{s}_3}(\zeta[j_0\nu])\,A^{*Ms}_{j_0}(\nu)\,. \qquad (2.59)$$

$A^{Ms}_{j_0}(\nu)$ can now be determined from the completeness relation of the states $|ps_3[Ms]\rangle$ and the orthogonality of $|jm[j_0\nu]\rangle$:

$$\sum_{s_3}\int\frac{d^3p}{2p_0}\langle j'm'[j_0'\nu']\,|\,ps_3[Ms]\rangle\,\langle ps_3[Ms]\,|\,jm[j_0\nu]\rangle = \delta_{j'j}\,\delta_{m'm}\,\delta_{j_0'j_0}\,\delta(\nu'-\nu)\,. \qquad (2.60)$$

Inserting Equ. (2.59) into this we get

$$\begin{aligned}&\sum_{\substack{s_3\\ \bar{s}_3\bar{s}_3'}}\int d\hat{p}\,D^{s*}_{s_3\bar{s}_3'}(\hat{p})\,D^{j'}_{m'\bar{s}_3'}(\hat{p})\,D^{s}_{s_3\bar{s}_3}(\hat{p})\,D^{j*}_{m\bar{s}_3}(\hat{p})\,\times\\ &\times\int\frac{p^2dp}{2p_0}\,B^{j's}_{\bar{s}_3'}(\zeta[j_0'\nu'])\,B^{js}_{\bar{s}_3}(\zeta[j_0\nu])^*\,|A^{Ms}_{j_0}(\nu)|^2 = \delta_{j'j}\,\delta_{m'm}\,\delta_{j_0'j}\,\delta(\nu'-\nu)\,.\end{aligned} \qquad (2.61)$$

The $\sum_{s_3}$ obviously leads to $\delta_{\bar{s}_3\bar{s}_3'}$ and the angular integration can be performed to give

$$\frac{4\pi}{2j+1}\,\delta_{j'j}\,\delta_{m'm}\,. \qquad (2.62)$$

All that remains it then:

$$\frac{4\pi}{(2j+1)}\sum_{s_3}\int\frac{p^2dp}{2p_0}\,B^{js}_{s_3}(\zeta[j_0'\nu'])\,B^{js*}_{s_3}(\zeta[j_0\nu])\,|A^{Ms}_{j_0}(\nu)|^2 = \delta_{j_0'j_0}\,\delta(\nu'-\nu)\,. \qquad (2.63)$$

We evaluate this integral for the simplest case $s = 0$. Then j_0, j_0' are necessarily zero. The functions B are in this case (see App. A 31)

$$B_0^{j0}(\zeta[0, \nu]) = N^{j0}(0 j_1)\, 2^{j+1/2} (j + 1/2)!\,(\gamma^2 - 1)^{-1/4} P_{j_1-1/2}^{-j-1/2}(\gamma) \qquad (2.64)$$

where $\gamma = \operatorname{ch}\zeta = [1 - v^2/c^2]^{1/2}$
with

$$N^{j0}(0 j_1) = 2^j \frac{j!}{(2j)!}\left[\frac{(1+\nu^2)\dots(j^2+\nu^2)}{2j+1}\right]^{1/2}. \qquad (2.65)$$

Using $j!/(2j)!\,(j + 1/2)! = [(2j+1)/2^{2j+1}]\,\pi^{1/2}$ this becomes

$$B_0^{j0}(\zeta[0,\nu]) = \frac{\pi}{\nu}\left[\frac{2j+1}{2\pi}\right]^{1/2}\left|\frac{(i\nu+j)!}{(i\nu-1)!}\right|(\gamma^2-1)^{-1/4} P_{i\nu-1/2}^{-j-1/2}(\gamma). \qquad (2.66)$$

Now

$$\int \frac{p^2\,dp}{2p_0} = \frac{1}{2}\int_1^\infty \sqrt{\gamma^2 - 1}\,d\gamma \qquad (2.67)$$

and since

$$\int_1^\infty d\gamma\, P_{i\nu-1/2}^{-j-1/2}(\gamma)\, P_{-i\nu'-1/2}^{-j-1/2}(\gamma) = \frac{(i\nu-1)!\,(-i\nu-1)!}{(i\nu+j)!\,(-i\nu+j)!}\,\delta(\nu-\nu') \qquad (2.68)$$

we find from (2.63)

$$|A_0^{M0}(\nu)| = \frac{\nu}{\pi}. \qquad (2.69)$$

The spherical harmonic of $O(3,1)$ for spin zero is then (from (2.59) and (2.69)) [6,1]

$$\langle p_0[M0]\,|\,jm[0,\nu]\rangle = D^{*j}_{M0}(\hat p)\left[\frac{2j+1}{2\pi}\right]^{1/2}\left|\frac{(i\nu+j)!}{(i\nu-1)!}\right|(\gamma^2-1)^{-1/4} P_{i\nu-1/2}^{-j-1/2}(\gamma) \qquad (2.70)$$

where the phases have been chosen such that $L_{\mu\nu} = 1/i\,(p_\mu\partial_\nu - p_\nu\partial_\mu)$ has the same matrix elements on the functions (2.70) as $L_{\mu\nu}$ in Equ. (A. 2).
These functions clearly can be obtained, apart from a normalization constant, from the $O(4)$ spherical harmonics (where j_1 = integer)

$$Y_{j_1}^{jm}(y) = \frac{1}{2}\left[(j_1+1)\,\frac{2j+1}{4\pi}\,\frac{(j_1+j+1)!}{(j_1-j)!}\right]^{1/2} D^{*j}_{m0}(\hat y)\, P_{j_1+1/2}^{-j-1/2}\left(\frac{y_0}{R}\right) \qquad (2.71)$$

by analytic continuation of j_1 into $i\nu$.

5. Current Conservation and Mass Spectrum

In Paragraph 3 of this chapter, when discussing the electromagnetic interactions in $O(3,1)$ theory, we have tacitly assumed that all currents calculated from the vector operator Γ^μ by means of Equ. (2.20) are conserved. It turns out that this is in general only true if the current is elastic, i.e. its initial and final particles are the same. If Γ^μ is to give the complete electromagnetic current also for inelastic processes, then current conservation fixes the whole mass spectrum uniquely. This can easily be shown: Current conservation implies:

$$(p' - p)^\mu I_\mu = 0 \tag{2.72}$$

if p', p denote the momentum of initial and final particle, respectively, i.e.:

$$p' = (M', 0, 0, 0), p = (p_0, \boldsymbol{p}). \tag{2.73}$$

Observe now that in terms of matrix elements (2.72) can be rewritten as:

$$M' \langle M' | \Gamma^0 e^{iM\vec{\xi}} | M \rangle - \langle M' | p^\mu \Gamma_\mu \, e^{iM\vec{\xi}} | M \rangle = 0 \tag{2.74}$$

but due to the vector transformation

$$e^{iM\vec{\xi}} M \Gamma_0 \, e^{-iM\vec{\xi}} = p^\mu \Gamma_\mu \tag{2.75}$$

this reduces to

$$M' \langle M' | \Gamma^0 \, e^{iM\vec{\xi}} | M \rangle - M \langle M' | e^{iM\vec{\xi}} \Gamma^0 | M \rangle = 0. \tag{2.76}$$

If now the particles are eigenstates of Γ^0 as in our $O(3,1)$ theory

$$\Gamma_0 | M \rangle = \gamma | M \rangle \tag{2.77}$$

we see that

$$(M'\gamma' - M\gamma) = 0 \tag{2.78}$$

is necessary for current conservation. For elastic currents this is automatically fulfilled. For inelastic ones, we find

$$M = \frac{M_0}{\gamma}.$$

In the $O(3,1)$ case, $\gamma = j + 1/2$ and the mass spectrum becomes

$$M = \frac{M_0}{j + 1/2} \tag{2.79}$$

which gets smaller with increasing spin and has no correspondence in particle spectroscopy. This mass spectrum has been found by Majorana in 1932 for the special case of $\nu = 0$ by using an infinite component wave equation [5]. We shall generalize this method and discuss the relation to our purely algebraic approach later in Ch. V.

In order to assign the Δ- and N-baryon towers (see p. 6) to an $O(3,1)$ multiplet and get the right mass spectrum without violating current conservation, we obviously would have to add new terms of the form (2.18), (2.19) to the current operator Γ^μ. In particular terms of the form $\Gamma_1^\mu = f(p'^\mu + p^\mu)$ with constant f have the nice property of reversing the j-dependence of the mass spectrum (2.79) to an increasing function of j. The consequences of such terms will need future investigation. Hopefully, the condition on the mass Spectrum will uniquely single out the correct current among all possible vector operators. With such terms the postulate that $O(3,2)$ as a dynamical group has to contain all current operators in the algebra is then, however, violated. To get around this difficulty, we may

either extend $O(3,2)$ and the currents (2.19) by more operators until they close to a much larger group or weaken our postulate. We shall not do either one but rather take the standpoint that the need of additional current terms like (2.19) expresses a weakness of the dynamical group we started out with and rather look for a better group. We basically would like to avoid any explicitly momentum dependent currents, which allow for too much freedom on the theory. The current operator should be purely algebraic and essentially fixed by a small group, while ad hoc terms like (2.18), (2.19) are obviously not.
Thus also from this point of view it is strongly suggested to use a larger dynamical group for electromagnetic interactions which will be done in Ch. VI for baryons.

6. Majorana Equation for an $O(3,1)$ Multiplet

The 4-vector operator Γ^μ used for the electromagnetic coupling can also be used to couple with the 4-momentum p^μ to a scalar, and we can write an infinite component Poincare invariant wave equation on an arbitrary $O(3,1)$ representation space $[j_0, \nu] = [^1/_2, \nu]$.

$$\left(p_\mu \Gamma^\mu + \varkappa(M^2, W^2)\right) \psi(x) = 0 \tag{2.80}$$

where $\varkappa$ is an arbitrary function of the Poincare invariants

$$M^2 = -\partial^\mu \partial_\mu$$

and

$$W^2 = (i\, \varepsilon^{\mu\nu\lambda\varkappa} L_{\nu\lambda}\, \partial_\varkappa)^2 . \tag{2.81}$$

By using the property of the Γ^μ matrices of transforming under the Lorentz transformation Λ as

$$U^{-1}(\Lambda)\, \Gamma^\mu\, U(\Lambda) = \Lambda^\mu{}_\nu \Gamma^\nu \tag{2.82}$$

where $U(\Lambda)$ is the representation of Λ in the doubled Hilbert space, we can write (2.80) as

$$\left(p_\mu \Gamma^\mu + \varkappa(M^2, W^2)\right) U(\Lambda)\, \psi(\Lambda^{-1}x) = U(\Lambda) \left(p_0 \Gamma^0 + \varkappa(M^2, W^2)\right) \psi(\Lambda^{-1}x) = 0 . \tag{2.83}$$

In momentum space we get from this on the state at rest

$$(j + {}^1/_2)\, M + \varkappa\left(M^2, M^2 j(j+1)\right) = 0 \tag{2.84}$$

which is an implicit formula for the masses

$$M = M(j) \tag{2.85}$$

Majorana used a constant $\varkappa$ and obtained

$$M = \frac{\varkappa}{j + {}^1/_2} . \tag{2.86}$$

What does the Majorana equation mean algebraically? Consider again the scalar product of Poincare and Lorentz states discussed in Sec. 4.

$$\langle p s_3 [M s] \,|\, j m [j_0 \nu] \rangle .$$

From what was said there it is clear that the Hilbert space $|jm[j_0\nu]\rangle$ can be spanned by the Poincare states at rest $|0s_3[ms]\rangle$ if one uses all spins $s \geqq j_0$. For every spin one may choose any mass M to go with it. Precisely this choice is fixed by the Majorana equation.
To see this we go to the momentum representation of the wave function $\psi(x)$. There we obtain

$$\big(p_\mu \Gamma^\mu + \varkappa(M^2, M^2 s(s+1))\big) \langle p s_3[Ms] | jm[j_0\nu]\rangle = 0. \tag{2.87}$$

In the rest frame this reduces to

$$\big(M(j + {}^1/_2) + (M^2, M^2 j(j+1))\big) \langle 0m[Mj] | jm[j_0\nu]\rangle = 0. \tag{2.88}$$

Thus we see that only those masses $M = M(s)$ are allowed to occur in the scalar product which follow the mass formula (2.84).
The Poincare group is thus coupled in a highly complicated way to the $O(3,1)$ group, and the Majorana equation gives the most concise way of describing this coupling.
In (11.5) we found that Γ^μ gives only then a completely conserved current if the mass spectrum is that of Majorana equation (2.86). This corresponds to the infinite component equation

$$(p_\mu \Gamma^\mu + M_0)\psi = 0. \tag{2.89}$$

Current conservation is now a consequence of this equation:

$$\langle p' | (p' - p)^\mu \Gamma_\mu | p\rangle = (M_0 - M_0) \langle p' | p\rangle = 0. \tag{2.90}$$

We see again that for general $\varkappa = \varkappa(M^2, W^2)$ only the diagonal current is conserved.
Assuming Γ^μ to give the electromagnetic interaction amounts now obviously to postulating minimal electromagnetic coupling in Equ. (2.89) through the substitution:

$$p_\mu \to p_\mu + eA_\mu. \tag{2.91}$$

Majorana's equation has been generalized by NAMBU [7] who uses a larger group D than the Lorentz group, to construct a vector Γ^μ and to write an infinite component equation like (2.80). In this case one can generally add more terms to the equation which are scalars in the group D under the Lorentz subgroup. This has the consequence that infinite mixtures of states can be eigenstates of Γ_0. We shall briefly dicsuss such an equation later in Ch. VI, after having investigated $O(4,2)$ as the dynamical group of electromagnetic interactions of the H-atom.

III. Minimal Breaking of Internal Symmetries

Until now we have neglected internal symmetries like isospin and strangeness completely. We shall assume here that for every spin level in $O(3,1)$ there exists some broken $SU(3)$ symmetry. The simplest way to include $SU(3)$ into a dynamical $O(3,1)$ theory is by postulating the direct product $O(3,1) \times SU(3)$ [9, 10] as a dynamical group. In an irreducible representation space this means that at every spin j in the $O(3,1)$ tower there lies a whole $SU(3)$ multiplet which is the same for every spin. Thus we get octet towers decuplet towers, etc., as particle multiplets. We may then assume that the Δ-resonances form in fact, together with $\Sigma(1385)$, (1770), (2035), (2260)? [8], $\Xi(1530)$, (1933)? a decuplet tower, while the nucleon octet and $\Sigma(1660)$, (1910), (2260)?, $\Lambda(1520)$, (1820), (2100), (2340)?, $\Xi(1815)$? may be assigned to an octet tower. On these Hilbert spaces we then assume the baryons to couple to a pseudoscalar meson octet by an interaction similar to (2.43)

$$A(\zeta) = \frac{G}{\sqrt{2}} \langle j' m [j_0 \nu], i | P \lambda^j e^{iM_3 \zeta} | j m [j_0 \nu], k \rangle \tag{3.1}$$

where i, j, k are $SU(3)$ indices and λ^i are the eight $SU(3)$ generators. Observe that the amplitude possesses exact $SU(3)$ symmetry if $\zeta = 0$, which happens for the invariant momentum transfer

$$t = (M_i - M_k)^2 \tag{3.2}$$

and is a different point of the mass shell for every external configuration of particles. There the amplitudes are just given, up to $G/\sqrt{2}$, by $SU(3)$ Clebsch Gordan coefficients (C.G.)

$$A = \frac{G}{\sqrt{2}} \times \text{C.G.} \tag{3.3}$$

For ζ different from zero this amplitude gets multiplied by the *universal function of the rapidity* $B_m^{\pm j' j}(\zeta [j_0, \nu])$, since $SU(3)$ commutes with the generators $\boldsymbol{M}$ of $O(3,1)$. The decay rates become now

$$\Gamma = \Phi \frac{G^2}{2} (\text{C.G.})^2 \sum_m |B_m^{\pm j' j}(\zeta [j_0 \nu])|^2 \tag{3.4}$$

with Φ being again the invariant phase space (2.45). If we normalize the Clebsch Gordan coefficient for $\Delta \to N\pi$ (or $N^* \to N\pi$) to one, then for the $\Delta(N^*)$ decays this calculation has to go over into the old one. Hence, the parameters are fixed to be $\nu = 3.5$, $G = 13.5$ (19). For those resonances, whose spin parity assignment is known and fits on our decuplet tower ($3/2^+, 5/2^+, 7/2^+$), we have compared the experimental amplitudes for the decay into nucleon octet and meson with the theoretical function $B_{1/2}^{\pm j' 1/2}(\zeta [1/2, 3.5])$ on Fig. 3 where also the $\Delta \to N\pi$ amplitudes are plotted. We see that the agreement is excellent, except for the $\Delta\pi$ mode of the $\Sigma(1770)$, which is off by a factor of 2. The (1770) is therefore probably not a member of our decuplet tower. The same thing would have to be done for the octet tower also. There the fact that 8×8 can couple to 8 in two ways will introduce an additional parameter. Also there one obtains good agreement with experiment (B. Hamprecht and H. Kleinert, to be published).
Observe that from Equ. (3.4) the decay widths exhibit definitely broken $SU(3)$ symmetry since different members of one multiplet decay with different rapidities. Only if $B(\zeta)$ would be constant in ζ the symmetry would be exact. Fig. 3 shows, however, that $B(\zeta)$ has quite a strong slope in the range of the experimental points and therefore the symmetry breaking is considerable. On the other hand, since the inclusion of $SU(3)$ has been algebraically the simplest one possible through the direct product, we shall call symmetry breaking arising this way "minimal." We want to remark that traditional $SU(3)$ calculations use ad hoc

the centrifugal barrier P/M_f instead of the amplitude $B(\zeta)$, which is just the first term in a Taylor series expansion of $B(\zeta)$. Since $B(\zeta)$ on Fig. 3 doesn't deviate too much from its first Taylor term yet, the results are similarly good. The advantage of our approach in this respect is that the way mass differences have to be taken into account is *uniquely* prescribed.

IV. The Exactly Soluble $O(4,2)$ Model of the H-Atom

1. Introduction

The conclusion of Chapter II has been that physical baryon states must be mixtures of irreducible representations of the Lorentz group. Such representation mixing occurs in a natural way if one goes to a larger dynamical group D to describe electromagnetic interactions. D will then contain in general infinitely many $O(3,1)$ subgroups. The booster may be chosen differently for every state, and the vector operator Γ^μ can be dependent on the states between which it acts. How can one find some order in this great amount of freedom of choice?
There is one system in nature which is an ideal object to study how representation mixing is done in a systematic way: The dynamics of the non-relativistic H-atom can be completely described in group theoretical language without any use of internal coordinates. An expression for the current can be found which allows us to calculate the transition form factors for all the radiative decays

$$H^* \to H + \gamma \tag{4.1}$$

and for which current conservation fixes the mass spectrum to the observed one

$$M_n = M - \frac{\varepsilon}{2n^2} \tag{4.2}$$

where $\varepsilon = \mu\alpha^2$, $M = m_p + m_e$, $\mu = m_e m_p/m_e + m_p$. We therefore shall discuss this case in some detail. Its algebraic structure will then be generalized and applied to the description of particle dynamics.

2. The Group of Quantum Numbers $O(4,1)$ of the H-Atom and its Extension to $O(4,2)$

Consider the representation of $O(4,1)$ given by

$$\begin{aligned}
L_{ij} &= {}^1/_2\,(a^+\sigma_k a + b^+\sigma_k b) \equiv L_k\,,\\
L_{i4} &= -{}^1/_2(a^+\sigma_i a - b^+\sigma_i b) \equiv R_i\,,\\
L_{i5} &= -{}^1/_2(a^+\sigma_i C b^+ - a C\sigma_i b)\,,\\
L_{45} &= 1/(2i)\,(a^+ C b^+ - a C b)
\end{aligned} \tag{4.3}$$

on the Hilbert space

$$|n_1 n_2 m\rangle = [n_1!(n_2+|m|)!n_2!(n_1+|m|)!]^{1/2} \begin{cases} a_1^{+n_2+m} a_2^{+n_1} b_1^{+n_1+m} b_2^{+n_2}|0\rangle & m>0 \\ & \text{for} \\ a_1^{+n_2} a_2^{+n_1-m} b_1^{+n_1} b_2^{+n_2-m}|0\rangle & m<0 \end{cases} \tag{4.4}$$

for all $n_1, n_2 \geqq 0$, where a_r^+, b_r^+ are creation operators satisfying the commutation rules

$$[a_r a_s^+] = \delta_{rs}\, [b_r b_s^+] = \delta_{rs}. \tag{4.5}$$

The commutation rules of $L_{\mu\nu}$ are

$$[L_{\mu\nu} L_{\mu\lambda}] = i g_{\mu\mu} L_{\nu\lambda} \tag{4.6}$$

with

$$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

The representation is obviously unitary and irreducible. Its Casimir operators have the values

$$C_2 \equiv L_{ab} L^{ab} = -4,$$
$$C_4 \equiv L_{ab} L^{bc} L_{cd} L^{da} = 0. \tag{4.7}$$

L_k, L_{i4} generate an $O(4)$ subgroup which keeps the total number of operators a, b or

$$N = {}^1/_2 (a^+ a + b^+ b + 2) \tag{4.8}$$

invariant. On the states $|n_1 n_2 m\rangle$ we find

$$N|n_1 n_2 m\rangle = n|n_1 n_2 m\rangle = (n_1 + n_2 + |m| + 1)|n_1 n_2 m\rangle. \tag{4.9}$$

The other diagonal operators are:

$$L_3|n_1 n_2 m\rangle = m|n_1 n_2 m\rangle, \tag{4.10}$$

$$M_3|n_1 n_2 m\rangle = (n_1 - n_2)|n_1 n_2 m\rangle. \tag{4.11}$$

If we identify the operators as

$$L = \text{orbital angular momentum},$$
$$N = \frac{1}{\sqrt{-2H}} \tag{4.12}$$

where H is the Hamiltonian of the H-Atom, we see that we have thus generated with $O(4,1)$ the complete bound state Hilbert space of the hydrogen atom in the parabolic basis which is used for the theory of the Stark effect. In position space, the wave functions are given by [*11*]

$$u_{n_1 n_2 m}(\xi, \eta, \varphi) = e^{im\varphi} N_{n_1 n_2 m}\, e^{-i\frac{\xi+\eta}{2}} \left(\frac{\xi\eta}{n^2}\right)^{\frac{|m|}{2}} \times L^{|m|}_{n_1+|m|}(\xi/n)\, L^{|m|}_{n_2+|m|}(\eta/n) \tag{4.13}$$

where

$$\xi = r + z,$$
$$\eta = r - z \tag{4.14}$$

and Φ is the azimutal angle and the normalization constant is:

$$N_{n_1 n_2 m} = \frac{(-)^{n_2+(|m|-m)/2}}{\sqrt{\pi} n^2} \left[\frac{n_1!\, n_2!}{(n_1+m)!^3\, (n_2+m)!^3} \right]^{1/2}. \tag{4.15}$$

In the position representation L and $L_{i4} = R_i$ (which is the well known Lenz Runge vector) are

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{4.16}$$

$$\boldsymbol{R} = {}^1/_2(\boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{p}) - \hat{r} \tag{4.17}$$

if one identifies $u_{n_1 n_2 m}(\xi, \eta, \Phi)$ with $|n_1 n_2 m\rangle$. [12]
One may then ask, what the usual wave functions $\psi_{nlm}(x)$, on which L^2 and L_3 are diagonal, look like on our representation space. For this one just has to observe that

$$L = J + K, \quad R = -J + K \tag{4.18}$$

defines L, R in terms of the commuting $O(3) \times O(3)$ generators of a- and b-spin:

$$J = {}^1/_2\, a^+ \sigma a, \quad K = {}^1/_2\, b^+ \sigma b. \tag{4.19}$$

On $|n_1 n_2 m\rangle$, $O(3) \times O(3)$ is diagonal since all $a's$, $b's$ commute among each other and therefore couple totally symmetrically to $j = k = (n_1 + n_2 + |m|)/2 = (n-1)/2$. One can also read easily off, by counting up- and down-states that:

$$j_3 = {}^1/_2(n_2 - n_1 + m), \quad k_3 = {}^1/_2(n_1 - n_2 + m). \tag{4.20}$$

Now one only has to use the fact that J and K commute. Then the basis on which L^2 is diagonal is just given by means of Wigner's $3-j$ symbols

$$|nlm\rangle = (-)^m (2l+1)^{1/2} \begin{pmatrix} \frac{n-1}{2} & \frac{n-1}{2} & l \\ {}^1/_2(n_2 - n_1 + m) & {}^1/_2(n_1 - n_2 + m) & -m \end{pmatrix} |n_1 n_2 m\rangle. \tag{4.21}$$

In x-space one has to identify $|nlm\rangle$ with

$$\Psi_{nlm}(\boldsymbol{x}) = N_{nl}\, e^{-\frac{r}{n}} \left(\frac{r}{n}\right)^l F\left(-n_r, 2l+2, 2\frac{r}{n}\right) \text{[11]} \tag{4.22}$$

where

$$N_{nl} = \frac{2^{l+1}}{(2l+1)!\, n^2} \left[\frac{(n+l)!}{n_r!}\right]^{1/2} \tag{4.23}$$

and n_r is the radial quantum number $n_r = n - l - 1$.

Observe now, that the group $O(4,1)$ can be extended to $O(4,2)$ on the same Hilbert space by introducing the additional (hermitian) operators

$$L_{i6} = \frac{1}{2i}\,(a^+\sigma_i C b^+ + a C \sigma_i b),$$
$$L_{46} = {}^1/_2(a^+ C b^+ + a C b),$$
$$L_{56} = N \tag{4.24}$$

which close with (4.3) under the commutation rules (4.6) with μ, etc. running now from 1 to 6 and the metric

$$g = \begin{pmatrix}1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1\end{pmatrix}. \tag{4.25}$$

This is, on the other hand, the maximal continuous extension of $O(4,1)$ on this Hilbert space, as one can easily see [3]. The only operations one can add are discrete ones, like parity, which we shall discuss in Ch. V.

The group $O(4,2)$ turns out to be the dynamical group of the H-atom with respect to electromagnetic transitions. To show this we shall make extensive use of the $O(2,1) \times O(2,1)$ subgroup of $O(4,2)$ containing only the generators:

$$L_{35}, L_{34}, L_{45}; L_{56}, L_{36}, L_{46}. \tag{4.26}$$

Define the operators

$$N_1^+ = -a_2^+ b_1^+,$$
$$N_1^- = -a_2 b_1,$$
$$N_1^3 = {}^1/_2(a_2^+ a_2 + b_1^+ b_1 + 1) = {}^1/_2(N_{a_2} + N_{b_1} + 1),$$
$$N_2^+ = a_1^+ b_2^+,$$
$$N_2^- = a_1 b_2,$$
$$N_2^3 = {}^1/_2(a_1^+ a_1 + b_2^+ b_2 + 1) \equiv {}^1/_2(N_{a_1} + N_{b_2} + 1) \tag{4.27}$$

and, as usual, their cartesian combinations

$$N_i^1 = {}^1/_2(N_i^+ + N_i^-),$$
$$N_i^2 = \frac{1}{2i}\,(N_i^+ - N_i^-). \tag{4.28}$$

Then we see that also these operators commute according to $O(2,1) \times O(2,1)$ rules:

$$[N^i_{\substack{1\\2}} N^j_{\substack{1\\2}}] = i g_{kk} N^k_{\substack{1\\2}} \quad (i, j, k = \text{cyclic, running from 1 to 3}),$$

$$[N_1^i, N_2^j] = 0, \quad g = \begin{pmatrix}1 & & \\ & 1 & \\ & & -1\end{pmatrix}. \tag{4.29}$$

In terms of N_i^j, the operators (4.26) are:

$$\begin{aligned}
L_{34} &= N_1^3 - N_2^3, \\
L_{35} &= N_1^1 - N_2^1, \\
L_{45} &= N_1^2 + N_2^2, \\
L_{36} &= -N_1^2 + N_2^2, \\
L_{46} &= N_1^1 + N_2^1, \\
L_{56} &= N_1^3 + N_2^3.
\end{aligned} \tag{4.30}$$

On the basis $|n_1 n_2 m\rangle$ the matrix elements of $N_{\frac{1}{2}}$ have a very simple representation, as can be seen by direct application of (4.27) onto (4.4):

$$\begin{aligned}
N_1^{\pm}|n_1 n_2 m\rangle &= -\left[\left(n_1 + \begin{Bmatrix}1\\0\end{Bmatrix}\right)\left(n_1 + m + \begin{Bmatrix}1\\0\end{Bmatrix}\right)\right]^{1/2} |n_1 \pm 1 n_2 m\rangle, \\
N_2^{\pm}|n_1 n_2 m\rangle &= \left[\left(n_2 + \begin{Bmatrix}1\\0\end{Bmatrix}\right)\left(n_2 + m + \begin{Bmatrix}1\\0\end{Bmatrix}\right)\right]^{1/2} |n_1 n_2 \pm 1 m\rangle, \\
N_r^{3}|n_1 n_2 m\rangle &= (2n_r + m)|n_1 n_2 m\rangle \quad (r = 1,2).
\end{aligned} \tag{4.31}$$

The whole $O(4,2)$ can be given in x-space representation if we find N_i^j as functions of x. Using (4.31) and

$$\begin{aligned}
L^m_{n_1+1+m}(\xi) &= \frac{n_1 + m + 1}{n_1 + 1}\left(\xi \frac{\partial}{\partial \xi} + n_1 + m + 1 - \xi\right) L^m_{n_1+m}(\xi), \\
L^m_{n_1-1+m}(\xi) &= \frac{1}{(n_1 + m)^2}\left(\xi \frac{\partial}{\partial \xi} - n_1\right) L^m_{n_1+m}(\zeta)
\end{aligned} \tag{4.32}$$

one easily finds before the states $u_{n_1 n_2 m}(\xi, \eta, \Phi)$

$$\begin{aligned}
N_1^{+} &= -D_{\frac{n}{n+1}}\left(\xi \frac{\partial}{\partial \xi} - \xi + \frac{\xi}{2n} + \frac{L_3}{2} + n_1 + 1\right)\left(\frac{n}{n+1}\right)^2 \\
N_1^{-} &= -D_{\frac{n}{n+1}}\left(\xi \frac{\partial}{\partial \xi} + \frac{\xi}{2n} - \frac{L_3}{2} - n_1\right)\left(\frac{n}{n+1}\right)^2
\end{aligned} \tag{4.33}$$

where $D_{\frac{n}{n+1}}$ is the dilatation operator defined by [*14*]:

$$D_a u(x) = u(ax). \tag{4.34}$$

3. The Dipole Operator in $O(4,2)$ Language

In order that $O(4,2)$ may be the dynamical group of the H-atom, we have to be able to express at least the diople operator $\boldsymbol{x}$ completely in terms of Lie algebra operators and group elements. This is indeed possible. Consider for example

$z = {}^1/_2(\xi - \eta)$. Using the recursion relation of Laguerre polynomials [15]

$$x L^m_{n_1+m}(x) = -\frac{n_1+1}{n_1+m+1} L^m_{n_1+m+1}(x) + (2n_1+m+1) L^m_{n_1+m}(x) - \\ - (n_1+m)^2 L^m_{n_1+m+1}(x) \tag{4.35}$$

we obtain

$$\frac{\xi}{n} u_{n_1 n_2 m} = -\frac{n_1+1}{n_1+m+1} \frac{N_{n_1 m_2 m}}{N_{n_1+1 n_2 m}} D_{\frac{n+1}{n}} u_{n_1+1 n_2 m} + \\ + (2n_1+m+1) u_{n_1 n_2 m} - \\ - (n_1+m)^2 \frac{N_{n_1 n_2 m}}{N_{n_1-1 n_2 m}} D_{\frac{n-1}{n}} u_{n_1-1 n_2 m} \tag{4.36}$$

and a similar equation for $\eta/n\, u_{n_1 n_2 m}$ where D is again the dilatation operator, (4.34). If one inserts $N_{\frac{1}{2}}$ from (4.31), we find

$$\xi u_{n_1 n_2 m} = \left(D_{\frac{n+1}{n}} N_1^+ \frac{(n+1)^2}{n} + M_3 + N + D_{\frac{n-1}{n}} N_1^- \frac{(n-1)^2}{n} \right) u_{n_1 n_2 m},$$

$$\eta u_{n_1 n_2 m} = \left(D_{\frac{n+1}{n}} N_2^+ \frac{(n+1)^2}{n} - M_3 + N + D_{\frac{n-1}{n}} N_2^- \frac{(n-1)^2}{n} \right) u_{n_1 n_2 m}. \tag{4.37}$$

Subtracting these equations and making use of the rotational symmetry and the relations (4.30), one obtains for the x-operator [14]:

$$x_i = D_{\frac{n+1}{n}} \frac{1}{2} (L_{i5} - i L_{i6}) \frac{(n+1)^2}{n} + M_i + D_{\frac{n-1}{n}} \frac{1}{2} (L_{i5} + i L_{i6}) \frac{(n-1)^2}{n},$$

$$r = D_{\frac{n+1}{n}} \frac{1}{2} (L_{46} + i L_{45}) \frac{(n+1)^2}{n} + N + D_{\frac{n-1}{n}} \frac{1}{2} (L_{46} - i L_{45}) \frac{(n-1)^2}{n}. \tag{4.38}$$

The dilatation operator in front turns out to be up to a factor a group element in $O(4,2)$. This permits us to write the x operator in the form

$$\langle n' l' m' | x_i | n l m \rangle = \frac{i}{\omega_{n'n}} \frac{1}{n' n} \langle n' l' m' | e^{-i\theta_{n'n} L_{45}} L_{i6} | n l m \rangle + \langle n' l' m' | L_{i4} | n l m \rangle \tag{4.39}$$

where $\omega_{n'n}$ is the Rydberg frequency for the transition $n \to n'$

$$\omega_{n'n} = -\frac{1}{2n^2} + \frac{1}{2n'^2} = \frac{1}{2} \frac{n^2 - n'^2}{n^2 n'^2} \tag{4.40}$$

and the angle $\theta_{n'n}$ is given by

$$\theta_{n'n} = \log \frac{n}{n'}. \tag{4.41}$$

The proof for this formula is given in Appendix C.

For the momentum operator p_i, the group theoretical form turns out to be very simple. Since L_{i4} commutes with the Hamiltonian

$$p_i = i[H, x_i] = i\left[-\frac{1}{2\,n^2}, x_i\right] \tag{4.42}$$

leads to

$$\langle n' l' m' | p_i | n l m\rangle = \frac{1}{n'\,n}\langle n' l' m' | e^{-i\theta_{n'n} L_{45}} L_{i6} | n l m\rangle . \tag{4.43}$$

We therefore can introduce the new non-orthogonal "tilted" basis into the Hilbert space

$$|\bar{n} l m\rangle = \frac{1}{n}\, e^{-i\theta_n L_{45}} | n l m\rangle \tag{4.44}$$

with the tilting angle

$$\theta_n = \log n a \tag{4.45}$$

in which the momentum operator becomes simply [*16*]

$$p_i = L_{i6} . \tag{4.46}$$

Remember that this holds for atomic units $\mu = 1$, $e = 1$, $\hbar = 1$. The parameter a in (4.45) is arbitrary. Thus we see that the conformal group indeed contains the whole algebra of observables of the H-atom in its Hilbert space, and dipole transitions can ce calculated by means of group and Lie algebra operations. Hence $O(4,2)$ is, according to our definition in the introduction, the dynamical group with respect to dipole transition. In the next section and Sec. 5 we shall show moreover, that the operators giving all electromagnetic form factors of bound states for arbitrary non-relativistic momentum transfers are also in the group. Thus $O(4,2)$ will prove to be in fact the dynamical group with respect to any electromagnetic bound-bound transition.

4. Complete Form Factors of the H-Atom

a) The Form Factor in $O(4,2)$ Language

The last section has provided us with an algebraic expression for the momentum operator p_i. From the correspondence principle we know that $1/m_e\, p_i$ is also the current operator of the electron in the H-atom, in the dipole approximation. In the notation of Ch. II we therefore identify the spatial part of the electromagnetic current Γ_i in this approximation with p_i or,

$$\Gamma_i = \frac{1}{m_e} L_{i6} \tag{4.47}$$

if it is understood (from (4.43)) that the physical states are represented by the tilted expressions:

$$|\bar{n} l m\rangle = \frac{1}{n}\, e^{-i\theta_n L_{45}} | n l m\rangle ; \qquad \theta_n = \ln n a . \tag{4.48}$$

Let's now recall how the complete non-relativistic form factor for all momentum transfers is calculated in Schrödinger theory. There we have to form the Fourier transform of electron charge and current distribution between an H-atom at rest and one that moves with momentum q. We obtain as the current operator

$$J^0 = 1\,, \qquad J^i = \Gamma^i + \frac{q^i}{2m_e} \tag{4.49}$$

giving the charge and current distribution of the electron by [11]

$$I^\mu = \int \psi^*_{n'l'm'}(\boldsymbol{x})\, J^\mu e^{-i\boldsymbol{q}\boldsymbol{x}_e}\, \psi_{nlm}(\boldsymbol{x})\, d^3\boldsymbol{x} \tag{4.50}$$

where q is the momentum transfer, and $x_e = \mu/m_e\, x$. $e^{-i\boldsymbol{q}\boldsymbol{x}_e}$ forms evidently a representation of the Galilei group on the space of bound state wave functions. In accordance with the discussion in Ch. (11.2) this is a dynamical way of representing the Galilei group, with the matrix elements giving transition probabilities under electromagnetic interactions. This case therefore provides us with an instructive physical example where group elements can indeed be used to give us complete transition form factors for all momentum transfers.
Here we *know* the answer to the interpretation problem of the matrix elements of the booster:

$$B = e^{-i\boldsymbol{q}\boldsymbol{x}_e} \tag{4.51}$$

How does one calculate I^μ as in (4.50) without referring to the x-representation using only group theoretical operations?
Under the above assumption, that $O(4,2)$ is the dynamical group of the H-atom, we have to find then Galilean generators $\boldsymbol{M}$ representing $\boldsymbol{x}$ inside the Lie algebra.
$\boldsymbol{M}$ has to satisfy the commutation rules of the Galilean group (which are satisfied by its $\boldsymbol{x}$-representation $\boldsymbol{M} = \boldsymbol{x}$ as one can easily verify)

$$[M_i M_j] = 0\,, \tag{4.52}$$

$$[L_i M_j] = i M_k \qquad (i,\, j,\, k \text{ cyclic, running form 1 to 3}). \tag{4.53}$$

What is now the complete current operator J^μ corresponding to (4.49) in group theoretical language? The algebraic part of J^i, Γ^i, has been fixed to give the correct dipole transitions by (4.47). But then also the zeroth component Γ^0 is determined from (4.49): Γ^μ as a Galilean current vector has to satisfy the commutation rules:

$$[M_i, \Gamma^0] = 0\,, \tag{4.54}$$

$$[M_i, \Gamma^i] = i\,\Gamma^0\,. \tag{4.55}$$

These equations (4.52, 3, 4, 5) have a unique solution in the Lie algebra once Γ_i has been fixed to be $\Gamma^i = (1/2m_e)\, L_{i6}$.
$\boldsymbol{M}$ has to be a vector from (4.53), hence its most general form is a combination of L_{i5}, L_{i6}, L_{i4}. Inserting this into (4.52) the combination is restricted to be

$$M_i = -m_p \frac{1}{a} (\cos\varkappa\, L_{i5} + \sin\varkappa\, L_{i6} - L_{i4})$$

with an arbitrary parameter a'.

Commuting this with Γ^i according to (4.55), it gives as a possible Γ^0

$$\Gamma^0 = -\frac{m_p}{m_e}\frac{1}{a'}(\cos\varkappa L_{56} - L_{46}) \tag{4.56}$$

which is, however, seen to fulfill (4.54) only for $\varkappa = 0$. Normalizing the charge to one finally fixes a' to be the tilting parameter a of equ. (4.48). Hence the only possible vector operator is

$$\begin{aligned} I^\mu &= \frac{m_p}{m_e}\left(-\frac{1}{a}(L_{56} - L_{46}), \frac{1}{m_e} L_{i6}\right) + \frac{(p'+p)^\mu}{2m_e}\frac{1}{a}(L_{56} - L_{46}) = \\ &= \left(\frac{1}{a}(L_{56} - L_{46}), \quad \frac{1}{m_e} L_{i6} + \frac{(p'+p)^i}{2m_e}\frac{1}{a}(L_{56} - L_{46})\right) \end{aligned} \tag{4.57}$$

with the Galilean generators

$$M_i = -\frac{m_p}{a}(L_{i5} - L_{i4}). \tag{4.58}$$

Thus we get for the prescription of calculating I^μ inside the $O(4,2)$ group, corresponding to the position space equation (4.50),

$$I^\mu = \langle \bar{n}' l' m' | J^\mu e^{i\frac{\boldsymbol{q}}{M}\cdot \boldsymbol{M}} \bar{n} l m\rangle. \tag{4.59}$$

We want to stress the fact that this expression is only that part of the total form factor of the H-atom which is due to the electronic orbit.
Remember that Equ. (4.59) has been obtained by postulating $O(4,2)$ to be a dynamical group for all momentum transfers q. Otherwise $\boldsymbol{M}$ would not have been uniquely specified; one could have added vectors in the enveloping algebra to it. Our postulate will be justified in Sec. 5, where we shall prove Equ. (4.59) by a direct transformation of (4.50) to the group theoretical form (4.59). First, this equation will be used to obtain the full bound-bound charge from factors for any transition

$$n' \to n + \gamma. \tag{4.60}$$

b) Evaluation of the Form Factors

We evaluate here Equ. (4.60) for the charge part [17]

$$I^0 = \langle \bar{n}' l' m' J^0 e^{i\frac{\boldsymbol{q}}{M}\boldsymbol{M}} | \bar{n} l m\rangle. \tag{4.61}$$

We furthermore use again the special frame of reference in which q points in the Z-direction, without loss of generality. Then I^0 is explicitly, absorbing tacitly

m_p/M into q:

$$I^0 = \frac{1}{a} \langle \bar{n}' | (L_{56} - L_{46}) e^{-i\frac{q}{a}(L_{35}-L_{34})} | \bar{n} \rangle =$$

$$= \frac{1}{a n n'} \langle n' | e^{i\theta_{n'} L_{45}} (L_{56} - L_{46}) e^{-i\frac{q}{a}(L_{35}-L_{34})} e^{-i\theta_n L_{45}} | n \rangle =$$

$$= \frac{1}{n} \langle n' | (L_{56} - L_{46}) e^{-i\theta_{n'n} L_{45}} e^{-iqn(L_{35}-L_{34})} | n \rangle \tag{4.62}$$

wehre $\theta_{n'n} = \ln(n/n')$, as in (4.41). Inserting a complete set of intermediate states, we can write:

$$I^0 = \frac{1}{n} \langle n' | \Gamma^0 | n'' \rangle \langle n'' | e^{-i\theta_{n'n} L_{45}} e^{-iqn(L_{35}-L_{34})} | n \rangle . \tag{4.63}$$

The matrix elements of Γ^0 are easily found to be (for $m \geqq 0$):

$$\begin{aligned} \langle n | \Gamma^0 | n \rangle &= n , \\ \langle n | \Gamma^0 | n_1' + 1, n_2', m \rangle &= {}^1/_2 \, [(n_1' + 1)\,(n_1' + m + 1)]^{1/2} , \\ \langle n | \Gamma^0 | n_1' - 1, n_2', m \rangle &= {}^1/_2 \, [n_1 (n_1 + m)]^{1/2} \end{aligned} \tag{4.64}$$

by using the representations (4.4) and (4.27). The finite transformation

$$G \equiv e^{-i\theta_{n'n} L_{45}} e^{-iqn(L_{35}-L_{34})} \tag{4.65}$$

can be evaluated if one observes that

$$K_1 = L_{45},\, K_2 = -L_{35},\, K_3 = L_{34}$$

close to an $O(2,1)$ subalgebra of the $O(2,1) \times O(2,1)$ algebra discussed in (4.30) satisfying:

$$[K_1, K_2] = -iK_3,\, [K_2, K_3] = iK_1,\, [K_3, K_1] = iK_2 . \tag{4.66}$$

K_3 is diagonal on the $|n_1 n_2 m\rangle$ basis with the eigenvalue $n_1 - n_2$. The other two operators can be written more explicitly in terms of the generators N_i^j as

$$\begin{aligned} K_1 &= N_1^2 + N_2^2 , \\ K_2 &= -(N_1^1 - N_2^1) . \end{aligned} \tag{4.67}$$

Consider now the representation of elements of the $O(2,1)$ groups formed by N_i^j. An irreducible representation of the N_1^i algebra, for instance, which contains a state $|n_1 n_2 m\rangle$ contains all the states

$$|0, n_2, m\rangle, \ldots, |\infty, n_1, n_2\rangle \tag{4.68}$$

with the eigenvalues of N_1^3 being $n_1 + |m| + 1/2$. Then in the notation of BARGMANN [12] (see App. A.39), the matrix element of $e^{-iN_1^3\beta}$ is just a v_{mn}^k function with the Casimir operator $k = m + 1/2$

$$\langle n_1' n_2' m' | e^{-iN_1^3\beta} | n_1 n_2 m\rangle = v^{\frac{m+1}{2}}_{n_1'+\frac{m+1}{2},\, n_1+\frac{m+1}{2}} \left(\operatorname{sh}\frac{\beta}{2}\right) \delta_{n_2' n_2} \delta_{m'm}. \tag{4.69}$$

Similarly, one obtains for $e^{-iN_2^3\beta}$

$$\langle n_1' n_2' m' | e^{-iN_2^3\beta} | n_1 n_2 m\rangle = v^{\frac{m+1}{2}}_{n_1'+\frac{m+1}{2},\, n_1+\frac{m+1}{2}} \left(-\operatorname{sh}\frac{\beta}{2}\right) \delta_{n_1' n_1} \delta_{m'm}$$

and therefore for $e^{-iK_1\beta}$

$$\langle n_1' n_2' m' | e^{-iK_1\beta} | n_1 n_2 m\rangle = v^{\frac{m+1}{2}}_{n_1'+\frac{m+1}{2},\, n_1+\frac{m+1}{2}} \left(+\operatorname{sh}\frac{\beta}{2}\right) v^{\frac{m+1}{2}}_{n_2'+\frac{m+1}{2},\, n_2+\frac{m+1}{2}} \left(-\operatorname{sh}\frac{\beta}{2}\right) \delta_{m'm}. \tag{4.70}$$

Knowing this we can find the matrix elements of the operator

$$G = e^{-i\theta_{n'n} L_{45}}\, e^{-iqn(L_{35}-L_{34})} \tag{4.71}$$

by parametrizing it in Euler form. Inserting K_i from (4.66), we have to find Euler angles α, β, γ such that

$$G = e^{-i\theta K_1}\, e^{-iqn(K_2+K_3)} = e^{-i\alpha K_3}\, e^{-i\beta K_1}\, e^{-i\gamma K_3} \tag{4.72}$$

One does this most easily in the 2×2 quaternion representation of $O(2,1)$, substituting

$$K_1 = \frac{i\sigma_1}{2}, \quad K_2 = \frac{i\sigma_2}{2}, \quad K_3 = \frac{\sigma_3}{2} \tag{4.73}$$

from which one finds for the left side of Equ. (4.72) the expression

$$e^{-i\theta K_1}\, e^{-inq(K_2+K_3)} = \operatorname{ch}\frac{\theta}{2} + \left[\sigma_1 - \frac{nq}{2}\left(1 - \operatorname{cth}\frac{\theta}{2}\right)(\sigma_2 - i\sigma_3)\right] \operatorname{sh}\frac{\theta}{2} \tag{4.74}$$

while the "Euler quaternion" is

$$e^{-i\alpha K_3} e^{-iK_1} e^{-i\gamma K_3} = \cos\frac{\alpha+\gamma}{2} \operatorname{ch}\frac{\beta}{2} +$$

$$+ \sigma_1 \cos\frac{\alpha-\gamma}{2} \operatorname{sh}\frac{\beta}{2} + \sigma_2 \sin\frac{\alpha-\gamma}{2} \operatorname{sh}\frac{\beta}{2} - i\sigma_3 \sin\frac{\alpha+\gamma}{2} \operatorname{ch}\frac{\beta}{2}. \tag{4.75}$$

Comparison of (4.74) with (4.75) gives the four equations:

$$\operatorname{ch}\frac{\theta}{2} = \cos\frac{\alpha+\gamma}{2} \operatorname{ch}\frac{\beta}{2}, \tag{4.76a}$$

$$\operatorname{sh}\frac{\theta}{2} = \cos\frac{\alpha-\gamma}{2} \operatorname{sh}\frac{\beta}{2}. \tag{4.76b}$$

$$\frac{nq}{2}\left(1-\operatorname{cth}\frac{\theta}{2}\right)\operatorname{sh}\frac{\theta}{2}=\sin\frac{\alpha-\gamma}{2}\operatorname{sh}\frac{\beta}{2}, \tag{4.76c}$$

$$\frac{nq}{2}\left(1-\operatorname{cth}\frac{\theta}{2}\right)\operatorname{sh}\frac{\theta}{2}=\sin\frac{\alpha+\gamma}{2}\operatorname{ch}\frac{\beta}{2}. \tag{4.76d}$$

Squaring b) and c) and adding them we obtain

$$\operatorname{sh}^2\frac{\beta}{2}=\operatorname{sh}^2\frac{\theta}{2}\left(1+\frac{n^2q^2}{4}\left(1-\operatorname{cth}\frac{\theta}{2}\right)^2\right). \tag{4.77}$$

If we insert θ from (4.41), we find

$$\operatorname{sh}\frac{\beta}{2}=\frac{1}{2\sqrt{n'n}}\left[(n'-n)^2+q^2n'^2n^2\right]^{1/2},$$

$$\operatorname{ch}\frac{\beta}{2}=\frac{1}{2\sqrt{n'n}}\left[(n'+n)^2+q^2n'^2n^2\right]^{1/2}. \tag{4.78}$$

For α we obtain then

$$\sin\alpha=\frac{-2n'n^2q}{\{[(n'-n)^2+q^2n'^2n^2]\,[(n'+n)^2+q^2n'^2n^2]\}^{1/2}}=-\frac{nq}{\operatorname{sh}\beta}$$

$$\cos\alpha=-\frac{n'^2-n^2+n'^2n^2q^2}{\{[(n'-n)^2+q^2n'^2n^2]\,[(n'+n)^2+q^2n'^2n^2]\}^{1/2}}. \tag{4.79}$$

For γ one has to exchange $n\to-n'$ in (4.78). The phase of β has been fixed such that in the limit $q\to+0$

$$\alpha\to\begin{Bmatrix}-\pi\\-\pi/2\\0\end{Bmatrix},\qquad \gamma\to\begin{Bmatrix}\pi\\\pi/2\\0\end{Bmatrix} \tag{4.80}$$

for $n'\gtreqless n$, respectively, as we can see from the Equ. (4.79).
With these angles, the matrix elements of the finite transformation G becomes in the $|n_1n_2m\rangle$ basis:

$$G^m_{n_1'n_2'n_1n_2}=\langle n_1'n_2'm\,|\,G(q^2)\,|\,n_1n_2m\rangle=$$

$$=e^{-i(n_1'-n_2')\alpha}\,e^{-i(n_1-n_2)\gamma}\times v^{\frac{m+1}{2}}_{n_1'+\frac{m+1}{2},\;n_1+\frac{m+1}{2}}\left(+\operatorname{sh}\frac{\beta}{2}\right)v^{\frac{m+1}{2}}_{n_2'+\frac{m+1}{2},\;n_2+\frac{m+1}{2}}\left(-\operatorname{sh}\frac{\beta}{2}\right)\delta_{m'm}. \tag{4.81}$$

Collecting the different terms in Equs. (4.63), (4.64) and changing to the $|nlm\rangle$ basis according to (4.21) we finally obtain $(m \geqq 0)$:

$$I^{0}_{n',l',m;\,n,l,m} = \frac{[(2l'+1)(2l+1)]^{1/2}}{n} \sum_{\substack{n_1' n_2' \\ n_1 n_2}} \begin{pmatrix} \frac{n'-1}{n} & \frac{n'-1}{2} & l' \\ \frac{m-n_1'+n_2'}{2} & \frac{m+n_1'-n_2'}{2} & -m \end{pmatrix} \times$$

$$\times \begin{pmatrix} \frac{n-1}{2} & \frac{n-1}{2} & l \\ \frac{m-n_1+n_2}{2} & \frac{m+n_1-n_2}{2} & -m \end{pmatrix} \times \{n' h^{0 l', l}_{n_1'-n_2',\, n_1-n_2} G^{m}_{n_1' n_2' n_1 n_2} +$$

$$+ [(n_1'+1)(n_1'+m+1)]^{1/2} h^{+n',n}_{n_1'-n_2',\, n_1-n_2} G^{m}_{n_1'+1, n_2' n_1 n_2} +$$

$$+ [n_1'(n_1'+m)]^{1/2} h^{-n',n}_{n_1'-n_2',\, n_1-n_2} G^{m}_{n_1'-1, n_2' n_1 n_2}\} \tag{4.82}$$

where:

$$h^{\left\{\substack{+\\0\\-}\right\} l',l}_{s',s} = \begin{bmatrix} \cos \\ -i \sin \end{bmatrix} \left[\left(s' + \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \right) \alpha + s\gamma \right] \quad \text{for} \quad (-)^{l'-l} = \begin{bmatrix} +1 \\ -1 \end{bmatrix}. \tag{4.83}$$

Every G contains a term $\operatorname{ch}^{-(n'+n)} \beta/2$ which has a singularity when

$$\operatorname{ch} \frac{\beta}{2} = 0. \tag{4.84}$$

From (4.78) we see that this happens at

$$q^2 = \frac{-(n'+n)^2}{n^2 n'^2} \tag{4.85}$$

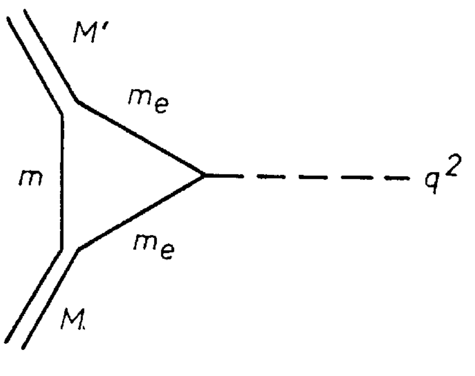
which can be written in terms of the binding energies of the states $|nlm\rangle$

$$B_n = -\frac{1}{2n^2} \tag{4.86}$$

as

$$q^2 = -2\left(\sqrt{B_n} + \sqrt{B_{n'}}\right)^2. \tag{4.87}$$

Observe that this position of the singularity coincides up to order $B/M \simeq 10^{-1}$ exactly with the anomalous threshold of the diagram



which can be calculated from the CUTKOSKY rules to lie at [18]

$$\cos\theta_1 = \cos(\theta_2 + \theta_3) \tag{4.88}$$

with

$$\cos\theta_1 = \frac{2m_e^2 - q^2}{2m_e^2},$$

$$\cos\theta_2 = \frac{m_e^2 + m^2 - M^2}{2m_e m},$$

$$\cos\theta_3 = \frac{m_e^2 + m^2 - M'^2}{2m_e m}.$$

Solving this we obtain

$$t \equiv q^2 = 2m_e^2 - \frac{1}{2m_e^2 m^2}\{(m_e^2 + m^2 - M^2)(m_e^2 + m^2 - M'^2) - \\ - [(2M^2(m_e^2 + m^2) - (m_e^2 - m^2)^2 - M^4)(2M'^2(m_e^2 = + m^2) - \\ - (m_e^2 - m^2) - M'^4)]^{1/2}\}. \tag{4.89}$$

For $M = M'$ this reduces to

$$t = 4m_e^2 - \frac{1}{m^2}[m_e^2 + m^2 - M^2]^2. \tag{4.90}$$

If, as in the case of the H-atom, M differs from $m + m_e$ only by a very small binding energy B,

$$M = m + m_e - B \tag{4.91}$$

and find up to order $\left(B/(m + m_e)\right)^2$

$$t = 2\,\frac{m_e}{m}(m_e + m)\left(\sqrt{B} + \sqrt{B'}\right)^2 \tag{4.92}$$

which reduces for $M = M'$ to

$$t = 8\,\frac{m_e}{m}(m_e + m)B. \tag{4.93}$$

We see that (4.85) coincides exactly with this since the units are chosen as atomic units and (4.85) holds actually for m/Mq instead of q.
Let's return to Eq. (4.82). In the ground state, one obtains for the charge distribution the double pole formula

$$I^0_{1,0,0;1,0,0} = \frac{1}{\left(1 - \frac{t}{4}\right)^2}. \tag{4.94}$$

This form factor falls off much faster than the $O(3,1)$ form factor for spin 0 discussed in Ch. II, p. 11.

$$I^0 = \frac{1}{\left(1 - \frac{t}{4M^2}\right)^{3/2}} \tag{4.95}$$

First the power in the denominator is different and second, the position of the pole has been shifted towards the origin by a factor of M, which is very large in terms of atomic units. This increases the slope dramatically.

There is another characteristic difference between the form factors (4.82) and our $O(3,1)$ results (2.20). While there the singularity lies for all transitions at the same place $4M^2$, it depends in the $O(4,2)$ group strongly on n (see (4.85)). Looking back at (4.77) we see that this effect comes from the tilting operation $e^{i\theta L_{45}}$, which doesn't exist in $O(3,1)$.

Observe that all form factors approach $\delta_{n'n}$ for $t \to 0$ as is clear from the Schrödinger form

$$I^0 = \int \Psi_{n'l'm}(x)\, e^{-iqx_e}\, \Psi_{nlm}(x)\, dx\,. \tag{4.96}$$

In the algebraic approach this means that

$$\left\langle n' \left| \frac{1}{n}\,(L_{46} - L_{56})\, e^{-i \log \frac{n}{n'} L_{45}} \right| n \right\rangle \tag{4.97}$$

is a unit matrix. We verify this not so obvious property in App. D.

5. Fock Sphere Representation

A natural unitary representation space for the spectrum generating group $O(4,1)$ can be given on the set of normed homogeneous functions $f(z_a)$ in the 5-dimensional parameter space of $O(4,1)$ [19]. The scalar product is defined as

$$(f, g) = 2 \int f^*(z)\, g(z)\, \delta(z^2)\, dz \tag{4.98}$$

which projects out the irreducible part of the representation that remains on the light cone.

A finite group element G of $O(4,1)$ transforms $f(z)$ into

$$f'(z) = U(G)\, f(z)\, U^{-1}(G) = f(zG) \tag{4.99}$$

G can be written as

$$G_{cd} = (e^{-i\alpha_{ab} L_{ab}})_{cd} \tag{4.100}$$

with

$$(L_{ab})_{cd} = i\,(g_{ac}g_{bd} - g_{ad}g_{ba}) \tag{4.101}$$

and its x-representation is:

$$L_{ab} = \frac{1}{i}\,(z_a\,\partial_b - z_b\,\partial_a), \tag{4.102}$$

$$U(G) = e^{-i\alpha_{ab} L_{ab}} \tag{4.103}$$

It is easy to verify that a function can be normed if and only if its degree of homogeneity N satisfies the conditions:

$$N = -\frac{3}{2} + i\nu, \quad \nu = \text{real} \left(= -\frac{p-1}{2} + i\nu \quad \text{for} \quad O(p,1) \right) \tag{4.104}$$

or

$$-3 < N < 0 \quad \left(-(p-1) < N < 0 \quad \text{for} \quad O(p,1) \right). \tag{4.105}$$

In such an irreducible representation, the parameter z_5 can be removed out of the operators and states defining new functions Φ by

$$f(z) = z_5^N \, \Phi(\xi_\mu) \tag{4.106}$$

where we have introduced homogenous coordinates

$$\xi = \frac{z_\mu}{z_5}. \tag{4.107}$$

On the functions $\Phi(\xi_\mu)$ the finite transformations are then given by the factor representation

$$U(G)\,\Phi(\xi)\,U^{-1}(G) = (\xi_\mu G_5^\mu + G_5^5)^N \,\Phi(\xi^* G) \tag{4.108}$$

where

$$(\xi^* G)_\mu \equiv U(G)\,\xi_\mu\,U^{-1}(G) = \frac{\xi_\nu G_\mu^\nu + G_\mu^5}{\xi_\nu G_5^\nu + G_5^5}. \tag{4.109}$$

The infinitesimal generators are then

$$L_{\mu\nu} = \frac{1}{i}\,(\xi_\mu \partial_\nu + \xi_\nu \partial_\mu), \tag{4.110}$$

$$L_{\mu 5} = -\frac{1}{i}\left[\xi_\mu\big(N - (\xi\,\partial)\big) + \partial_\mu\right] \tag{4.111}$$

and the scalar product becomes

$$(\Phi', \Phi) \equiv (f', f) = \int dz_5 z_5^{2N+3}\,\Phi'^*(\xi)\,\Phi(\xi)\,d\Omega \tag{4.112}$$

where

$$d\Omega = 2\delta(\xi^2 - 1)\,d\xi. \tag{4.113}$$

z_5 can in general not be eliminated out of the scalar product. A complete orthogonal set of functions in the representation space is given by

$$f_{N,n,\alpha} = z_5^{N-n+1}\, y_{N,n-1,\alpha}(z_\mu) \tag{4.114}$$

where the $y_{N,n-1,\alpha}$'s are homogeneous polynomials of degree $n-1$ in z_μ, i.e., they can be written in terms coordinate tensors

$$y_{N,n-1,\alpha}(z_\mu) = z_{\mu_1} \cdot \ldots \cdot z_{\mu_{n-1}} \tag{4.115}$$

and in terms of 4-dimensional spherical harmonics as

$$y_{N,n-1,\alpha}(z_\mu) = (z_5)^{n-1}\, y_{N,n-1,\alpha}(\xi_\mu). \tag{4.116}$$

Observe that $y_{N,n-1,\alpha}$ solves the potential equation in four dimensions:

$$\partial^2 y_{N,n-1,\alpha}(z_\mu) = 0 \tag{4.117}$$

which is therefore invariant under the operations of $O(4,1)$ for every N.
The Casimir operators of $O(4,1)$ are:

$$C_2 = L_{ab}L^{ab} = 2[\big(3 + (z\,\partial)\big)(z\,\partial)] = 2[(p-1+z\,\partial)(z\,\partial)] \quad \text{for} \quad O(p,1) \tag{4.118}$$

and

$$C_4 = L_{ab}L^{bc}L_{cd}L^{da} = 0 \tag{4.119}$$

which give in front of functions of homogeneity N, from the Euler equation:

$$(z\,\partial)\,f(z) = N f(z) \tag{4.120}$$

$$C_2 = 2N(3+N) \tag{4.121}$$

$$C_4 = 0. \tag{4.122}$$

We see that $N = -2$ gives the same representation as the one used in (4.3) (see (4.7)). This is also clear from another reason. On the H-atom representation there exists a 5-vector $(L_{\mu 6}, L_{56})$ (Equ. 4.24) transforming like z_a in the z-representation of $O(4,1)$. But the operator z_a raises the degree of homogeneity from N to $N+1$. Hence, it can only then exist inside a single representation if N and $N+1$ are equivalent. From the Casimir operators C_2, C_4 we see that this is only the case for $N = -2\big(-p/2$ for $O(p,1)\big)$.
The functions $f_{N,n,\alpha}$ can be chosen to be eigenstates of L^2, L_3. They also satisfy

$$R^2 f_{N,n,\alpha} = \frac{n^2-1}{2}\, f_{N,n,\alpha}. \tag{4.123}$$

Hence, with the labels $\alpha = l, m$; $N = -2$ we have the correspondence:

$$f_{-2,n,l,m}(z_a) = \langle z\,|\,n\,l\,m\rangle. \tag{4.124}$$

Therefore we can formally introduce a completeness relation:

$$\int d^5z\,\delta(z^2)\,|z\rangle\langle z| = 1. \tag{4.125}$$

Observe now, that since there exists for any $O(p,1)$ an operator Γ_5 on the representation space with $N = -p/2$ which transforms like z_5 and is completely expressible as a function of ξ, we can define a new scalar product replacing (4.112) by

$$(\Phi', \Phi) = \int \Phi'^*(\xi)\,\Gamma_5^{2N+3}\,\Phi(\xi)\,d\Omega_\xi \tag{4.126}$$

and eliminate in this way the fifth variable z_5 completely out of the representation. It is only in this case, that one can represent $O(p,1)$ unitarily on a space with p

variables. Γ_5^{2N+3} is the metric which makes the non-hermitian operators $L_{\mu 5}$ equivalent to hermitian ones. For the particular case of the H-atom representation, the invariant scalar product then becomes

$$(\Phi', \Phi) = \int \Phi'^*(\xi)\, \Gamma_5^{-1}\, \Phi(\xi)\, d\Omega_\xi$$
$$= \int \Phi'^*(\xi)\, L_{56}^{-1}\, \Phi(\xi)\, d\Omega_\xi . \tag{4.127}$$

Analogously to what we did in Equs. (4.123) and (4.124) we can now introduce the special homogeneous functions in ξ: $\Phi_{-2,n,l,m}(\xi)$ and identify them with the H-atom states $|nlm\rangle$ through the formal definition:

$$\Phi_{nlm}(\xi) \equiv \Phi_{-2,n,l,m} = \langle \xi | nlm \rangle \tag{4.128}$$

with the completeness relation

$$\int d\Omega_\xi\, |\xi\rangle\, L_{56}^{-1}\, \langle \xi | = 1 . \tag{4.129}$$

The explicit connection between hydrogen wave functions and $\Phi(\xi)'s$ can now easily be given. FOCK [*20*] observed that the n-dependent stereographic projection of the wave functions $\psi_{nlm}(p)$ in momentum space onto the surface of a sphere in 4-dimension with unit radius defined by

$$\boldsymbol{\xi} = \frac{2 p_n \boldsymbol{p}}{p^2 + p_n^2}, \qquad \xi_4 = \frac{p^2 - p_n^2}{p^2 + p_n^2}, \qquad p_n = \frac{1}{n} \tag{4.130}$$

or its inverse

$$\boldsymbol{p} = p_n \frac{\boldsymbol{\xi}}{1 - \xi_4}, \qquad p^2 = p_n^2 \frac{1 + \xi_4}{1 - \xi_4} \tag{4.131}$$

and

$$\Phi_{nlm}(\boldsymbol{\xi}) = \frac{1}{4} \frac{(p_n^2 + p^2)^2}{p_n^{3/2}}\, \psi_{nlm}(\boldsymbol{q}) = p_n^{3/2} (1 - \xi_4)^{-2}\, \psi_{nlm}\left(p_n \frac{\boldsymbol{\xi}}{1 - \xi_4}\right) \tag{4.132}$$

transforms the Schrödinger equation

$$\left(\frac{p^2}{2} - E\right) \psi_{nlm} = \frac{1}{2\pi^2} \int d\boldsymbol{q}\, |\boldsymbol{q} - \boldsymbol{p}|^{-2}\, \psi_{nlm}(\boldsymbol{q}) \tag{4.133}$$

into the integral equation for 4-dimensional spherical harmonics

$$\Phi_{nlm}(\xi) = \frac{1}{2\, p_n \pi^2} \int d\, \Omega_n (\eta - \xi)^{-2}\, \Phi_{nlm}(\eta) , \tag{4.134}$$

which therefore are just our $\Phi_{nlm}(\xi)$ defined in Equ. (4.128). The physical scalar product is not equal to the invariant one. For equal principal quantum numbers it is found to be

$$(\psi_n', \psi_n)_{\text{phys}} = \int d\Omega_\xi\, \Phi_n'^*(\xi)\, \Phi_n(\xi) \tag{4.135}$$

since the (n-dependent) spherical angle is

$$d\Omega_\xi^n = 2\delta(\xi^2 - 1)\, d\xi = \frac{(2p_n)^3}{(p^2 + p_n^2)^3}\, d^3p, \tag{4.136}$$

giving together with (4.132)

$$(\psi'_n, \psi_n)_{\text{phys}} = \int d^3p\, \psi'^*_n \psi_n. \tag{4.137}$$

For $n' \neq n$, the scalar product (4.135) obviously gives zero, since the functions Φ', Φ are spherical harmonics of different degree. In this case $d\Omega_\xi$ becomes a more involved function of the momentum and $p_{n'} p_{n'}$ than (4.136).
The quantum mechanical operators $\boldsymbol{x}$ and $\boldsymbol{p}$ have a complicated n-dependent form before the functions just as discussed in Ch. IV. 3. But like there, we can introduce the alternative states

$$\overline{\Phi}_{nlm}(\xi) = \frac{1}{4}\, \frac{(p^2 + a^2)^2}{a^2}\, \psi_{nlm}(p) \tag{4.138}$$

together with the fixed stereographic projection which one obtains by substituting $p_n \equiv a$ in Equs. (4.130), (4.131). The physical scalar product then becomes

$$(\psi', \psi)_{\text{phys}} = \frac{1}{a} \int d\Omega_\xi\, \overset{*}{\overline{\Phi'}}(\xi)\, (1 - \xi_4)\, \overline{\Phi}(\xi) =$$

$$= \int d\Omega_\xi\, \overset{*}{\overline{\Phi'}}(\xi)\, \frac{1}{L_{56}}\, \frac{1}{a}\, (L_{56} - L_{46})\, \overline{\Phi}(\xi) \tag{4.139}$$

or in terms of the invariant scalar product (4.127)

$$(\psi', \psi)_{\text{phys}} = \left(\overline{\Phi'}, \frac{1}{a}\, (L_{56} - L_{46})\, \overline{\Phi}\right). \tag{4.140}$$

Thus the physical scalar product corresponds to using the Galilean charge operator (4.56) $\Gamma^0 = 1/a(L_{56} - L_{46})$ in the invariant scalar product. For the operators x_i and p_i we find

$$x_i = i\, \frac{\partial}{\partial p_i} = i\, \frac{\partial \xi_\mu}{\partial p_i}\, \frac{\partial}{\partial \xi_\mu} = \frac{1}{a}\, (L_{i5} - L_{i4}) \tag{4.141}$$

and from (4.131)

$$p_i = a\, \frac{\xi}{1 - \xi_4} = a\, (L_{56} - L_{46})^{-1}\, L_{i6} \tag{4.142}$$

whose physical matrix elements are from (4.139) expressible in terms of the invariant scalar product as $(\psi', p_i \psi)_{\text{phys}} = (\overline{\Phi}', L_{i6} \overline{\Phi})$. Comparing this with (4.59) we find that we can identify the states $\overline{\Phi}_{nlm}(\xi)$ with $1/n\, |\overline{n} l m\rangle$ of (4.45). Therefore, the connection with $\Phi(\xi)$ must be again given by the tilting operation

$$T_n\, n\, \overline{\Phi}_{nlm}(\xi) = \Phi_{nlm} \tag{4.143}$$

with

$$T_n = e^{i\theta_n L_{45}}, \qquad \theta_n = \ln na. \tag{4.144}$$

Applying this to $\bar{\Phi}(\xi)$ we indeed find

$$\begin{aligned} e^{i\theta_n L_{45}} \bar{\Phi}_n(\xi) &= (\text{ch} \ln na - \text{sh} \ln na\, \xi_4)^{-2} \Phi_n(\xi^* T_n) = \\ &= \frac{4}{n^2 a^2} \left[(1 - \xi_4) \left(1 + \frac{p_n^2}{a^2} p^2 \right) \right]^{-2} \bar{\Phi}_n(\xi^* T_n) = \\ &= \frac{p_n^{3/2}}{\sqrt{n}} (1 - \xi_4)^{-2} \psi_n \left(p_n \frac{\xi_i}{1 - \xi_4} \right). \end{aligned} \tag{4.145}$$

But the functions on the right side are according to (4.128) just the spherical harmonics in ξ up to a factor $1/\sqrt{n}$. Hence,

$$T_n n \bar{\Phi}_n(\xi) = \sqrt{n}\, \Phi_n(\xi). \tag{4.146}$$

We see from this equation that $T_n n \bar{\Phi}_n(\xi)$ has the norm one in the invariant scalar product (4.127) in agreement with Sec. 2, where this state is $|nlm\rangle$ and the invariant scalar product $\langle n' l' m' | nlm \rangle$. $\bar{\Phi}$ itself has the invariant norm n^{-2} like the state (4.44).

Observe that the tilter dilates the p in the wave function by p_n/a, as has been discussed in x-space when deriving Equ. (4.39) (see App. C).

We have thus established a one to one correspondence between our states $|nlm\rangle$ and functions in the 4-dimensional ξ space. This correspondence can be summarized by:

$$\sqrt{n} \Phi_{nlm}(\xi) = \langle \xi | nlm \rangle, \tag{4.147}$$

$$\bar{\Phi}_{nlm}(\xi) = \langle \xi | \bar{n} lm \rangle, \tag{4.148}$$

$$\int d\Omega_\xi |\xi\rangle L_{56}^{-1} \langle \xi | = 1. \tag{4.149}$$

6. Galilean Majorana Equation

The Schrödinger equation for bound states can be written without the use of x coordinates in a purely algebraic form corresponding to the Majorana equation for an $O(3,1)$ multiplet. Since

$$L_{56} |nlm\rangle = n |nlm\rangle$$

we find for $|\bar{n} lm\rangle$ by tilting

$$\frac{n}{2a} \left(\left(\frac{1}{n^2} + a^2 \right) L_{56} - \left(\frac{1}{n^2} - a^2 \right) L_{46} \right) |\bar{n} lm\rangle = n |\bar{n} lm\rangle \tag{4.150}$$

or, using the energy $E = -1/(2n^2)$

$$\left[\left(E - \frac{a^2}{2} \right) L_{56} - \left(E + \frac{a^2}{2} \right) L_{46} + a \right] |\bar{n} lm\rangle = 0. \tag{4.151}$$

Remember now that the Galilean transformed states $e^{-iqx}\psi_{nlm}(x)$, whose scalar product with $\psi_{n'l'm'}(x)$ gives the charge distribution $I^0_{n'l'm',nlm}$ (see (4.82)), are algebraically represented by

$$e^{-i\frac{q^i}{a}(L_{i5}-L_{i4})}\,|\bar{n}lm\rangle.$$

For these states we can therefore also find a Schrödinger equation by transforming (4.151) according to

$$e^{i\frac{q}{a}(L_{i5}-L_{i4})}\,L_{56}\,e^{-i\frac{q}{a}(L_{i5}-L_{i4})} = L_{56} - \frac{q_i}{a}\,L_{i6} + \frac{q^2}{2a^2}\,(L_{56}-L_{46}),$$

$$e^{i\frac{q}{a}(L_{i5}-L_{i4})}\,L_{46}\,e^{-i\frac{q}{a}(L_{i5}-L_{i4})} = L_{46} - \frac{q_i}{a}\,L_{i6} + \frac{q^2}{2a^2}\,(L_{56}-L_{46}),$$

$$e^{i\frac{q}{a}(L_{i5}-L_{i4})}\,L_{i6}\,e^{-i\frac{q}{a}(L_{i5}-L_{i4})} = L_{i6} - \frac{q_i}{a}\,(L_{56}-L_{46}). \tag{4.152}$$

This gives

$$\left[\left(E-\frac{q^2}{2}\right)(L_{56}-L_{46}) - \frac{a^2}{2}\,(L_{56}+L_{46}) - aq_iL_{i6} + a\right]|\bar{n}lm\rangle = 0. \tag{4.153}$$

Electromagnetic interactions with the electron can now be introduced through the minimal substitution $\boldsymbol{q}\to\boldsymbol{q}-\boldsymbol{A}$. The resulting additional term corresponds exactly to the same current as that given by equ. (4.57).

7. Representation Mixing

The reason why we went to $O(4,2)$ as a dynamical group was to prescribe mixing of irreducible representations of the boosting group in a definite way, as was discussed in Ch. II. 5. In the case of the H-atom, this boosting group is Galilean. It is interesting to see now, how the hydrogen atom mixes representations of the Galilean group. We can see this best if we go to the x-representations of $\psi_{nlm}(x)$ of $O(4,2)$. Then the Galilean transformations leading to the form factor I^0 are simply obtained using the Galilean generators $M_i = -x_i$. The boosted states are (see 5.52)

$$\langle x\,|\,nlm,\,q\rangle = e^{-iqx}\,\psi_{nlm}(x) \tag{4.154}$$

and the Casimir operators of this representation of the Galilei group are

$$C_1 = \boldsymbol{xL} = 0 \tag{4.155}$$

which follows from parity invariance of the states $\psi_{nlm}(x)$, and

$$C_2 = x^2 = r^2. \tag{4.156}$$

Hence, the projections of the states $\psi(x)$ onto a fixed radial shell are the basis of an irreducible representation. Every state can be expanded into the irreducible states by using the radial part of the wave function $R_{nl}(r)$ as

$$|nlm\rangle = \int dr\, R_{nl}(r) \int dx\, Y_{nlm}(\hat{x})\, |\boldsymbol{x}\rangle . \tag{4.157}$$

Therefore, $R_{nl}(r)$ is just the spectral distribution of irreducible representations over the state $|nlm\rangle$ which correspond to $\psi^{j}_{j_0}(\nu)$ defined in (2.39). Thus the idea of a wave function governing the representation mixing turns out to have a quantum mechanical example.

V. Current Conservation, Mass Spectra, and Infinite Component Wave Equations

For the $O(3,1)$ currents we have seen in Ch. II. 5 that, given a vector operator Γ^μ, the requirement of Γ^μ leading to a conserved current is sufficient to specify the mass spectrum. Remember that in that case the physical states were eigenstates of Γ^0. In a larger group this is not necessarily so, as we saw in the discussion of the H-atom. The physical states are in general tilted in the form

$$e^{iS_n}|n\rangle \tag{5.1}$$

where $|n\rangle$ denotes the basis of the representation space and S_n an arbitrary rotational scalar in the Lie algebra. Assume that a vector operator Γ^μ as well as a Lorentz booster $\boldsymbol{M}$ have been specified in the Lie algebra. The assumption that Γ^μ gives the electromagnetic interaction then turns out to determine S_n uniquely for all n given its value for one state (say the ground state). After the S_n have been fixed, current conservation then specifies the masses of all particles (if it can be achieved at all for all transitions).
We first show how the S_n are determined. Since the zeroth component of the current consists of the charge of the system, which in turn is proportional to the physical scalar product between initial and final state, we require that between particles of charge one α and α' at zero 3-momentum transfer

$$\langle n'|J^0|n\rangle = \delta_{n'n} \times q_n \tag{5.2}$$

where q_n is the charge of the state n. In terms of the scalars S_n and Γ_0 this becomes

$$\langle n'|e^{-iS_{n'}}\, \Gamma_0 e^{iS_n}|n\rangle = \delta_{n'n} . \tag{5.3}$$

In the case of the H-atom this has been shown to hold in App. D, if (and, as one can easily see, only if)

$$S_n = -\theta_n\, L_{45} , \qquad \theta_n = \log na \tag{5.4}$$

which contains one free parameter a that can be fixed by prescribing Θ_1, for example.
Given these S_n, let's now require the electromagnetic current Γ_μ to be conserved. Like in Ch. II.5, this implies again

$$\langle n'|e^{-iS_{n'}}\, [(M' - p^0)\, \Gamma^0 + p^3 \Gamma^3]\, e^{iM_3\xi}\, e^{iS_n}|n\rangle = 0 \tag{5.5}$$

with the same notation as there and this gives the condition:

$$M' I^0_{n'n}(\zeta) = M' \langle n' | e^{-iS_{n'}} \Gamma^0 e^{iM_3\zeta} e^{iS_n} | n \rangle =$$
$$= M \langle n' | e^{-iS_{n'}} e^{iM_3\zeta} \Gamma^0 e^{iS_n} | n \rangle = M I^{0*}_{nn'}(-\zeta). \tag{5.6}$$

For diagonal transitions, $n = n'$, $m = m'$, and we see that the current is conserved if $I^0(\zeta)$ is even in ζ (since it has to be real anyhow from general arguments). For nondiagonal transitions, Equ. (5.6) gives a rather complicated system of equations for the masses. Obviously the solution is only determined up to a multiplicative constant.

Sometimes it may be easier to require current conservation directly from (5.5) instead of (5.6). Then one has to see that

$$\frac{I^3}{I^0} = \frac{p_0 - M'}{p_3} \tag{5.7}$$

if the boosted particle moves in the third direction. In the Galilean limit of the H-atom this leads, for example, to the condition:

$$\frac{I^3}{I^0} = \frac{E_n - E_{n'} + \dfrac{q^2}{2M}}{q}. \tag{5.8}$$

We may now calculate I^0, I^3 from (4.59), insert them into Equ. (5.8), and determine E_n to be

$$E_n = -\frac{1}{2n^2}. \tag{5.9}$$

In special cases it may be possible that one can obtain a simpler relation between current conservation and mass spectra by means of infinite component wave equations, as it happened in the $O(3,1)$ theory. Suppose the physical states

$$\psi(p) = \langle p s_3 [Ms] | e^{iM\zeta} e^{iS_n} | n \rangle \tag{5.10}$$

can be written as solutions of the very general Poincare invariant infinite component wave equation:

$$[p\Gamma + \alpha(M^2, W^2) S + \varkappa(M^2, W^2)] \psi(p) = 0 \tag{5.11}$$

with the same notation as in Ch. II.6, except that a term S has been added where S is now some Lorentz scalar operator in the Lie algebra that exists only for groups larger than $O(3,1)$ (except for the trivial one, the Casimir operator $L^2 - M^2$). This equation again projects the mass contents out of the scalar products (5.10). The current can now obviously be conserved for all transitions if α and $\varkappa$ are constants. This is indeed the case for the Galilean equivalent equation of the H-atom (4.147) as can easily be verified.

The algebraic structure of the infinite component equation (5.11) is very similar to that of the Majorana equation in Ch. II.6. There occurs, however, one new feature in groups larger than $O(3,1)$: It may sometimes be possible to express the content of Equ. (5.11) purely algebraically on the Hilbert space of states $|n\rangle$

without going to the wave functions $\psi(p)$ in (5.11). This happens if the momentum operator can be given in terms of the group generators. As remarked in Ch. II.6, for $O(3,1)$ such a relation cannot exist since Γ^μ is the only vector on the Hilbert space. An example for this is supplied by BÖHM's [21] construction of momentum operator on an $O(4,1)$ Hilbert space: He shows that

$$L_{\mu 5} = \frac{1}{\lambda} P_\mu + \frac{1}{2} \frac{\{P^\nu, L_{\nu\mu}\}}{(P_\mu P^\mu)^{1/2}} \tag{5.12}$$

leads to mass content

$$M^2 = \lambda^2 \alpha^2 - 9/4\, \lambda^2 + \lambda^2\, s(s+1) \tag{5.13}$$

in an $O(4,1)$ representation where the value of the Casimir operator

$$C_2 = 1/2 \quad L_{\mu\nu} L^{\mu\nu} \tag{5.14}$$

is α. From (5.12) we can obtain a Majorana equation like (5.11) for $\psi(p)$ by simply contracting (5.12) with P^μ. If $O(4,1)$ with $C_2 = \alpha$ is assumed to be the dynamical group of the electromagnetic interactions of the masses (5.13), $L_{\mu 5}$ has to be the electromagnetic current operator Γ_μ. It would be interesting to see whether this current is conserved with the mass spectrum (5.13).

VI. $O(4,2)$ Dynamics in Particle Physics

1. Description of the Model

After we have learned how nature does its representation mixing in the exactly soluble case of the H-atom, we may try to apply similar methods to the case of particle dynamics. The first problem is to find a possible group of quantum numbers. Then one may use this group, or an extension of it, to do group dynamics.

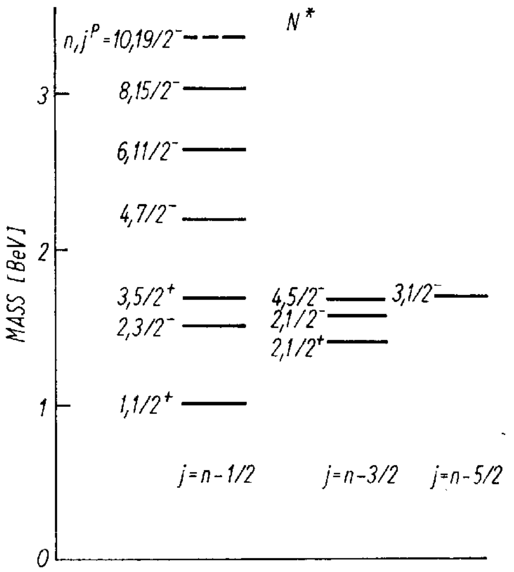


Fig. 5. The levels of the isospin 1/2 baryons are shown and a possible assignment to the quantum numbers of the Dirac hydrogen atom n, j^P is given (which have the ordering $j^P = 1/2\pm, 3/2\pm, \ldots, (n-1/2)^{(-)^{n-1}}$ for every $n = 1,2,3,\ldots$)

Inspection of the baryon quantum numbers shows that the levels of the isospin 12 baryon resonances look like the spectrum of relativistic hydrogen atom with some spin-orbit coupling. We have shown the observed levels on Fig. 5, and tried to assign the quantum numbers $|njm, \eta\rangle$ of the relativistic H-atom, where η is the parity.

Up to $n = 2$ the assignments is very good and above that, many of the states with $j = n - 1/2$ do exist. It will be interesting to see whether the others will be found in the future.

There is a simple representation of $O(4,2)$ containing all these fermions. It can be given in explicit form by the states:

$$|njm\pm\rangle = (-)^m \left[n+j+\frac{1}{2}\right]^{-1/2} (2j+1)^{1/2} \sum_{r=1,2} \begin{pmatrix} 1/2 & j-1/2 & j \\ 3/2-r & m+r-3/2 & -m \end{pmatrix} \times$$

$$\times\ (a_r^+ \pm (-)^{j-1/2}\, i b_r^+)\, |n, j-1/2, m+r-3/2\rangle \qquad (6.1)$$

where $|nlm\rangle$ are the hydrogenic states used in Ch. IV, and parity is again defined by $a \to b$, $b \to -a$ and for the ground state by $|0\rangle \to i|0\rangle$. For the lowest states, we find explicitly

$$\left|1\,\frac{1}{2}\,r^\pm\right\rangle = \frac{1}{\sqrt{2}}\,(a_r^+ + i b_r^+)\,|0\rangle;\quad r = 1{,}2$$

$$\left|2\,\frac{3}{2}\,\frac{3}{2}\,\pm\right\rangle = \frac{1}{2}\,(a_1^+ \mp i b_1^+)\, a_1^+ b_1^+\,|0\rangle$$

$$\left|2\,\frac{3}{2}\,\frac{1}{2}\,\pm\right\rangle = \frac{1}{2}\left[\frac{1}{\sqrt{3}}\,(a_2^+ \mp i b_2^+)\, a_1^+ b_1^+ + \frac{1}{\sqrt{3}}\,(a_1^+ \mp b_1^+)\,(a_1^+ b_2^+ + a_2^+ b_1^+)\right]\left|0\right\rangle$$

$$\left|2\,\frac{1}{2}\,\frac{1}{2}\,\pm\right\rangle = \frac{1}{\sqrt{6}}\,(a_1^+ \pm i b_1^+)\,(a_1^+ b_2^+ - a_2^+ b_1^+)\,|0\rangle. \qquad (6.2)$$

This representation unfortunately contains more than just the states of the relativistic hydrogen atom. For every n, the highest j occurs in both parities, in the H-atom only once with parity $(-)^{n-1}$ (See Fig. 5). Since $O(4{,}2)$ dynamics is now, however, easily calculable after all the preliminary work on the H-atom, we shall assume as a first approximation that the effect of this deficiency is small, or that the missing particles just haven't been seen yet. The results will indeed be very good: One obtains for ground state a double pole formula as magnetic form factor and finds for the magnetic moment $\mu = -1/6$. [23, 24]
Since isospin has been neglected throughout the approach we shall again, as in Ch. II, identify the results as isoscalar properties of the nucleons. Then the shape of the magnetic form factor agrees very well with experiment while the magnitude is slightly off. $\left(\mu_s = (\mu_p^+ + \mu_n)/2 = 0.44\right)$
It is probable that by inclusion of $SU(3)$ the magnitude of μ will change without affecting the good result for the shape.
There also exist other extensions of $O(4{,}2)$ by parity. As we shall show, however, in Sec. 3, the one given here is the only one which leads to a nontrivial theory with $O(4{,}2)$ as the dynamical group.

2. The Most General Electromagnetic Interaction

Using the electromagnetic theory of the H-atom as a guide we construct the most general possible theory of the same type on the fermion representation space. Putting the initial particle again at rest and boosting the final one into the Z-direction with rapidity ζ, the form factor has the structure:

$$I^\mu(\zeta) = \langle n|\, e^{-iS_2}\, I^\mu\, e^{+iM_3\zeta}\, e^{iS_1}\,|n\rangle \qquad (6.3)$$

where $M_i (i = 1, 2, 3)$ are Lorentz generators in the Lie algebra under which I^μ transforms like a four-vector, while S_1, S_2 are arbitrary rotational scalars. We shall assume in this work that I^μ is a purely algebraic vector operator Γ^μ like in the $O(3,1)$ discussion. According to the dynamical group philosophy, Γ^μ and S_1, S_2 have to be also elements of the Lie algebra.
Since M_i must be a vector under the rotation subgroup of $O(4,2)$, it can at most be a linear combination of L_{i4}, L_{i5}, L_{i6} and the commutation rules

$$[M_i, M_j] = -iL_{ij} \tag{6.4}$$

fix this combination to be of the form

$$M_i = \text{ch}\,\varepsilon(\cos\tau\, L_{i5} + \sin\tau\, L_{i6}) + \text{sh}\,\varepsilon\, L_{i4}. \tag{6.5}$$

Observe now that this M_i can be rotated by operators of the form e^{iS} into L_{i5}, namely

$$L_{i5} = e^{iL_{45}\varepsilon}\, e^{-iL_{56}\tau}\, M_i e^{iL_{56}\tau}\, e^{-iL_{45}\varepsilon}. \tag{6.6}$$

We therefore can assume without loss of generality that

$$M_i = L_{i5} \tag{6.7}$$

otherwise we could bring M_i to this form by changing S and Γ^μ appropriately. Next, we can assume S to contain only L_{45} and L_{46}

$$S_{\frac{1}{2}} = \theta_{\frac{1}{2}} L_{45} + \Delta_{\frac{1}{2}} L_{46}, \tag{6.8}$$

the only other possible term L_{56} could always be taken out of the matrix element (6.3) giving only an overall phase change, since L_{56} is diagonal on the Hilbert space. θ and Δ are tilting angles which we allow, as in the case of the H-atom, to depend on n.
Finally, the most general current operator Γ^μ must be a linear combination of

$$\Gamma_1^\mu = (L_{56}, L_{i6}),$$

$$\Gamma_2^\mu = (L_{45}, -L_{i4}), \tag{6.9}$$

say

$$\Gamma^\mu = a\Gamma_1^\mu + b\Gamma_2^\mu. \tag{6.10}$$

Define now the tilted operators O' as

$$O' = e^{-iS_2} O e^{iS_2} \tag{6.11}$$

then the form factor I^μ can be written as

$$I^\mu = \langle n' | \Gamma^\mu e^{-iS_2} e^{iM_3\zeta} e^{iS_1} | n\rangle = \sum_{n''} \langle n' | \Gamma^{\mu'} | n''\rangle \langle n'' | G(\zeta) | n\rangle \tag{6.12}$$

where we have inserted a complete set of states and defined

$$G(\zeta) \equiv e^{-iS_2} e^{+iM_3\zeta} e^{iS_1}. \tag{6.13}$$

We first bring G to a form in which its matrix elements can easily be calculated. We write G explicitly, inserting (6.7) and (6.8) into (6.13):

$$G(\zeta) = e^{-i(\Delta_2 L_{46} + \Theta_2 L_{45})} e^{iL_{35}\zeta} e^{i(\Delta_1 L_{46} + \Theta_1 L_{45})} \tag{6.14}$$

Now observe that the operators N_1^i and N_2^i $(i = 1, 2, 3)$ defined by:

$$\begin{aligned} N_1^1 &= {}^1/_2 (L_{35} + L_{46}), N_1^2 = {}^1/_2 (L_{45} - L_{36}), N_1^3 = {}^1/_2 (L_{56} + L_{34}), \\ N_2^1 &= {}^1/_2 (-L_{35} + L_{46}), N_2^2 = {}^1/_2 (L_{45} + L_{36}), N_2^3 = {}^1/_2 (L_{56} - L_{34}) \end{aligned} \tag{6.15}$$

form an $O(2,1) \times O(2,1)$ algebra with the commutation rules

$$\begin{aligned} &[N_{\substack{1\\2}}^i, N_{\substack{1\\2}}^j] = i g_{kk} N_{\substack{1\\2}}^k, \quad g = \begin{pmatrix}1 & & \\ & 1 & \\ & & -1\end{pmatrix} \\ &[N_1^i, N_2^j] = 0. \end{aligned} \tag{6.16}$$

Then G can be brought to Euler angle form of $O(2,1) \times O(2,1)$

$$G(\zeta) = e^{-i(\alpha_1 N_1{}^3 - \alpha_2 N_2{}^3)} e^{-i(\beta_1 N_1{}^2 + \beta_2 N_2{}^2)} e^{-i(\gamma_1 N_1{}^3 - \gamma_2 N_2{}^3)} \tag{6.17}$$

by a simple parameter transformation. The outer factors can be taken out of the matrix elements of G as phases, since L_{34} and L_{56} are diagonal in the representation. All that remains is the product

$$\langle n | e^{-i(\beta_1 N_1{}^2 + \beta_2 N_2{}^2)} | 1 \rangle = \sum_{n'} \langle n | e^{-\beta_1 N_1{}^2} | n' \rangle \langle n' | e^{-i\beta_2 N_2{}^2} | 1 \rangle \tag{6.18}$$

but every factor is again just a global representation of $O(2,1)$ $v_{mn}^k(\text{sh}\, \beta/2)$ given by Bargmann, which has been extensively used in the Chs. II and IV (See App. A. 39).

The Euler angles are readily evaluated. To simplify $G(\zeta)$ take, in Equ. (6.14), the right tilter to the left. Then we obtain:

$$G(\zeta) = e^{-i(\Delta_2 L_{46} + \Theta_2 L_{45})} e^{i(\Delta_1 L_{46} + \Theta_1 L_{45})} e^{-i\zeta(uL_{35} + vL_{36} + wL_{34})} \tag{6.19}$$

with:

$$u = 1 + \frac{\theta^2}{\nu^2} (\text{ch}\, \nu - 1), \tag{6.20a}$$

$$v = (\text{ch}\, \nu - 1) \frac{\Delta\theta}{\nu^2}, \tag{6.20b}$$

$$w = \frac{\theta}{\nu} \text{sh}\, \nu. \tag{6.20c}$$

Inserting the $O(2,1) \times O(2,1)$ operators (6.15) we find

$$G(\zeta) = e^{-i(\Delta_2 L_{46} + \Theta_2 L_{45})} e^{i(\Delta_1 L_{46} + \Theta_1 L_{45})} e^{+i\zeta[u(N_1{}^1 - N_2{}^1) - v(N_1{}^3 - N_2{}^2) + w(N_1{}^3 - N_2{}^3)}$$

where the second factor can be separated into the product of two commuting operators $G_1(\zeta)$, $G_2(\zeta)$ in the form:

$$G_1(\zeta) \cdot G_2(\zeta) \equiv e^{+i\zeta[uN_1{}^1 - vN_1{}^2 + wN_1{}^3]} e^{i\zeta[-uN_2{}^1 + vN_2{}^3 - wN_2{}^3]} . \tag{6.21}$$

In this work we shall confine our attention to the form factors of the $1/2^+$ ground state. As we saw in the general considerations of Ch. V, the tilting operator S for this state contains among the angles Δ, θ one free parameter (corresponding to a in the H-atom). For the higher states Δ_n and θ_n may then be determined from the relations (6.2), for example. We have not done this yet; therefore we also cannot make any statements about the mass spectrum following from current conservation. We shall leave these problems open for the time being.

For the ground state then, call

$$\theta \equiv \theta_1 = \theta_2, \qquad \Delta = \Delta_1 = \Delta_2$$

and only $G = G_1(\zeta)\, G_2(\zeta)$ remains in (6.21).

We can now calculate the Euler angles $\alpha \equiv \alpha_1, \beta \equiv \beta_1, \gamma \equiv \gamma_1$ for the first factor in the same way as in Ch. IV, by going to the 2×2 quaternion representation:

$$N_1^1 = i\,\frac{\sigma_1}{2}, \quad N_1^2 = i\,\frac{\sigma_2}{2}, \quad N_1^3 = \frac{\sigma_3}{2}. \tag{6.22}$$

The quaternion for G_1 becomes

$$G_1(\zeta) = \operatorname{ch}\frac{\zeta}{2} - (u\sigma_1 - v\sigma_2 - iw\sigma_3)\operatorname{sh}\frac{\zeta}{2} \tag{6.23}$$

which has to be compared with the Euler quaternion of

$$e^{-i\alpha N_1{}^3} e^{-i\beta N_1{}^2} e^{-i\gamma N_1{}^3}$$

which is

$$G_1(\zeta) = \cos\frac{\alpha+\gamma}{2}\operatorname{ch}\frac{\beta}{2} - \sin\frac{\alpha-\gamma}{2}\operatorname{sh}\frac{\beta}{2}\,\sigma_1,$$

$$+ \cos\frac{\alpha-\gamma}{2}\operatorname{sh}\frac{\beta}{2}\,\sigma_2 - i\sin\frac{\alpha+\gamma}{2}\operatorname{ch}\frac{\beta}{2}\,\sigma_3. \tag{6.24}$$

This yields the four equations:

$$\cos\frac{\alpha+\gamma}{2}\operatorname{ch}\frac{\beta}{2} = \operatorname{ch}\frac{\zeta}{2}, \tag{6.25a}$$

$$\sin\frac{\alpha-\gamma}{2}\operatorname{sh}\frac{\beta}{2} = u\operatorname{sh}\frac{\zeta}{2}, \tag{6.25b}$$

$$\cos\frac{\alpha-\gamma}{2}\operatorname{sh}\frac{\beta}{2} = v\operatorname{sh}\frac{\zeta}{2}, \tag{6.25c}$$

$$\sin\frac{\alpha+\gamma}{2}\operatorname{ch}\frac{\beta}{2} = -w\operatorname{sh}\frac{\zeta}{2}. \tag{6.25d}$$

One can easily convince oneself that the same equations hold also for α_2, β_2, γ_2. From (6.25b) and (6.25c) we find

$$\operatorname{sh}\frac{\beta}{2} = \sqrt{u^2+v^2}\,\operatorname{sh}\frac{\zeta}{2}, \quad \operatorname{ch}\frac{\beta}{2} = \sqrt{1+\operatorname{sh}^2\frac{\beta}{2}} \tag{6.26}$$

such that

$$\sin\frac{\alpha-\gamma}{2} = \frac{u}{\sqrt{u^2+v^2}}, \quad \cos\frac{\alpha-\gamma}{2} = \frac{v}{\sqrt{u^2+v^2}}. \tag{6.27}$$

From (6.25a) and (6.25b) we see that for $\zeta \to 0$, $\alpha \to -\gamma$, therefore $(\alpha-\gamma)/2$ is equal to the angle α (or $-\gamma$) for $\zeta = 0$. We do not give the general solution of (6.25) since only the following special combination of Euler angles, apart from $(\alpha-\gamma)/2$, $(\alpha+\gamma)/2$, will be needed for the form factor of the ground state:

$$\cos\frac{3\alpha+\gamma}{2} = \cos\frac{\alpha-\gamma}{2} + 2\left(\sin^2\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2} + \sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2}\right) \tag{6.28}$$

which becomes due to (6.25)

$$\cos\frac{3\alpha+\gamma}{2} = +\frac{v}{\sqrt{u^2+v^2}} - 2\left[\frac{w^2 v}{(u^2+v^2)^{3/2}}\operatorname{th}^2\frac{\beta}{2} + \frac{uw}{u^2+v^2}\operatorname{th}\frac{\beta}{2}\frac{\operatorname{ch}\frac{\zeta}{2}}{\operatorname{ch}\frac{\beta}{2}}\right]. \tag{6.29}$$

Consider now the current I^μ generated by $\Gamma^{\mu\prime} = a\Gamma_1^\mu + b\Gamma_2^\mu$. For the electric form factor we need I^0, for the magnetic one I^1 in order to apply Equ. (2.31). According to Equ. (6.12), Γ^μ has to be tilted to $\Gamma^{\mu\prime}$ before we can use it. We find

$$\Gamma_1^{0\prime} = L_{56}' = \operatorname{ch}\nu\, L_{56} + \operatorname{sh}\nu\left(\frac{\theta}{\nu}L_{46} - \frac{\Delta}{\nu}L_{45}\right),$$

$$\Gamma_2^{0\prime} = L_{45}' = L_{45} - \frac{\Delta}{\nu}\left[\operatorname{sh}\nu\, L_{56} + (\operatorname{ch}\nu - 1)\left(\frac{\theta}{\nu}L_{46} - \frac{\Delta}{\nu}L_{45}\right)\right], \tag{6.30}$$

$$\Gamma_1^{1\prime} = L_{16}' = L_{16} + \frac{\Delta}{\nu}\left[\operatorname{sh}\nu\, L_{14} + (\operatorname{ch}\nu - 1)\left(\frac{\theta}{\nu}L_{15} + \frac{\Delta}{\nu}L_{16}\right)\right],$$

$$\Gamma_2^{1\prime} = -L_{14}' = -\operatorname{ch}\nu\, L_{14} - \operatorname{sh}\nu\left(\frac{\theta}{\nu}L_{15} + \frac{\Delta}{\nu}L_{16}\right) \tag{6.31}$$

and therefore need the currents associated with the operators L_{ab}:

$$I_{ab} \equiv \langle n|L_{ab}|n'\rangle\langle n'|G(\zeta)|n\rangle. \tag{6.32}$$

The operator representations of the L's occurring in (6.30) are given by (4.30) and those in (6.31) follow from (4.3) to be, explicitly:

$$L_{14} = -{}^1/_2\,(a_1^+ a_2 + a_2^+ a_1 - b_1^+ b_2 - b_2^+ b_1)$$

$$L_{15} = {}^1/_2\,(a_1^+ b_1^+ - a_2^+ b_2^+ + a_1 b_1 - a_2 b_2)$$

$$L_{16} = -\frac{1}{2i}\,(a_1^+ b_1^+ - a_2^+ b_2^+ - a_1 b_1 + a_2 b_2)\,. \tag{6.33}$$

Inserting the L's into (6.30), (6.31) we find, calculating $G(\zeta)$ from (6.17) in the same way as in (4.81):

$$I_{46} = -\frac{1}{2}\left(\cos\frac{\alpha-\gamma}{2} + 2\cos\frac{3\alpha+\gamma}{2}\right)\frac{\operatorname{sh}\frac{\beta}{2}}{\operatorname{ch}\frac{4\beta}{2}},$$

$$I_{45} = i I_{46},$$

$$I_{56} = \frac{3}{2}\cos\frac{\alpha+\gamma}{2}\,\frac{1}{\operatorname{ch}^3\frac{\beta}{2}},$$

$$I_{16} = -\frac{1}{2}\sin\frac{\alpha-\gamma}{2}\,\frac{\operatorname{sh}\frac{\beta}{2}}{\operatorname{ch}^4\frac{\beta}{2}},$$

$$I_{15} = i I_{16},$$

$$I_{14} = -\frac{i}{2}\sin\frac{\alpha+\gamma}{2}\,\frac{1}{\operatorname{ch}^3\frac{\beta}{2}}\,. \tag{6.34}$$

We collect now all these terms to obtain for the total currents I^0, I^1

$$I^0 = \left(a\operatorname{ch}\nu - b\frac{\Delta}{\nu}\operatorname{sh}\nu\right)\frac{3}{2}\operatorname{ch}\frac{\zeta}{2} - $$
$$-\left[a\frac{\theta - i\Delta}{\nu}\operatorname{sh}\nu - \mathrm{b}\frac{\Delta(\theta - i\Delta)}{\nu^2}(\operatorname{ch}\nu - 1) + ib\right]\times$$
$$\times\left[\frac{3}{2}\frac{v}{\sqrt{u^2+v^2}} - 2\left(\frac{w^2 v}{(u^2+v^2)^{3/2}}\operatorname{th}^2\frac{\beta}{2} - \frac{uw}{u^2+v^2}\operatorname{th}\frac{\beta}{2}\frac{\operatorname{ch}\frac{\zeta}{2}}{\operatorname{ch}\frac{\beta}{2}}\right)\right]\times$$
$$\times\operatorname{sh}\frac{\beta}{2}\,\frac{1}{\operatorname{ch}^4\frac{\beta}{2}}, \tag{6.35}$$

$$I^1 = \mu \frac{\operatorname{sh}\frac{\zeta}{2}}{\operatorname{ch}^4\frac{\beta}{2}} \tag{6.36}$$

with the magnetic moment being

$$\mu = -\frac{i}{2} u \left[a \frac{\Delta(\theta - i\Delta)}{\nu^2} (\operatorname{ch}\nu - 1) - b \frac{\theta - i\Delta}{\nu} \operatorname{sh}\nu - ia\right] + $$
$$+ \frac{i}{2} w \left[a \frac{\Delta}{\nu} \operatorname{sh}\nu - b \operatorname{ch}\nu\right] \tag{6.37}$$

if the charge, i.e. $I^0(\zeta = 0)$, has been normalized to 1. Comparing (6.36) with (2.31) we obtain the first important result: The magnetic form factor is

$$G_M = \frac{\mu}{\operatorname{ch}^4\frac{\beta}{2}} \tag{6.38}$$

which becomes with (6.26)

$$G_M = \frac{\mu}{\left(1 + (u^2 + v^2)\operatorname{sh}^2\frac{\zeta}{2}\right)^2} \tag{6.39}$$

and introducing from (2.21) the invariant momentum transfer through

$$t = q^2 = -4M^2 \operatorname{sh}^2\frac{\zeta}{2} \tag{6.40}$$

we find:

$$G_M(t) = \frac{\mu}{\left(1 - \frac{u^2 + v^2}{4M^2} t\right)^2}. \tag{6.41}$$

Hence, the magnetic form factor has the shape of a double pole formula with a singularity at:

$$t = \frac{4M^2}{u^2 + v^2}. \tag{6.42}$$

Since $u^2 = v^2 = 1 + w^2 \geqq 1$ the pole position corresponds in general to an anomalous threshold, just as we observed it for the H-atom. We see that the singularity degenerates to a normal threshold at $t = 4M^2$ if $w = 0$, hence, $\theta = 0$. It is, therefore, the L_{45} tilter which is responsible for the shift of the pole position. L_{46} doesn't affect it at all. The shape (6.41) is in excellent agreement with experiment, which is usually fitted best by [22]

$$G_M = \frac{\mu}{(1 - t/0.71)^2}, \quad \left(t \text{ in units } (\mathrm{GeV})^2\right). \tag{6.43}$$

What are the electric form factor and the value of μ given by this theory? Except from the charge normalization, we have to impose two physical conditions upon the currents.

a) They have to be conserved.

b) They have to be real from general dispersion theoretical arguments.

These conditions restrict the theory to have only two solutions. As we discussed in Ch. V, current conservation forces I^0 to be even in ζ. But from (6.35) we see that then necessarily $v = 0$. This is fulfilled if either $\theta = 0$ or $\Delta = 0$. In the first case (6.20) gives

$$u = 1, v = 0, w = 0$$

and the form factors become

$$G_E = \frac{1}{\left(1 - \frac{t}{4M^2}\right)^2},$$

$$G_M = \frac{\mu}{\left(1 - \frac{t}{4M^2}\right)^2},$$

$$\mu = -{}^1/_3. \tag{6.44}$$

In the second case we obtain

$$G_E = \left[1 + \frac{4}{3}\,\mathrm{th}^2\,\theta\,\frac{\mathrm{ch}^2\,\theta\,\frac{t}{4M^2}}{\left(1 - \mathrm{ch}^2\,\theta\,\frac{t}{4M^2}\right)}\right]\frac{1}{\left(1 - \mathrm{ch}^2\,\theta\,\frac{t}{4M^2}\right)^2},$$

$$G_M = \frac{\mu}{\left(1 - \mathrm{ch}^2\,\theta\,\frac{t}{4M^2}\right)^2},$$

$$\mu = -{}^1/_3. \tag{6.45}$$

As in Ch. II we may again tentatively identify the result as applying to the isoscalar properties of the nucleons, i.e.

$$\frac{G_E}{2} = G_E^S,$$

$$\frac{G_M}{2} = G_M^S,$$

$$\frac{\mu}{2} = \mu^S.$$

The first solution reproduces then very well the observed symmetry

$$\frac{G_M}{\mu} = G_E .$$

But the magnitude of μ and the shape of G_E and G_M are far off the observed curve (6.43), since $4M^2 = 3.5$ and $\mu^S = 0.44$. The second solution is far better. Choosing θ such that

$$\mathrm{ch}^2\theta = 5$$

we reproduce the correct singularity at $t = 0.71$ and obtain explicitly

$$G_E = \left[1 + 0.8 \times \frac{4}{3} \frac{\frac{t}{0.71}}{1 - \frac{t}{0.71}}\right] \frac{1}{\left(1 - \frac{t}{0.71}\right)^2},$$

$$\frac{G_M}{\mu} = \frac{1}{\left(1 - \frac{t}{0.71}\right)^2}. \tag{6.46}$$

We have plotted these functions onto the experimental data on Fig. 6 and find a good agreement. The magnitude of the magnetic moment is, however, slightly wrong. Proper inclusion of $SU(3)$ will probably correct this defect.

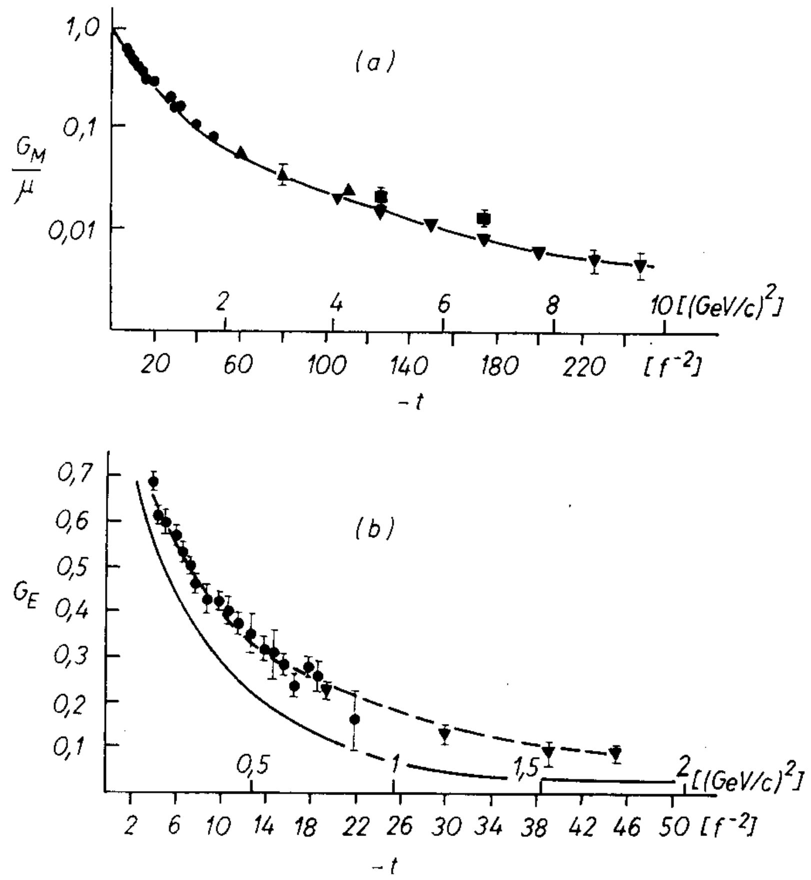


Fig. 6. The electromagnetic form factors obtained form $O(4,2)$ theory are shown and compared with those of the nucleons, The dashed line in (b) shows the best fit employed by the experimentalists [22] assuming $G_E = G_M\mu$ In (a) that best fit coincides with our curve

3. Other Possible Extensions by Parity

Starting from an irreducible representation of $O(4,2)$ built up on the states

$$|njm\pm\rangle = (-)^m \left[n + j + \frac{1}{2}\right]^{-1/2} [2(2j+1)]^{1/2}$$
$$\sum_{r=1,2} \begin{pmatrix} 1/2 & j-1/2 & j \\ 3/2-r & m+r-3/2 & m \end{pmatrix} a_r^+ \left| n, j - \frac{1}{2}, m + r - \frac{3}{2} \right\rangle \qquad (6.47)$$

we may ask, how many other extensions by parity one could construct on this space giving possibly different theories. According to the philosophy of the dynamical group approach, $O(4,2)$ has to contain the current operators Γ^μ, the Lorentz generators M_i, and the tilter S as Lie algebra elements. Under these conditions, the group extension chosen in (6.1) turns out to be unique.

On the table 2 we have listed all the possibilities of how the L_{ab}'s could transform under parity. We have restricted L_{ij} to be an axial vector because its physical meaning as angular momentum is fixed. The parities of L_{i4}, L_{i5}, L_{i6} can be chosen freely while those of L_{45}, L_{46}, L_{56} are then determined.

Table 2
Possible Reflection Properties of L_{ab} under Parity.

Case	L_i	L_{i4}	L_{i5}	L_{i6}	L_{45}	L_{46}	L_{56}	Parity Operation
(1)	+	−	−	−	+	+	+	$a \to b, b \to -a$
(2)	+	−	−	+	+	−	−	
(3)	+	−	+	−	−	+	−	
(4)	+	−	+	+	−	−	+	$a \to b, b \to a$
(5)	+	+	−	−	−	−	+	$a \to ia, b \to ib$
(6)	+	+	−	+	−	+	−	
(7)	+	+	+	−	+	−	−	
(8)	+	+	+	+	+	+	+	$a \to ia, b \to -ib$

In order to construct now a form factor of the structure (3.1), we need a tilter S and a Γ_0, both scalars under parity and rotation. Therefore only the cases (1) and (8) on the table are possible. Since we need moreover a Lorentz generator M_i, which is odd under parity, case (8) can also be excluded. Therefore only (1) remains and parity can be represented with doubling of the states by the prescription (6.1):

That doubling is really needed to represent parity in case (1) can easily be seen in the following way: If there is no doubling, every state picks up at most a phase under parity:

$$\Pi|njm\rangle = \eta(-)^{f(n,j)}|njm\rangle. \qquad (6.48)$$

$f(n,j)$ cannot depend on m since L_i is an axial vector. From (6.47) we see that L_{56} is necessarily a scalar. We also can easily prove that L_{i4} has to be an axial vector.

Suppose it were a vector. Then L_{i4} applied to $|njm\rangle$ has to change parity. On the other hand, L_{i4} conserves n and has matrix elements between equal as well as different j's. But this is impossible; there is no choice of $f(n,j)$ that can make L_{i4} a vector.

Consider now the case that L_{i4} is an axial vector. Then obviously $f(n, j)$ can only be a function of n. Two cases can now be distinguished: Either $L_{46} + iL_{45} = a^+Cb^+$ is a scalar (L_{46} and L_{45} have the same parity since L_{56} is parity even), then $f(n) = 0$, since a^+Cb^+ changes n but not the parity, or $L_{46} + iL_{45}$ is a pseudoscalar, then by the same kind of argument $f(n) \equiv 0$.
Hence, only the cases (8) and (5) on the table can be verified without doubling of the representation space. They are explicitly given by

$$\Pi|njm\rangle = \eta|njm\rangle \tag{6.49}$$

$$\Pi|njm\rangle = \eta(-)^n|njm\rangle \tag{6.50}$$

and can operationally be defined by

$$a \to ia, \quad b \to -ib, \quad |0\rangle \to i|0\rangle \tag{6.51}$$

$$a \to ia, \quad b \to ib, \quad |0\rangle \to i|0\rangle \tag{6.52}$$

respectively.
All other cases need doubling which can be achieved by using the direct product of L_{ab} with σ_0 or σ_3, according to whether they are scalars or pseudoscalars. The cases (1) and (4) also allow for an operational definition of parity by

$$a \to b, \quad b \to -a, \quad |0\rangle \to i|0\rangle, \tag{6.53}$$

$$a \to b, \quad b \to a, \quad |0\rangle \to i|0\rangle \tag{6.54}$$

respectively.
If we allow a Γ_0 from outside the Lie algebra we only need one scalar as a tilter S and the theory is much richer. $O(4,2)$ is then, however, not any more the dynamical group of the system. Such a model is briefly discussed in Sec. 4.

4. An $O(4,2)$ Model with a Current outside the Lie Algebra

On the $O(3,1) \sim \Pi$ Hilbert space we found that the physically best electromagnetic currents were reproduced by the theory if the algebraic current operator closes with the Lie algebra of $O(3,1)$ to a representation of $O(3,2)$ on the same space. A similar thing happened with the exactly soluble model of the H-atom: On the $O(4,1)$ Hilbert space the algebraic part of the correct electromagnetic current extends the group to $O(4,2)$ without increasing the number of states. These facts supported our postulate in Ch. I that algebraic current operators should be in the Lie algebra of some group on the space of physical states. In this section we want to test the postulate once more. We consider a model theory with the same structure as before but assume that the current operator is given by the vector

$$\Gamma^\mu = \left\{\frac{1}{2i}(a^+b - b^+a), \frac{1}{4}(a^+\sigma_i Ca^+ - aC\sigma_i a + b^+\sigma_i Cb^+ - bC\sigma_i b)\right\} \tag{6.55}$$

which lies outside the $O(4,2)$ algebra. It is important to notice Γ^μ cannot be used to extend the group $O(4,2)$ without increasing the Hilbert space. In fact, operating with it on our $O(4,2)$ states (6.1) it generates a representation space of the group $C(4)$, whose generators are, apart from the $O(4,2)$ generators (4.3) and (4.24)

$$
\begin{aligned}
L'_{ij'} &= \frac{1}{2}\,(a^+\sigma_k b + b^+\sigma_k a)\,,\\
L'_{i4'} &= -\frac{1}{2i}\,(a^+\sigma_i b - b^+\sigma_i a)\,,\\
L^a_{i5} &= -\frac{1}{2}\,(a^+\sigma_i C a^+ - a C\sigma_i a)\,,\\
L^b_{i5} &= -\frac{1}{2}\,(b^+\sigma_i C b^+ - b C\sigma_i b)\,,\\
L^a_{i6} &= \frac{1}{2i}\,(a^+\sigma_i C a^+ + a C\sigma_i a)\,,\\
L^b_{i6} &= \frac{1}{2i}\,(b^+\sigma_i C b^+ + b C\sigma_i b)\,,\\
L^{+\prime}_{56} &= \frac{1}{2}\,(a^+ b + b^+ a)\,,\\
L^{-\prime}_{56} &= \frac{1}{2i}\,(a^+ b - b^+ a)\,,\\
L^-_{56} &= \frac{1}{2i}\,(a^+ a - b^+ b)\,.
\end{aligned}
\tag{6.56}
$$

Altogether there are 36 generators which is the maximal number of bilinear operators one can form with the eight creation and annihilation operators a_r^+, b_r^+; a_r, b_r $(r = 1, 2)$.

For this current we obtain using methods just like in Sec. 2 that the electromagnetic form factors are (in the $\Delta = 0$ case)

$$
\begin{aligned}
G^E &= \frac{1}{(1 - \mathrm{ch}^2\theta\, t/4M^2)} = \frac{1}{(1 - t/0.71)^2}\\
G^M &= \frac{\mu + 2\,\mathrm{sh}^2\theta\,\mathrm{th}^2\beta/2}{1 - \mathrm{ch}^2\theta\, t/4M^2} = \frac{\mu - 8\,\dfrac{t/0.71}{1 - t/0.71}}{[1 - t/0.71]^2}\\
\mu &= -\mathrm{ch}\,2\theta = -9
\end{aligned}
\tag{6.57}
$$

which is a result much worse than that of Sec. 2. Thus, our postulate proves again useful, excluding this theory from consideration.

VII. Epilogue

The idea that group dynamics governs particle interactions has passed its first tests quite well considering the simplicity of the models discussed. The coupling of baryons to pseudoscalar mesons seems to follow $O(3,1) \times SU(3)$ symmetry, at least in the range of low momentum transfer in which the corresponding decays have been observed. For higher momentum transfers the amplitudes start oscillating and we don't expect them to be a good approximation anymore. One will need a larger dynamical group to describe this region, probably $O(4,2)$. The group $O(4,2)$ has been used for a theory of the electromagnetic form factor of the $1/2^+$ ground state, which is in good agreement with experiment, except that the magnitude of the magnetic moment is somewhat off. A non-trivial inclusion of internal symmetries will probably be needed to change this value. Before one can do this correctly it does not make sense to try to predict the electromagnetic form factors of the higher resonances as yet, which are in principle all determined. Such predictions will be the next goal in this approach.

Once one can describe three-particle vertices properly, the investigation of scattering problems by similar methods will be desirable. Some attempts in this direction have already been made [*26*], with quite encouraging results. The amplitudes are, however, all real and correspond to a contact interaction. Unitarity will probably have to be included by summing diagrams and one might have to formulate a quantum field theory with infinite component wave equations. How far one will be able to go in this direction is not yet clear.

The strength of the theory is that it can connect processes involving many different spins and treat whole towers of particles on the same footing, with amplitudes analytic in the external spins. Its weakness has been, until now, that the finding of the possible dynamical group from the (only partially observed) particle spectra is, to some extent, guesswork, and that the identification of the interaction operators is often not unique. One will need more stringent principles to find rigorous identification rules.

The point of view of our approach has been to understand the simplest models of group dynamics as well as possible, rather than look at more realistic models, whose structure we cannot yet understand. Indeed our models give far better results than others, which start out with larger groups, $SL(6, C)$, for example [*25*]. Simplicity again proves to be a good guide when trying to uncover dynamical structures. There still remains much to be learned.

Appendix A:
The Irreducible Representations of the Lorentz Group

1. Infinitesimal Representations

The Lie algebra of $O(3,1)$ consists of the angular momentum and boosting vectors L and M which close like:

$$[L_i L_j] = i L_k,\ [L_i, M_j] = i M_k,\ [M_i, M_j] = -i L_k. \tag{A.1}$$

An irreducible representation is given by a tower of spins between j_0 and $j_1 - 1$ and the matrix elements are

$$L_\pm |jm\rangle = [(j \mp m)(j \pm m + 1)]^{1/2} |j, m \pm 1\rangle,$$

$$L_3|jm\rangle = m|jm\rangle,$$

$$M_\pm|jm\rangle = \pm[(j\mp m)(j\mp m-1)]^{1/2}\; C_j|j-1, m\pm 1\rangle -$$
$$-[(j\mp m)(j\pm m+1)]^{1/2} A_j|j, m\pm 1\rangle \pm$$
$$\pm[(j\pm m+1)(j\pm m+2)]^{1/2} C_{j+1}|j+1, m\pm 1\rangle,$$

$$M_3|jm\rangle = [j^2-m^2]^{1/2}\; C_j|j-1, m\rangle - mA_j|j, m\rangle -$$
$$-[(j+1)^2-m^2]^{1/2} C_{j+1}|j+1, m\rangle, \tag{A.2}$$

where:

$$A_j = -\frac{ij_0j_1}{j(j+1)} = \frac{j_0\nu}{j(j+1)}; \quad C_j = \frac{i}{j}\left[\frac{(j^2-j_0^2)(j^2-j_1^2)}{4j^2-1}\right]^{1/2}. \tag{A.3}$$

We see, if j_1 is not an integer (or half-integer) number, the representation doesn't break off and the tower becomes infinitely high.
In $O(3,1)$ tensor notation, when

$$[L_{\mu\nu}, L_{\mu\lambda}] = -ig_{\mu\mu}L_{\nu\lambda};\ \mu = 0,1,2,3;\ g = \begin{pmatrix}1 & -1 & -1 & -1\end{pmatrix} \tag{A.4}$$

we can identify:

$$L_{ij} \equiv L_k, \quad L_{i0} \equiv M_i. \tag{A.5}$$

The Casimir operators are then

$$C_1 = {}^1/_2\, L_{\mu\nu}L^{\mu\nu} = L^2 - M^2 = j_0^2 + j_1^2 - 1,$$
$$C_2 = -{}^1/_4\, \varepsilon^{\mu\nu\varrho\sigma} L_{\mu\nu}L_{\varrho\sigma} = L\cdot M = +ij_0j_1. \tag{A.6}$$

We see that parity changes the sign of C_2 and therefore $j_1 \to -j_0$ or $j_1 \to -j_1$, and that $[j_0, j_1]$ is equivalent to $[-j_0, -j_1]$. The representations are unitary for

a) $j_0 = 0, j_1 = 1$ (trivial representation),

b) $j_0 =$ integer or half-integer (for $j_0 = 0, \nu \geqq 0$ only)
$j_1 = i\nu, -\infty < \nu < \infty$ (principal series),

c) $j_0 = 0, 0 < j_1 < 1$ (supplementary series) (A.7)

and are all infinite dimensional (except a). For j_0, j_1 both integer or half-integer, the representations are non-unitary and equivalent to the well known $D^{(s,s')}$ representation with

$$j_0 = |s-s'|, \quad j_1 = \operatorname{sgn}(s-s')[s+s'+1]. \tag{A.8}$$

In this case L, iM represent unitarily $O(4)$. s, s' are just the spins in the $SU(2)\times$ $\times SU(2)$ diagonalization defined by

$$J = {}^1/_2\,(L - iM),$$
$$K = {}^1/_2\,(L + iM). \tag{A.9}$$

2. Global Representations

$O(3,1)$ is isomorphic to $SL(2,C)$ through the relation:

$$A^+\sigma^\mu A = \Lambda^\mu_\nu\,\sigma^\nu;\quad \sigma^\mu = (\sigma^0, \boldsymbol{\sigma}),\quad \boldsymbol{\sigma} = \text{Pauli matrices} \tag{A.10}$$

from which follows:

$$\Lambda^\mu_\nu(A) = {}^1/_2\,\mathrm{tr}\,[A^+\sigma^\mu A\,\tilde{\sigma}^\nu] \tag{A.11}$$

$$\pm A = \frac{1}{N}\,\Lambda^\mu_\nu\sigma^\nu\tilde{\sigma}_\mu, \tag{A.12}$$

$$N^2 = \Lambda^\mu_\nu\Lambda^\lambda_\varkappa\sigma_\mu\tilde{\sigma}^\nu\sigma_\lambda\tilde{\sigma}^\varkappa.$$

Then we can obtain the unitary irreducible representations $[j_0\nu]$ of

$$A = \begin{pmatrix} a_1^1 a_1^2 \\ a_2^1 a_2^2 \end{pmatrix} \tag{A.13}$$

on the space of all normed functions $f(z, z^*)$ on the complex plane by defining:

$$U(A)\,f(z,z^*) = (a_1^2 z + a_2^2)^{*i\nu-j_0-1}\,(a_1^2 z + a_2^2)^{i\nu+j_0-1} \times f\left(\frac{a_1^1 z + a_2^1}{a_1^2 z + a_2^2}, \left(\frac{a_1^1 z + a_2^1}{a_1^2 z + a_2^2}\right)^*\right) \tag{A.14}$$

with the scalar products

$$(f, g) = \int dx\,dy\,f^*(z,z^*)\,g(z,z^*) \tag{A.15}$$

for the principal series $(z = x + iy)$ and

$$(f, g) = \int \frac{dx\,dy\,dx'\,dy'}{[(x-x')^2 + (y-y')^2]^{1+i\nu}}\,f^*(z,z^*)\,g(z'z^{*\prime}) \tag{A.16}$$

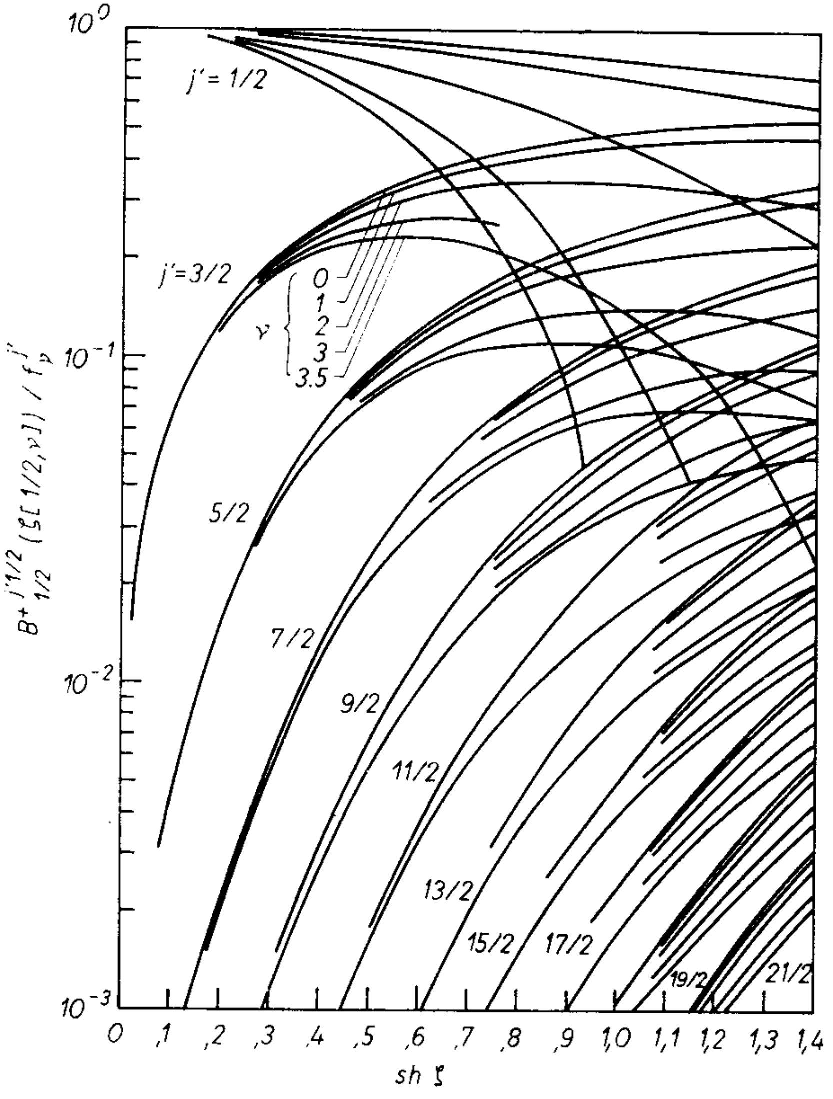
for the supplementary series. On this space the infinitesimal representation become:

$$\begin{aligned} J_+ &= -z^2\,\partial_z - {}^1/_2\,(-i\nu - j_0 + 1)z, \\ J_- &= \partial_z, \\ J_3 &= z\,\partial_z + {}^1/_2\,(-i\nu - j_0 + 1), \\ K_+ &= z^{*2}\,\partial_{z^*} + {}^1/_2\,(-i\nu + j_0 + 1)z^*, \\ K_- &= -\,\partial_{z^*}, \\ K_3 &= -z^*\,\partial_{z^*} - {}^1/_2\,(-i\nu + j_0 + 1). \end{aligned} \tag{A.17}$$

The basis functions, which diagonalize L^2 and L_3 on this space, are:

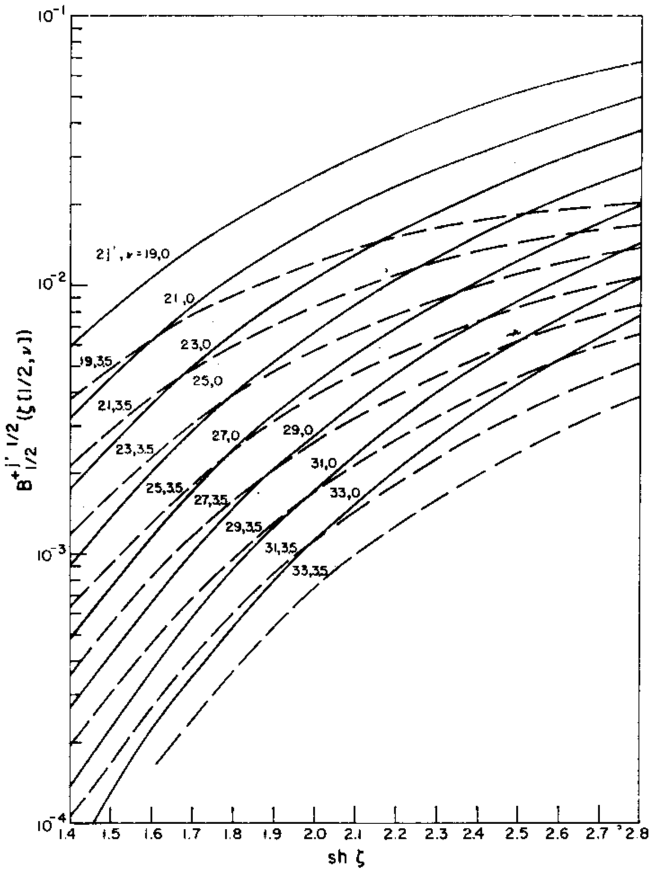
$$f_{jm}(z,z^*) \equiv \langle z, z^*\,[j_0\nu]\,|\,jm[j_0\nu]\rangle = \left[\frac{2j+1}{\pi}\,\frac{(-i\nu+j)!\,(j+m)!\,(j-m)!}{(+i\nu+j)!\,(j+j_0)!\,(j-j_0)!}\right]^{1/2} \times$$

$$\times\,(-)^j\,z^{m+j_0}(1 - zz^*)^{-i\nu-m-1} \times P_{j-m}^{(m+j_0\;m-j_0)}\left(\frac{1-zz^*}{1+zz^*}\right) \tag{A.18}$$



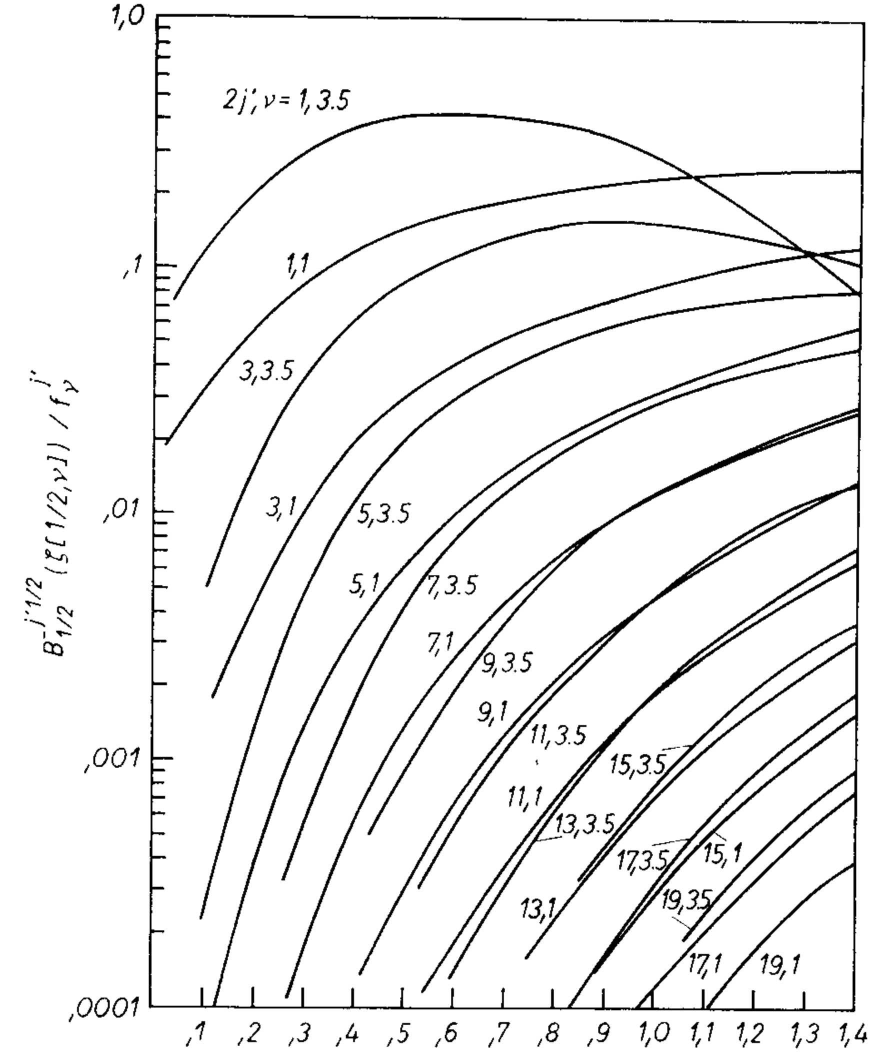
10^0
10^{-1}
10^{-2}
10^{-3}
$B^{+\,j'1/2}_{1/2}(\zeta[1/2,\nu])\,/\,f^{j'}_{\nu}$
j' = 1/2
j'=3/2
ν
0
1
2
3
3.5
5/2
7/2
9/2
11/2
13/2
15/2
17/2
19/2
21/2
0 ,1 ,2 ,3 ,4 ,5 ,6 ,7 ,8 ,9 1,0 1,1 1,2 1,3 1,4
sh ζ

7a

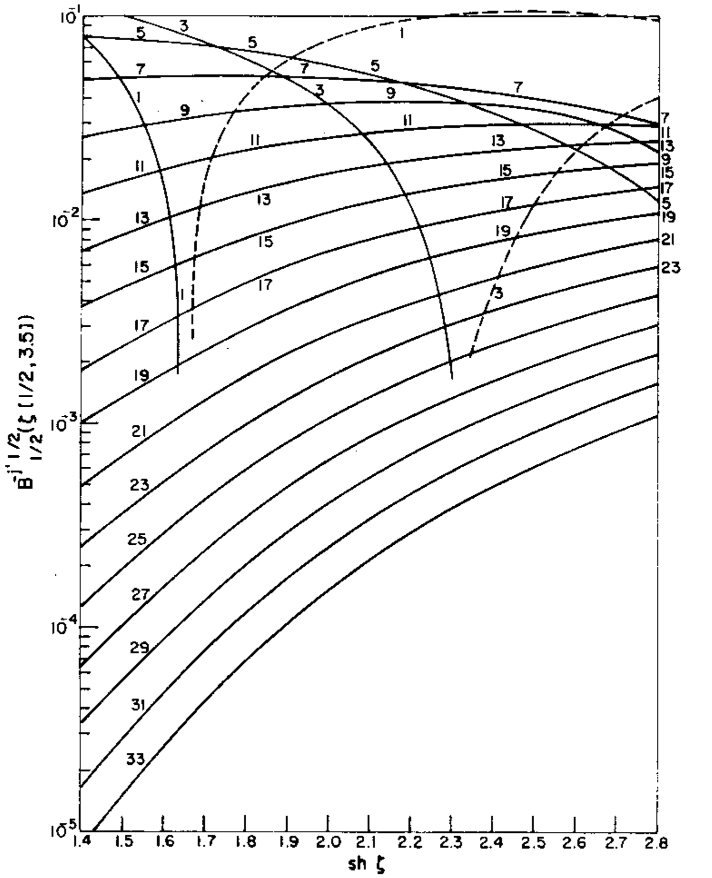


10^{-1}
10^{-2}
10^{-3}
10^{-4}
$B^{+\,j'\,1/2}_{1/2}(\zeta[1/2,\nu])$
2j', ν = 19,0
21,0
23,0
25,0
27,0
29,0
31,0
33,0
19,35
21,35
23,35
25,35
27,35
29,35
31,35
33,35
1.4 1.5 1.6 1.7 1.8 1.9 2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8
sh ζ

7b



7c



7d

Fig. 7(abcd). The theoretical amplitudes $B^{j'\,1/2}_{1/2}\,(\zeta[{}^1/_2, \nu])$ of the $O(3,1)$ theory are shown for different j' and ν values in the range $P/M_f = \operatorname{sh}\zeta = 0, \ldots, 2.8$. A factor has been divided out

$$f^{j'}_{\nu} = \frac{\sqrt{(j'^2 + \nu^2) \cdot \ldots \cdot (9/4 + \nu^2)}}{j' \cdot \ldots \cdot 3/2}$$

to allow for a better graphical representation. Add the factor $\left(f^{j'}_{\nu}\right)^{-1}$ in the denotation of the ordinate axes of fig. 7b and fig. 7b.

where $P_n^{(\alpha,\beta)}$ is Jacobi's Polynomial. On this basis, the representation (A. 14) leads then to:

$$\langle j' m[j_0\nu] | e^{iM_3\zeta} | j m[j_0\nu]\rangle \equiv B_m^{j'j}(\zeta[j_0\nu]). \tag{A.19}$$

B is given explicitly in (2.20) for $j = m$ and plotted on Fig. 7 for $j = 1/2$. B has the symmetry properties

$$B_m^{j'j}(\zeta[j_0\nu]) = (-)^{j'-j} B_m^{jj'}(\zeta[j_0\nu]), \quad B_m^{j'j}(\zeta[j_0\nu]) = B_m^{j'j}(\zeta[j_0 - \nu]). \tag{A.20}$$

For $m \neq j$ one obtains a much more complicated result. The reader is referred ro STRÖM s paper [3] for a discussion of this case.
The boosters in an arbitrary direction are obtained from (A. 19) by using the formula

$$\langle j' m'[j_0\nu] | e^{i\vec{M}\vec{\zeta}} | j m[j_0\nu]\rangle = \sum_{\overline{m}} D_{m'\overline{m}}^{j'}(\hat{\zeta})\, B_{\overline{m}}^{j'j}(\zeta[j_0\nu])\, D_{\overline{m}m}^{j^{-1}}(\hat{\zeta}). \tag{A.21}$$

For $j_0 = 0$ the hypergeometric function in (A. 19) has the special arguments

$$F(j' + 1 - j_1, j' + 1, 2j' + 2, 1 - e^{-2\zeta})$$

i.e. has the form $F(a, b, 2b, z), z = 1 - e^{-2\zeta}$. By means of the identity

$$F(a, b, 2b, z) = (1 - z)^{-a/2} F(a, 2b - a, b + 1/2, x) \tag{A.22}$$

with

$$x = -\frac{(1 - \sqrt{1 - z})^2}{4\sqrt{1 - z}} = -\mathrm{sh}^2\zeta/2 \tag{A.23}$$

we get for F:

$$F = e^{\zeta(j'+1-j_1)} F\left(j' + 1 - j_1, j' + 1 + j_1, j' + \frac{3}{2}, -\mathrm{sh}^2\frac{\zeta}{2}\right) \tag{A.24}$$

and using

$$F(a, b, c, x) = (1 - x)^{c-a-b} F(c - a, c - b, c, x) \tag{A.25}$$

we find

$$F = \left(\mathrm{ch}\frac{\zeta}{2}\right)^{-2j'-1} e^{\zeta(j'+1-j_1)} F\left(\frac{1}{2} + j_1, \frac{1}{2} - j_1, j' + \frac{3}{2}, -\mathrm{sh}^2\frac{\zeta}{2}\right). \tag{A.26}$$

Hence, we get the alternative form for B

$$B_j^{j'j}(\zeta[0, \nu]) = N^{j'j}(0, j_1)\frac{(\mathrm{sh}\,\zeta)^{j'-j}}{\left(\mathrm{ch}^2\frac{\zeta}{2}\right)^{j'+1/2}} F\left(\frac{1}{2} + j_1, \frac{1}{2}, -j_1, j' + \frac{3}{2}, -\mathrm{sh}^2\frac{\zeta}{2}\right) \tag{A.27}$$

$$N^{j'j}(0, j_1) = 2^{j'-j}\frac{j'!}{j!}\left[\frac{(j' + j)!\,(2j + 1)!}{(j' - j)!\,(2j' + 1)!}\,\frac{[(j + 1)^2 + \nu^2]\cdot\ldots\cdot[j'^2 + \nu^2]}{(2j')!}\right]^{1/2}. \tag{A.28}$$

$F(1/2 + j_1, 1/2 - j_1, j' + 3/2, -\mathrm{sh}^2\,\zeta/2)$ can finally also be expressed in terms of Legendre functions of 1st kind:

Using the relation

$$F(a, 1-a, c, z) = \Gamma(c)\left(\frac{1-z}{-z}\right)^{\frac{c-1}{2}} P^{1-c}_{-a}(1-2z) \tag{A.29}$$

we find

$$F\left(\frac{1}{2}+j_1, \frac{1}{2}-j_1, j'+\frac{3}{2}, -\operatorname{sh}^2\frac{\zeta}{2}\right) = \left(j'+\frac{1}{2}\right)!\left(\operatorname{th}\frac{\zeta}{2}\right)^{-j'-1/2} P^{-j'-1/2}_{j_1-1/2}(\operatorname{ch}\zeta) \tag{A.30}$$

and therefore

$$B^{j'j}_j(\zeta[0,\nu]) = N^{j'j}(0,j_1)\,2^{j'-j}\left(j'+\frac{1}{2}\right)!\left(\operatorname{sh}\frac{\zeta}{2}\operatorname{ch}\frac{\zeta}{2}\right)^{-j-1/2} P^{-j'-1/2}_{j_1-1/2}(\operatorname{ch}\zeta). \tag{A.31}$$

For $\nu = 0, j_0 \neq 0$, replace in the functional part j_1 by $-j_0$ and use $N(j_0, 0)$ as normalization constant.

3. The Degenerate Cases $[j_0\nu] = [0, 1/2]$ and $[1/2, 0]$

Here there exists convenient explicit representation of L, M in terms of creation and annihilation operators

$$L_i = 1/2\,(a^+\sigma^i a),$$
$$M_i = -\frac{i}{4}(a^+\sigma^i C a^+ + a C \sigma^i a), \quad C = i\sigma^2 \tag{A.32}$$

on the Hilbert space

$$|jm\rangle = [(j+m)!\,(j-m)!]^{1/2}\, a_1^{+\,j+m} a_2^{+\,j-m}|0\rangle \tag{A.33}$$

where for $[0, 1/2]$, j runs through all bosons $0, 1, 2, \ldots$ and for $[1/2, 0]$ through all fermions $1/2, 3/2, 5/2, \ldots$. M_3 can be written in terms of an $O(2,1)$ algebra formed by

$$K^+ = -\frac{i}{2}a^+\sigma^1 a^+, \quad K^- = \frac{i}{2}a\sigma^1 a, \quad K^3 = \frac{1}{2}(a^+a+1) \tag{A.34}$$

satisfying the commutations rules

$$[K^+, K^-] = -2K^3, \tag{A.35}$$

$$[K^3, K^\pm] = \pm K^\pm \tag{A.36}$$

as

$$M_3 = 1/2\,(K^+ + K^-) = K^1 \tag{A.37}$$

and therefore the booster $B(\zeta)$ is

$$B(\zeta) = e^{iK^1\zeta} \tag{A. 38}$$

whose matrix elements are the representation of $O(2,1)$ found by BARGMANN.

$$\langle j'm|B(\zeta)|jm\rangle = v^{m+1/2}_{j'+1/2,\,j+1/2}(a) \tag{A.39}$$

where $v(a)$ is the representation of the quaternion

$$a = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad |a| = 1 \tag{A.40}$$

given for $m \geqq n$ by

$$v^k_{mn}(a) = \theta_{mn}\, \bar{\alpha}^{-(m+n)}\, \beta^{m-n}\, F\left(k - n,\, 1 - n - k,\, 1 + m - n,\, -\beta\bar{\beta}\right) \tag{A.41}$$

with

$$\theta_{mn} = \frac{1}{(m-n)!}\left[\frac{(m-k)!\,(m+k-1)!}{(n-k)!\,(n+k-1)!}\right]^{1/2}. \tag{A.42}$$

m, n denote the eigenvalues of K_3 on the states and k is the lowest of them which characterizes the representation (which is D^+_k in BARGMANN's notation). For $m \leqq n$ use v^T and $\beta \to -\beta^*$. In our case the quaternion of 38 is obtained using $K_1 = -i\sigma_1$ as

$$B(\zeta) = \begin{pmatrix} \mathrm{ch}\,\frac{\zeta}{2} & \mathrm{sh}\,\frac{\zeta}{2} \\ \mathrm{sh}\,\frac{\zeta}{2} & \mathrm{ch}\,\frac{\zeta}{2} \end{pmatrix}, \tag{A.43}$$

and we see that K^3 has the eigenvalues $j + {}^1/_2$. Hence,

$$\begin{aligned} v^{m+1/2}_{j'+1/2,\, j+1/2}(a) &= v^{m+1/2}_{j'+1/2,\, j+1/2}\left(\mathrm{sh}\,\frac{\zeta}{2}\right) = \\ &= \frac{1}{(j'-j)!}\left[\frac{(j'-m)!\,(j'+m)!}{(j-m)!\,(j+m)!}\right] \frac{\left(\mathrm{sh}\,\frac{\zeta}{2}\right)^{j'-j}}{\left(\mathrm{ch}\,\frac{\zeta}{2}\right)^{j'+j+1}} \times \\ &\times F\left(m - j,\, -m - j,\, 1 + j' - j,\, -\mathrm{sh}^2\,\frac{\zeta}{2}\right). \end{aligned} \tag{A.44}$$

We compare this with (A. 27) for $j_1 = + i/2$ and find

$$B^{j'j}_j\left(\zeta\left[0, \frac{i}{2}\right]\right) = v^{j+1/2}_{j'+1/2,\, j+1/2}\left(\mathrm{sh}\,\frac{\zeta}{2}\right) \tag{A.45}$$

as expected.

4. Limits of the B-Functions

a) $\zeta \to 0$

For small ζ, $F(a, b, c, z) \simeq 1 + (ab/c)\, z$, hence,

$$F(j' + 1 - j_1,\, j' + 1 + j_0,\, 2j' + 2,\, 1 - e^{-2\zeta}) \simeq 1 + \frac{(j' + 1 - j_1)\,(j' + j + j_0)}{j' + 1}\,\zeta. \tag{A. 46}$$

Then to lowest order in ζ:

$$B_j^{+j'j}(\zeta[j_0\nu]) = N^{j'j}(\mathrm{sh}\,\zeta)^{j'-j}\,(1 + O(\zeta)) \tag{A.47}$$

$$B_j^{-j'j}(\zeta[j_0\nu]) = N^{j'j}(\mathrm{sh}\,\zeta)^{j'-j} \times \frac{1}{2}\left[(1 + j_1\zeta)\left(1 + \frac{(j'+1-j_1)(j'+1+j_0)}{j'+1}\,\zeta\right) - \right.$$

$$\left. - (1 - j_1\zeta)\left(1 + \frac{(j'+1+j_1)(j'+1+j_0)}{j'+1}\,\zeta\right)\right] =$$

$$= N^{j'j} j_1\left(1 - \frac{j'+1+j_0}{j'+1}\right)\zeta \quad = N^{j'j}\,\frac{(-j_0 j_1)}{j'+1}\,\zeta. \tag{A.48}$$

b) $\zeta \to \infty$

For large ζ, we make use of the formula:

$$F(a, b, c, z) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,F(a, b, a+b-c+1, 1-z) +$$

$$+ (1-z)^{c-a-b}\,\frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\,F(c-a, c-b, c-a-b+1, 1-z) \tag{A.49}$$

which gives for $z \to 1$ and $\mathrm{Re}(c-a-b) < 0$

$$F(a, b, c, z) \simeq \frac{(c-1)!\,(a+b-c-1)!}{(a-1)!\,(b-1)!}\,(1-z)^{c-a-b}. \tag{A.50}$$

Hence

$$F(j'+1-i\nu, j'+1+j_0, 2j'+2, 1-e^{-2\zeta}) = \frac{(2j'+1)!\,(j_0-1-i\nu)!}{(j'-i\nu)!\,(j'+j_0)!}\,e^{2\zeta(j_0-i\nu)}$$

and

$$B_j^{\pm j'j} = \frac{1}{2^{j'-j}}\,N^{j'j}\left|\frac{(2j'+1)!\,(j_0-1-i\nu)!}{(j'+j_0)!\,(j'-i\nu)!}\right| e^{-(j-j_0+1)\zeta}\begin{cases}\cos(\nu\zeta + \Phi_\nu)\\ -i\sin(\nu\zeta + \Phi_\nu)\end{cases} \tag{A.51}$$

where the phase angle is given by:

$$\begin{Bmatrix}\cos\,2\Phi_\nu\\ i\cdot\sin 2\Phi_\nu\end{Bmatrix} = \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\frac{\dfrac{(j_0-1+i\nu)!}{(j'+i\nu)!}}{\dfrac{(j_0-1-i\nu)!}{(j'-i\nu)!}}, \qquad \tan\Phi_\nu = \frac{\mathrm{Im}\,\dfrac{(j_0-1+i\nu)!}{(j'+i\nu)!}}{\mathrm{Re}\,\dfrac{(j_0-1+i\nu)!}{(j'+i\nu)!}}. \tag{A.52}$$

Observe that the factor in front reduces to

$$N^{j'j}\,\frac{(2j'+1)!}{(j'+j_0)!}\,\frac{1}{\sqrt{(j_0^2+\nu^2)\ldots(j'^2+\nu^2)}} =$$

$$= \left[\frac{(2j+1)!}{(j+j_0)!\,(j-j_0)!}\,\frac{(j'+j)!}{(j'-j)!}\,\frac{2j'+1}{(j'+j_0)\ldots(j'-j_0+1)}\,\frac{1}{(j_0^2+\nu^2)\ldots(j^2+\nu^2)}\right]^{1/2} \tag{A.53}$$

such that we get for the simplest cases

$$B^{\pm 1/2\, 1/2}_{1/2}\left(\zeta\left[\frac{1}{2}, \nu\right]\right) = \frac{2}{\sqrt{\frac{1}{4}+\nu^2}}\, e^{-\zeta} \begin{cases} \cos(\nu\zeta - \operatorname{arc\,tan} 2\nu) \\ -i\cdot\sin(\nu\zeta - \operatorname{arc\,tan} 2\nu) \end{cases} \tag{A.54}$$

or, explicitly

$$B^{+1/2\, 1/2}_{1/2}\left(\zeta\left[\frac{1}{2}, \nu\right]\right) = \frac{1}{\frac{1}{4}+\nu^2}\, e^{-\zeta}\,(\cos\nu\zeta + 2\nu\sin\nu\zeta),$$

$$B^{-1/2\, 1/2}_{1/2}\left(\zeta\left[\frac{1}{2}, \nu\right]\right) = \frac{-i}{\frac{1}{4}+\nu^2}\, e^{-\zeta}\,(\sin\nu\zeta - 2\nu\cos\nu\zeta) \tag{A.55}$$

and

$$B^{\pm 3/2\, 1/2}_{1/2}\left(\zeta\left[\frac{1}{2}, \nu\right]\right) = \frac{2\sqrt{2}}{\sqrt{\frac{1}{4}+\nu^2}}\, e^{-\zeta} \begin{Bmatrix} \cos \\ -i\sin \end{Bmatrix} \left(\nu\zeta - \operatorname{arc\,tan}\frac{2\nu}{\left(\frac{3}{4}-\nu^2\right)}\right),$$

$$B^{+3/2\, 1/2}_{1/2}\left(\zeta\left[\frac{1}{2}, \nu\right]\right) = \frac{2\sqrt{2}}{\frac{1}{4}+\nu^2}\,\frac{1}{\sqrt{\frac{9}{4}+\nu^2}}\, e^{-\zeta}\left(\left(\frac{3}{4}-\nu^2\right)\cos\nu\zeta + 2\nu\sin\nu\zeta\right),$$

$$B^{-3/2\, 1/2}_{1/2}\left(\zeta\left[\frac{1}{2}, \nu\right]\right) = \frac{-i2\sqrt{2}}{\frac{1}{4}+\nu^2}\,\frac{1}{\sqrt{\frac{9}{4}+\nu^2}}\, e^{-\zeta}\left(\left(\frac{3}{4}-\nu^2\right)\sin\nu\zeta - 2\nu\cos\nu\zeta\right). \tag{A.56}$$

Observe that I^3 in this range is (from (2.20))

$$I^3 = \frac{1}{\frac{1}{4}+\nu^2}\, e^{-\zeta}\left[-\frac{2}{3}\nu(\sin\nu\zeta - 2\nu\cos\nu\zeta) + \frac{4}{3}\left(\left(\frac{3}{4}-\nu^2\right)\cos\nu\zeta + 2\nu\sin\nu\zeta\right)\right] =$$

$$= \frac{1}{\frac{1}{4}+\nu^2}\, e^{-\zeta}\,(\cos\nu\zeta + 2\nu\sin\nu\zeta) = I^0 \tag{A.57}$$

as it should be from current conservation (2.31).

Appendix B

Existence of a Vector Operator

Suppose there is a vector Γ^μ, then it has to fulfill:

$$[M_i \Gamma^0] = i\Gamma^i, \tag{B.1}$$

$$[M_i \Gamma^i] = i\Gamma^0. \tag{B.2}$$

Let's abbreviate $|jm[j_0j_1]\rangle$ by $|jm\tau\rangle$. Given a set of irreducible representations $|jm\tau\rangle$, which ones can be connected by a vector Γ^μ? The answer is, as one expects from the nonunitary selection rules:

$$D^{1/2\,1/2}D^{(s,s')} = D^{(s+1/2,s')} + D^{(s-1/2,s')} + D^{(s,s'+1/2)} + D^{(s,s'-1/2)} \tag{B.3}$$

that

$$(j_0'j_1') = (j_0+1, j_1), (j_0, j_1+1), (j_0-1, j_1), (j_1, j_1-1)$$

and in this case the reduced matrix elements of Γ^0 defined by

$$\Gamma_0|jm\tau\rangle = |jm\tau'\rangle\, \gamma_{\tau'\tau} \tag{B.4}$$

are [2]

$$\gamma^j_{\tau'\tau} = \gamma_{\tau'\tau}\sqrt{(j+j_0+1)(j-j_0)} \quad \text{if} \quad j_0' = j_0+1,$$

$$\gamma^j_{\tau'\tau} = \gamma_{\tau'\tau}\sqrt{(j+j_1+1)(j-j_1)} \quad \text{if} \quad j_1' = j_1+1. \tag{B.5}$$

In the particular case that τ and τ' are parity conjugate of each other

$$\tau = (+j_0j_1), \quad \tau' = (-j_0j_1) \quad \text{or} \quad \tau' = (j_0j_1). \tag{B.6}$$

Γ^μ exists only if $\tau = (1/2, i\nu)$ or $(0, 1/2)$. The only hermitian Γ^0 is

$$\Gamma^0 = ({}_1{}^1)\,\gamma i \quad \text{or} \quad ({}_{-i}{}^{i})\,\gamma j \tag{B.7}$$

and since parity is $P = ({}_1{}^1)$, only the first one generates a vector, the other an axial vector. Using (B.1), one finds that on the states

$$|jm1\rangle \equiv |jm[1/2, i\nu]\rangle,$$

$$|jm2\rangle \equiv |jm\,[1/2, -i\nu]\rangle, \tag{B.8}$$

the vector Γ is

$$\Gamma_0|jm1\rangle = (j+1/2)\,|jm2\rangle,$$

$$i\Gamma^\pm|jm1\rangle = \pm\sqrt{(j\mp m)(j\mp m-1)}\,C_j\,|j-1, m\pm 1, 2\rangle +$$
$$+\sqrt{(j\mp m)(j\pm m+1)}\,(2j+1)\,A_j|jm\pm 1, 2\rangle \mp$$
$$\mp\sqrt{(j\pm m+1)(j\pm m+2)}\,C_{j+1}\,|j+1, m\mp 1, 2\rangle,$$

$$i\Gamma^3|jm1\rangle = \sqrt{j^2-m^2}\,C_j|j-1, m\,2\rangle + m(2j+1)\,A_j\,|jm2\rangle +$$
$$+\sqrt{(j+1)^2-m^2}\,C_{j+1}\,|j+1, m\,2\rangle, \tag{B.9}$$

with

$$C_j = \frac{i}{2j}\sqrt{j^2+\nu^2}, \quad A_j = \frac{\nu/2}{j(j+1)} \tag{B.10}$$

and on the parity eigenstates $|\pm\rangle = |1\rangle \pm |2\rangle$ one finally gets equations (2.13).

For $(^1/_2, 0)$ or $(0, ^1/_2)$ there is a a^+, a representation of Γ^μ corresponding to (A.32) which is

$$\Gamma^0 = {}^1/_2\,(a^+a + 1) = K_3\,,$$
$$\Gamma^i = + {}^1/_4\,(a^+\sigma_i\, c a^+ + - a c \sigma_i a)\,. \tag{B.11}$$

In this case, and, as we can verify from (B. 9), only in this case (except for the Dirac limit $j_1 = 3/2$), Γ^μ close under commutation rules among each other, giving

$$[\Gamma^i, \Gamma^j] = -iL_{ij}$$
$$[\Gamma^i, \Gamma^0] = -iM_i \tag{B.12}$$

hence, L_{ij}, $L_{i4} \equiv M_i$, $L_{i5} = \Gamma^i$ and $L_{45} = \Gamma^0$ generate together $O(3,2)$ with the commutation rules

$$[L_{\mu\nu} L_{\mu\lambda}] = -g_{\mu\mu} L_{\nu\lambda}\,, \quad g = \begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & -1\end{pmatrix} \tag{B.13}$$

Appendix C:

Group Theoretical Structure of the Dipole Operator

If one inserts intermediate states into (4.34) and observes that $L_{i5} \mp L_{i6}$ raises (lowers) n by one unit, one obtains

$$\begin{aligned}\langle n'l'm'|x_i|nlm\rangle &= \langle n'l'm'\left|D_{\frac{n+1}{n}}\right| n+1, l'm'\rangle \times \\ &\times \langle n+1, l', m'|L_{i5} - iL_{i6}|nlm\rangle \frac{(n+1)^2}{2n} + \\ &+ \langle n'l'm'\left|D_{\frac{n-1}{n}}\right| n-1, l', m'\rangle\langle n-1, l', m'|L_{i5} + iL_{i6}|nlm\rangle \frac{(n-1)^2}{2n} + \\ &+ \langle n'l'm'|L_{i4}|nlm\rangle\,.\end{aligned} \tag{C.1}$$

Hence we have only to find group theoretical expressions for the matrix elements of $D_{\frac{n\pm1}{n}}$. From the definition (4.29) it follows that these matrix elements can be expressed in terms of the integrals over hypergeometric functions:

$$\langle n'lm\left|D_{\frac{n\pm1}{n}}\right| n\pm1, lm\rangle \frac{(n\pm1)^2}{n} = \sqrt{n(n\pm1)}\; d^{n,n'}_{n\pm1,l;n',l}\,, \tag{C.2}$$

$$d^{\tau\tau'}_{n,l;n',l} = \left(\frac{nn'}{\tau\tau'}\right)^{3/2} \frac{N_{nl}N_{n'l}}{\tau^l\tau'^l}\, J^{(1,0)}_{2l+1}\left(n_r, \frac{2}{\tau}; n'_r, \frac{2}{\tau'}\right), \tag{C.3}$$

$$N_{nl} = \frac{2^{l+1}}{n^2(2l+1)!}\left[\frac{(n+l)!}{n_r!}\right]^{1/2}, \tag{C.4}$$

$$J^{(\sigma,\tau)}_\varrho(n_r, k; n'_r, k') = \int_0^\infty e^{-\frac{k+k'}{2}\xi}\, \xi^{\varrho+\tau} F(-n_r, \varrho+1, k\xi)\, F(-n'_r, \varrho+1-\tau, k'\xi)\,. \tag{C.5}$$

These integrals have been evaluated by GORDON [27] recursively with the result

$$J^{(1,0)}_{2l+1}(n, k; n'_r, k') = -\frac{4}{k'^2 - k^2} (kr - k'r') J^{(0,0)}_{2l+1}(n, k; n_r, k'),$$

$$J^{(0,0)}_{2l+1}(n_r, k; n'_r, k') = \frac{(2l+1)!^2 n_r!}{(n-n')!(l+n')!} \frac{1}{(kk')^{l+1}} v^{l+1} u^{n-n'} \times$$
$$\times F(-n'_r, n+l+l, 1+n-n', u^2),$$

$$u = \frac{k'-k}{k'+k}, \quad v = 2\frac{\sqrt{kk'}}{k'+k}. \tag{C.6}$$

For $n < n'$ one has to exchange n and n' in (C. 6) and $u \to -u$. If we use the identity

$$F(a, b, c; z) = \frac{1}{(1-z)^2} F\left(a, c-b, c; \frac{z}{z-1}\right)$$

and

$$\sinh\frac{\theta}{2} = \frac{u}{\sqrt{1-u^2}}, \quad \cosh\frac{\theta}{2} = \frac{1}{\sqrt{1-u^2}}$$

to rewrite $J^{(1,0)}_{2l+1}$ in (C.6) and define the functions

$$v^{l+1}_{nn'}(\theta) \equiv \theta_{nn'}\left(\cosh\frac{\theta}{2}\right)^{-(n+n')}\left(\sinh\frac{\theta}{2}\right)^{(n-n')} F\left(-n'_r, -n'-l, 1+n-n'_r, -\sinh\frac{2\theta}{2}\right) \tag{C.7}$$

with

$$\theta_{nn'} = \frac{1}{(n-n')!}\left[\frac{n_r!(n+l)!}{n'_r!(n'+l)!}\right]^{1/2}$$

we obtain finally

$$d^{\tau\tau'}_{n,l;n'l} = \left(\frac{\tau\tau'}{nn'}\right)^{1/2}\left(\frac{n'}{\tau'} - \frac{n}{\tau}\right)\frac{1}{\sinh\theta_{\tau'\tau}} v^{l+1}_{nn'}(\theta_{\tau'\tau}) \tag{C.8}$$

where the angle $\theta_{\tau'\tau}$ is determined by

$$\sinh\frac{\theta_{\tau'\tau}}{2} = \frac{\tau-\tau'}{2\sqrt{\tau\tau'}}, \quad \cosh\frac{\theta_{\tau'\tau}}{2} = \frac{\tau+\tau'}{2\sqrt{\tau\tau'}}.$$

In particular then Equ. (C.2) becomes

$$\langle n'lm \left| D_{\frac{n\pm1}{n}} \right| n\pm1, lm\rangle \frac{(n\pm1)^2}{n} = \mp\frac{1}{\sinh\theta_{n'n}} v^{l+1}_{n\pm1,n'}(\theta_{n'n}). \tag{C.9}$$

There is good reason of introducing the functions $v_{nn'}$ in (C.7). Consider the non-compact group generators $K_1 = L_{45}, K_2 = -L_{46}$ and $K_3 = L_{56} = N$ of $O(4,2)$. They form the algebra of $O(2,1)$ subgroup. The action of the raising and lowering operators $K^\pm = K_1 \pm K_2$ is $\Delta l = 0$, $\Delta m = 0$ and $\Delta m = \pm 1$.

The states $|nlm\rangle$, for fixed l and m, form therefore a basis for an irreducible representation D_k^+ of this $O(2,1)$, the transition group, characterized by the lowest eigenvalue of $K_3 = N$ which is clearly $n = l + 1$. The matrix elements for D_+ of the finite $O(2,1)$ transformations are given by

$$e^{-i\theta k_1}|n'lm\rangle = |nlm\rangle\, v_{nn'}^{l+1}(\theta) \tag{C.10}$$

where $v_{nn'}^{l+1}$ is precisely the function introduced in (C. 7). Therefore (C. 9) becomes

$$\langle n'lm \left| D_{\frac{n\pm 1}{n}} \right| n \pm 1, lm\rangle \frac{(n \pm 1)^2}{n} = \mp \frac{1}{\sinh \theta_{n'n}} \langle n'lm | e^{i\theta_{n'n}K_1} | n \pm 1, lm\rangle. \tag{C.11}$$

This equation inserted into (C. 1) finally gives Equ. (4. 36).

Appendix D:

Proof of the Orthogonality Relation (4.97)

For $n' = n$ the expression (4.97) has a value one, since L_{46} has only non-diagonal elements. Let's assume that $n' > n$. Inserting intermediate states we get five terms which consist of combinations of

$$v^{\frac{m+1}{2}}_{n_1'+\frac{m+1}{2},\, n_1+\frac{m+1}{2}}\left(+\,\mathrm{sh}\,\frac{\theta}{2}\right) v^{\frac{m+1}{2}}_{n_2'+\frac{m+1}{2},\, n_2+\frac{m+1}{2}}\left(-\,\mathrm{sh}\,\frac{\theta}{2}\right)$$

with $\theta = \log n/n$.

Expressing them in terms of hypergeometric functions according to App. (A.41) we can write the different contributions as

$$\begin{aligned}
&\frac{1}{2}\,\frac{(n_1'+1)\,(n_1'+m+1)}{n_1'+1-n_1}\,\frac{\mathrm{sh}\,\theta/2}{\mathrm{ch}\,\theta/2}\, v^{(+)}_{n_1'n_1} v_{n_2'n_2} \\
&+\frac{1}{2}\,\frac{(n_2'+1)\,(n_2'+m+1)}{n_2'+1-n_2}\,\frac{\mathrm{sh}\,\theta/2}{\mathrm{ch}\,\theta/2}\, v_{n_1'n_1} v^{(+)}_{n_2'n_2} \\
&+\frac{1}{2}\,(n_1'-n_1)\,\frac{\mathrm{ch}\,\theta/2}{\mathrm{sh}\,\theta/2}\, v^{(-)}_{n_1'n_1} v_{n_2'n_2} \\
&+\frac{1}{2}\,(n_2'-n_2)\,\frac{\mathrm{ch}\,\theta/2}{\mathrm{sh}\,\theta/2}\, v_{n_1'n_1} v^{(-)}_{n_2'n_2} \\
&+ n'\, v_{n_1'n_1}\, v_{n_2'n_2}
\end{aligned} \tag{D.1}$$

where $v^{(\pm)}$ indicates that a v function like (A. 41) has to be used, which is modified such that the third argument in the hypergeometric function is $1 + m - n \pm 1$ instead of $1 + m - n$. One can then collect $v^{(+)}$ and $v^{(-)}$ together through the identity

$$c(c-1)(z-1)F(a,b,c-1,z)+c[(c-1)-(2c-a-b-1)z]F(a,b,c,z)+ \\ +(c-a)(c-b)z\,F(a,b,c+1,z)=0 \tag{D.2}$$

and finds that the first four terms in (D. 1) indeed cancel the last one.

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Notes added in proof

1) (s. p. 6) In the Language of infinite component fields (see Ch. V) every representation specifies a different local interaction Lagrangian leading exactly to the transition amplitudes (2.6). One may say: A dynamical boost is such a representation of the Lorentz group which makes a given interaction local.

2) (see p. 9) Such terms have meanwhile been employed to obtain quite good electromagnetic form factors for the nucleons (see A. O. BARUT, D. CORRIGAN and H. KLEINERT, Phys. Rev. Letters **20**, 167 (1968)). The authors apply the ideas developped in Ch. V. to fix the contributions of the different terms to the current from the observed baryon mass spectrum.

3) (see p. 15) A factor

$$f_\nu^{j'} = \frac{\sqrt{[(3/2)^2 + \nu^2] \cdots [j'^2 + \nu^2]}}{3/2 \cdots j'}$$

has been divided out of all amplitudes $B_{1/2}^{j'\,1/2}$ in order to permit a better comparison of curves with different ν values (see also the Figs. 7a—d).

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