DOUBLE GAUGE THEORY OF STRESSES AND DEFECTS

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Elasticity and plasticity can both be formulated in a geometric language. In linear approximation, there are two gauge theories. We exhibit the dual relationship between the two and present a coupled unified double gauge theory of both stresses and defects.

The continuum theory of defects and stresses harbors two important applications of geometry (for a review see ref. [1]). One goes back to Kondo's work [2] and is based on a rather direct relation between dislocation and disclination densities $\alpha_{\alpha\beta}, \Theta_{\alpha\beta}$ with torsion $S_{\alpha\beta}{}^{\gamma} = \frac{1}{2}(\Gamma_{\alpha\beta}{}^{\gamma} - \Gamma_{\beta\alpha}{}^{\gamma})$ and Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R_{\gamma}{}^{\gamma}$, respectively, where

$$\Gamma_{\alpha\beta}{}^{\gamma} = \{{}_{\alpha}{}^{\gamma}{}_{\beta}\} + K_{\alpha\beta}{}^{\gamma} = \{{}_{\alpha}{}^{\gamma}{}_{\beta}\} + S_{\alpha\beta}{}^{\gamma} - S_{\beta}{}^{\gamma}{}_{\alpha} + S^{\gamma}{}_{\alpha\beta}$$

is the connection [‡1] and $R_{\beta\gamma} = R_{\alpha\beta\gamma}{}^{\alpha}$ with $R_{\alpha\beta\gamma}{}^{\delta}$ being the covariant curl of curvature tensor (= covariant curl of the connection) $\partial_{\alpha}\Gamma_{\beta\gamma}{}^{\delta} - \Gamma_{\alpha\gamma}{}^{\tau}\Gamma_{\beta\tau}{}^{\delta} - (\alpha \leftrightarrow \beta)$. This geometry is introduced by considering an ideal reference crystal with atoms positioned at cartesian places x^{a}. Under a deformation, the atoms move to $x^{\alpha} = [x^{a} + u^{a}(x^{\alpha})]\delta^{a\alpha}$ and the new coordinates are characterized by tangential vectors $e_{\alpha}^{a} = \partial_{\alpha}u^{a}(x^{\alpha})$. These define a connection $\Gamma_{\alpha\beta}{}^{\gamma} = e_{a}^{\gamma}\partial_{\alpha}e_{\beta}^{a}$ with a metric $g_{\alpha\beta} = e_{\alpha}^{a}e_{\beta}^{a}$. The length $ds = (g_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta})^{1/2}$ is invariant under elastic deformations. It measures the distance two points would have in the ideal reference crystal, i.e. ds counts the crystalline atoms when following the distorted lattice directions. Parallel displacement of a vector v^{β} in the reference crystal amounts to $D_{\alpha}v^{\beta} = 0$ in the distorted crystal. Plastic deformations introduce defects and, as a consequence, derivatives in front of u^{a} no longer commute. In linear approximation,

$$\alpha_{\alpha\beta} \equiv \epsilon_{\beta\gamma\delta}\partial_{\gamma}\partial_{\delta}u_{\alpha}, \quad \Theta_{\alpha\beta} \equiv \epsilon_{\beta\gamma\delta}\partial_{\gamma}\partial_{\delta}\omega_{\alpha}$$

are defined as *dislocation* and *disclination densities*, respectively, where ω_{α} is the local rotation field $\frac{1}{2}\epsilon_{\alpha\beta\gamma}\partial_{\beta}u_{\gamma}$. A related quantity is the incompatibility of the strain tensor

$$\eta_{\alpha\beta} \equiv \epsilon_{\alpha\gamma\delta}\epsilon_{\beta\sigma\tau}\partial_{\gamma}\partial_{\sigma}u_{\delta\tau} \tag{1}$$

called the *defect density*. It combines $\alpha_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ in the form

$$\eta_{\alpha\beta} = \Theta_{\alpha\beta} - \epsilon_{\beta\gamma\delta}\partial_{\gamma}K_{\delta\alpha}, \tag{2}$$

where $K_{\delta\alpha} \equiv -\alpha_{\delta\alpha} + \frac{1}{2}\delta_{\delta\alpha}\alpha_{\sigma\tau}$ is called the contortion tensor.

Now, in linear approximation $\Gamma_{\alpha\beta}{}^{\gamma} \sim \partial_{\alpha}\partial_{\beta}u^{\gamma}, g_{\alpha\beta} \sim \delta_{\alpha\beta} + 2u_{\alpha\beta}$ and we can verify that $\alpha_{\delta\gamma} \sim \frac{1}{2}\epsilon_{\delta\alpha\beta}\Sigma_{\alpha\beta,\gamma}$,

[‡1] Covariant derivatives: $D_{\alpha}v_{\beta} \equiv \partial_{\alpha}v_{\beta} - \Gamma_{\alpha\beta}{}^{\gamma}v_{\gamma}, D_{\alpha}v^{\beta} \equiv \partial_{\alpha}v^{\beta} + \Gamma_{\alpha\gamma}{}^{\beta}v^{\gamma}$.

where $\Sigma_{\alpha\beta,\gamma} \equiv 2(S_{\alpha\beta\gamma} - S_{\beta\gamma\alpha} + S_{\gamma\alpha\beta})$ is the standard spin density of a gravitational field [3] while [‡2] $\Theta_{\alpha\beta} \sim G_{\alpha\beta}$, where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R_{\gamma}{}^{\gamma}$ is the Einstein tensor. Relation (2) is the linearized version of Belinfante's formula [3] for the symmetric energy–momentum tensor $\eta_{\alpha\beta}$ when expressed in terms of the canonical tensor, here $G_{\alpha\beta}$, and the spin densities $\Sigma_{\alpha\beta,\gamma}$:

$$\eta_{\alpha\beta} = G_{\alpha\beta} - \tfrac{1}{2}(D_\gamma + 2S_{\gamma\sigma}{}^{\sigma})(\Sigma^{\alpha\beta,\gamma} - \Sigma^{\beta\gamma,\alpha} + \Sigma^{\gamma\alpha,\beta}) . \tag{3}$$

Hence we see that plastic deformations introduce spin and curvature into this geometry.

Elastic deformations, on the other hand, correspond to the usual Einstein coordinate transformations $u^a(x) \to u^a(x) + \xi^a(x)$ with *smooth* $\xi^a(x)$ under which all tensor equations are form invariant. In linear approximation, the strain tensor changes as follows

$$u_{\alpha\beta}(x) \to u_{\alpha\beta}(x) + \tfrac{1}{2}[\partial_\alpha \xi_\beta(x) + \partial_\beta \xi_\alpha(x)] . \tag{4}$$

This is a typical gauge transformation. Under it, the defect tensor (1) remains obviously invariant, as it should on physical grounds, a fact which will be referred to as *defect gauge invariance*.

The other type of gauge theory has a more formal appearance: Since the stress tensor $\sigma_{\alpha\beta}$ is symmetric and divergenceless it can be considered as the linearized Einstein tensor of an auxiliary riemannian space with a metric $\chi_{\alpha\beta}$, known as the Beltrami–Schäfer stress potential [2,4]. The linearized relation

$$\sigma_{\alpha\beta} = \epsilon_{\alpha\gamma\delta}\,\epsilon_{\beta\sigma\tau}\,\partial_\gamma \partial_\sigma \chi_{\delta\tau} \tag{5}$$

is gauge invariant under $\chi_{\alpha\beta} \to \chi_{\alpha\beta} + \partial_\alpha \Lambda_\beta + \partial_\beta \Lambda_\alpha$. The stress energy

$$E_{\text{stress}} = \int d^3x \, \frac{1}{4\mu} \left(\sigma^2_{\alpha\beta} - \frac{\nu}{1+\nu}\, \sigma^2_{\alpha\alpha} \right)$$

(with μ = shear module, ν = Poisson ratio) can be viewed geometrically [5,6] as the linearized version of a Weyl gravitational theory [7]. This *stress gauge invariance* has recently been exploited for the construction of a gauge theory of defect melting [8,9]. The significance of the gauge field lies in its having a *local* coupling with the defects. Since random chains of defect lines can be represented by a complex scalar disorder field (Higgs field), the resulting theory is completely local and has the form of the Ginzburg–Landau theory of superconductivity. The phase transition of melting is signalized by the complex disorder field taking a nonzero ground-state expectation value.

The question arises as to the relationship between the two gauge invariances. In addition, it is desirable to understand the role of the first *defect gauge invariance* within the existing *stress gauge theory* in which defects appear as disorder fields. It is the purpose of this note to clarify these questions, thereby arriving at a *double gauge theory of defects and stresses*. If desired, this can be generalized to a doubly geometric theory as proposed in ref. [6]. Our discussion will shed light on recent, more formal attempts [10] [‡3] at constructing field theories of defects and stresses, in which the defects appear as gauge fields while stress is described by complex matter fields.

Starting point is the partition function as given in ref. [9].

$$Z \propto \prod_{x,\alpha,\beta} \int_{-\infty}^{\infty} d\hat{\chi}_{\alpha\beta}(x)\, \delta(\bar{\nabla}_\alpha \hat{\chi}_{\alpha\beta}) \exp\left[-\tau \sum_x \left(\hat{\sigma}^2_{\alpha\beta} - \frac{\nu}{1+\nu}\hat{\sigma}^2_{\alpha\alpha}\right)\right] \sum_{\bar{\eta}_{\alpha\beta}} \delta_{\bar{\nabla}_\alpha \bar{\eta}_{\alpha\beta},0} \exp\left(2\pi i \sum_x \bar{\eta}_{\alpha\beta}\hat{\chi}_{\alpha\beta}\right) \tag{6}$$

where $\tau \equiv \pi^2 T/\mu a^3$ (with a = lattice spacing, T = temperature), $\hat{\sigma}_{\alpha\beta} = (a^3/2\pi T)\,\sigma_{\alpha\beta}$ is a dimensionless version of $\sigma_{\alpha\beta}$, $\hat{\chi}_{\alpha\beta}$ is the corresponding stress potential, and the symbols ∇_α, $\bar{\nabla}_\alpha$ denote the usual lattice derivatives [‡4]. The quantity $\bar{\eta}_{\alpha\beta}(x)$ is the dimensionless version of the defect density. It forms an integer-valued symmetric tensor

‡2 To derive this, notice that $R_{\alpha\beta\gamma}{}^{\delta} \equiv e^{a\delta}(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha)\, e_{a\gamma}$.

‡3 The present author disagrees with those theoretical constructions, which have no relation with the physics to be described.

‡4 $\nabla_\alpha \varphi(x) = \varphi(x+e_\alpha) - \varphi(x)$, $\bar{\nabla}_\alpha \varphi(x) = \varphi(x) - \varphi(x-e_\alpha)$, $e_\alpha \equiv$ lattice basis vector.

field. The condition $\overline{\nabla}_\alpha \overline{\eta}_{\alpha\beta} = 0$ can be interpreted geometrically. With each $\overline{\eta}_{\alpha\beta}$ we may associate three sets of field lines $(\overline{\eta}_\alpha)_\beta$, one for every β. These describe defect lines running through the crystal. The condition $\overline{\nabla}_\alpha \overline{\eta}_{\alpha\beta} = 0$ means that the lines are closed. The symmetry of $\overline{\eta}_{\alpha\beta}$ imposes certain constraints. The sum $\Sigma_{\overline{\eta}_{\alpha\beta}} \delta_{\overline{\nabla}_\alpha \overline{\eta}_{\alpha\beta},0}$ in (6) collects all such closed field lines.

Notice that the partition function (6) contains only pure stress energies of defects. In general, non-linearities of the crystalline forces will produce additional core energies which we may parameterize as

$$\frac{1}{T} E_{\text{core}} = \sum_{x,x'} \overline{\eta}(x)_{\alpha\beta}\, G(x-x')_{\alpha\beta,\alpha'\beta'}\, \overline{\eta}(x')_{\alpha'\beta'} \,. \tag{7}$$

Due to the constraint $\overline{\nabla}_\alpha \overline{\eta}_{\alpha\beta} = 0$, the partition function (6), with (7) added in the exponent, permits the introduction of a defect gauge field $\overline{u}_{\sigma\tau}$ with the double curl representation

$$\overline{\eta}_{\alpha\beta} = \epsilon_{\alpha\gamma\delta}\, \epsilon_{\beta\sigma\tau}\, \overline{\nabla}_\gamma \overline{\nabla}_\sigma\, \overline{u}_{\delta\tau} \,. \tag{8}$$

This relation is invariant under arbitrary gauge transformations

$$\overline{u}_{\delta\tau} \to \overline{u}_{\delta\tau} + \nabla_\delta\, \xi_\tau + \nabla_\tau\, \xi_\delta \,. \tag{9}$$

Notice that since $\overline{\eta}_{\alpha\beta}$ is integer valued, $\overline{u}_{\delta\tau}$ can also be chosen to be so. For example, in the axial gauge $\overline{u}_{\delta 3} = \overline{u}_{3\delta} = 0$ (not, however, in the gauge $\nabla_\alpha \overline{u}_{\alpha\beta} = 0$). The gauge transformations (9) can bring this to any non-integer gauge equivalent configuration. By comparing (8) with (3), the gauge fields $\overline{u}_{\delta\tau}$ have a very simple geometric interpretation. Let us introduce an integer-valued displacement field $\overline{u}_\alpha = a^{-1} u_\alpha$ where u_α are lattice vectors. This field $\overline{u}_\alpha$ represents possible jumps of atoms between the periodic equilibrium configurations of the crystal. We may identify the gauge field $\overline{u}_{\sigma\tau}$ with the strain field associated with $\overline{u}_\alpha$, i.e.

$$\overline{u}_{\sigma\tau} = (\nabla_\sigma \overline{u}_\tau + \nabla_\tau \overline{u}_\sigma)/2 \,. \tag{10}$$

Then we see that the defect field $\overline{\eta}_{\alpha\beta}$ measures the incompatibility of these jumps, i.e. it measures how bad the displaced atoms fail to form a perfect crystal. The gauge transformation (9) is an elastic distortion which does not change the defect pattern. The strain configuration (10) may be viewed as a singular local gauge transformation associated with the discrete translational symmetry of the crystal.

Using this integer gauge field representation of defects we can now write the partition function (6), including the defect core energies (7), as

$$Z = \prod_{x,\alpha,\beta} \int \mathrm{d}\chi_{\alpha\beta}(x)\, \delta(\overline{\nabla}_\alpha \chi_{\alpha\beta}) \sum_{\overline{u}_{\alpha\beta}(x)} \delta_{\overline{u}_{3\beta},0} \exp\left[-\tau \sum_x \left(\sigma_{\alpha\beta}^2 - \frac{\nu}{1+\nu}\, \sigma_{\alpha\alpha}^2\right)\right.$$

$$\left. - \frac{1}{2} \sum_{x,x'} \overline{\eta}(x)_{\alpha\beta}\, G(x-x')_{\alpha\beta}\, \overline{\eta}_{\alpha\beta}(x') + 2\pi \mathrm{i} \sum_x \overline{\eta}_{\alpha\beta}\, \chi_{\alpha\beta}\right] . \tag{11}$$

Note that there are two gauge fixing factors, one for the continuous *stress gauge field* and one for the integer valued *defect gauge field*. The first gauge is rather arbitrary, and the second must be of the axial type, or related to it by an integer-valued gauge transformation, in order to be compatible with the integer valuedness of $u_{\alpha\beta}$.

This is the proper *double gauge theory of stresses and defects* on the lattice.

By taking this theory to the continuum limit, the sum over discrete $\overline{\eta}_{\alpha\beta}$ becomes an integral and we arrive at a field theory involving two *continuous* gauge fields $\chi_{\alpha\beta}$ and $\overline{u}_{\alpha\beta}$ which are linearly coupled with each other. This coupling corresponds to the linear approximation of the geometric theory proposed earlier in ref. [6].

Notice the way this double gauge theory is related to the previously studied stress gauge theory in which the defects were represented by a complex disorder field. That can be obtained by turning the sum over $\overline{\eta}_{\alpha\beta}$ with $\overline{\nabla}_\alpha \overline{\eta}_{\alpha\beta} = 0$ into a functional integral over complex scalar fields ψ. In this way the grand-canonical ensemble of closed $(\overline{\eta}_\beta)_\alpha$ field lines, the defect lines, corresponds directly to the Feynman loop diagrams of the complex field. This is the statistical analogue to the equivalence of N-particle orbits and quantized field theory [11].

It is worth pointing out that due to the symmetry of (10) in σ and η it is possible to convert also the sum over $\sigma_{\alpha\beta}$ fields with $\overline{\nabla}_\alpha \sigma_{\alpha\beta} = 0$ into a complex field ψ stress using the same techniques. In this case, the result would be a gauge theory of defects in which the closed field lines of stress $(\sigma_\beta)_\alpha = \sigma_{\alpha\beta}$ with $\overline{\nabla}_\alpha \sigma_{\alpha\beta} = 0$ would correspond to the closed loop Feynman diagrams of the complex field. In this way one would obtain a correct version of a gauge theory attempted in ref. [10].

The specific form of the defect core energy requires more investigation. Experimentally, it certainly contains local quadratic terms in $\alpha_{\alpha\beta'}{}^{\gamma}$, just as the torsion terms expected in Einstein's theory in the presence of spinning matter.

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