FIELD THEORY OF SELF-AVOIDING RANDOM SURFACES

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We present and analyze a simple field theory whose partition function sums up all self-avoiding random surfaces with m internal degrees of freedom ("colors"). The field theory suggests that for $m = 1$ the critical indices are Ising like, while for $m = 2$ the transition falls into the universality class of the U(1) lattice gauge theory and the surfaces proliferate smoothly, the only phase transition lying at infinite temperature.

It has been recognized some time ago that for a theoretical understanding of microemulsions [1], entropy plays an important role [2]. Since the droplet surfaces in such emulsions are formed by surfactants, they are basically self-avoiding. This imposes severe constraints on the configuration sum. It is the purpose of this note to develop a simple field theory whose partition function correctly counts those configurations. The field theory can be studied by mean field techniques, and fluctuation corrections are easy to perform. The theory focuses on entropy and leaves out the important effects of curvature energies, van der Waal's attraction, and steric repulsion [3]. These must eventually be included for a comparison with experimental data. We do, however, allow for the possibility of forming different mutually exclusive walls, which we distinguish by a "color" label.

The partition function to be calculated is

$$Z = \sum_{\{S^a\}} \exp\left(-\sum_{a=1}^{m} \beta_a^{\mathrm{sur}} A_a\right), \tag{1}$$

where $\{S^a\}$ denotes all self-avoiding closed surface configurations of m colors, labeled by $a = 1, \ldots, m$, A_a is the surface area of color a, and β_a^{sur} the energy per unit area divided by the thermal energy $k_B T$. The counting of all configurations becomes easiest by considering a simple cubic lattice in D dimensions, with integer coordinates $\boldsymbol{x}$ and D oriented link vectors $\boldsymbol{i} = (0, \ldots, 1, \ldots, 0)$ whose plaquettes $i < j$ can be empty or occupied by a surface element of colors a. Let $n_{ij}^a(\boldsymbol{x})$ be the corresponding occupation numbers extended to form a symmetric matrix by defining $n_{ij}^a(\boldsymbol{x}) = n_{ji}^a(\boldsymbol{x})$ for $i > j$. The self-avoiding requirement implies that on plaquettes around a common link, the sum over all $n_{ij}^a(\boldsymbol{x})$ must be zero or two. This is assured by the constraint

$$\prod_{x,i,a} \sum_{z_i^a(x)=0,2} \delta_{\sum_{j\neq i}[n_{ij}^a(x-j)+n_{ij}^a(x)],\, z_i^a(x)} \prod_{x,i} \sum_{z_i(x)=0,2} \delta_{\sum_a z_i^a(x),\, z_i(x)} \,. \tag{2}$$

The first factor provides for self-avoidance within each color, the second for mutual avoidance of different colors. For convenience, we shall insert one more redundant condition

$$\prod_{x,i<j} \sum_{z_{ij}(x)=0,1} \delta_{\sum_a n_{ij}^a(x),\, z_{ij}(x)} \,, \tag{2'}$$

which is guaranteed already by (2) but will be useful for bringing Z to a more attractive form. If $C^m[n_{ij}^a]$ denotes the product of (2) and (3), the partition function (1) reads

$$Z = \sum_{\{n_{ij}^a(x)\}} C^m [n_{ij}^a] \beta_a^{n_{ij}^a(x)} , \qquad (3)$$

where we have introduced $\beta_a \equiv \exp(-\beta_a^{\rm sur})$. We now proceed as in our work on self-avoiding random chains [4] and introduce angular variables $\theta_i^a(x)$, $\delta_i(x)$, $\delta_{ij}(x)$ which allow writing the constraint as follows

$$C^m [n_{ij}^a] = \prod_{x,i,a} \int_{-\pi}^{\pi} \frac{d\theta_i^a(x)}{2\pi} \prod_{x,i} \int_{-\pi}^{\pi} \frac{d\delta_i(x)}{2\pi} \prod_{x,i<j} \int_{-\pi}^{\pi} \frac{d\delta_{ij}(x)}{2\pi} \exp\Bigg\{ i \sum_{x,i} \Bigg[\theta_i^a(x)$$

$$\times \Big(\sum_{j \neq i} [n_{ij}^a(x-j) + n_{ij}^a(x)] - z_i^a(x) \Big) + \delta_i(x) \Big(\sum_a z_i^a(x) - z_i(x) \Big) + \sum_{j<i} \delta_{ij}(x) \Big(\sum_a n_{ij}^a(x) - z_{ij}(x) \Big) \Bigg] \Bigg\} . \qquad (4)$$

If we call $U^a(x) \equiv \exp[i\theta_i^a(x)]$, $V_i(x) \equiv \exp[i\delta_i(x)]$, $V_{ij}(x) \equiv \exp[i\delta_{ij}(x)]$, the integrand becomes [‡1]

$$\prod_{x,i<j,a} [U_i^a(x) U_j^a(x+i) U_i^a(x+j) U_j^a(x) V_{ij}(x)]^{n_{ij}^a(x)} \prod_{x,i,a} [U_i^{a*}(x) V_i(x)]^{z_i^a(x)} \prod_{x,i} [V_i^*(x)]^{z_i(x)}$$

$$\times \prod_{x,i<j} [V_{ij}^*(x)]^{z_{ij}(x)} . \qquad (5)$$

Multiplying by $\beta_a^{n_{ij}^a(x)}$ and summing over all $n_{ij}^a(x) = 0, 1; z_i^a(x) = 0, 2; z_{ij}(x) = 0,1$yields the partition function

$$Z = \prod_{x,i,a} \int_{-\pi}^{\pi} \frac{d\theta_i^a(x)}{2\pi} \prod_{x,i} \int_{-\pi}^{\pi} \frac{d\delta_i(x)}{2\pi} \prod_{x,i<j} \int_{-\pi}^{\pi} \frac{d\delta_{ij}(x)}{2\pi} \prod_{x,i<j} [1 + V_{ij}^*(x)] \prod_{x,i,a} \{1 + [U_i^{a*}(x)]^2 V_i^2(x)\}$$

$$\times \prod_{x,i} \{1 + [V_i^*(x)]^2\} \prod_{x,i<j,a} [1 + \beta_a U_i^a(x) U_j^a(x+i) U_i^a(x+j) U_j^a(x) V_{ij}(x)] . \qquad (6)$$

By expanding the first and last products, the integrals over δ_i and δ_{ij} can be executed with the simple result

$$Z = \prod_{x,i,a} \int_{-\pi}^{\pi} \frac{d\theta_i^a(x)}{2\pi} \prod_{x,i} \Big(1 + \sum_a [U_i^{a*}(x)]^2 \Big) \prod_{x,i<j} \Big(1 + \sum_a \beta_q U_i^a(x) U_j^a(x+i) U_i^a(x+j) U_j^a(x) \Big) . \qquad (7)$$

This is the lattice field theory of self-avoiding random surfaces of m colors expressed in terms of pure phase variables. It is possible to go over to real fields $u_i^a(x) \in (-\infty, \infty)$ and rewrite Z as

$$Z = \prod_{x,i,a} \int_{-\infty}^{+\infty} du_i^a(x) \delta(u_i^a(x)) \prod_{x,i} \Big(1 + \frac{1}{2} \sum_a \frac{\partial^2}{\partial u_i^a(x)^2} \Big) \prod_{x,i<j} \Big(1 + \sum_a \beta_a u_i^a(x) u_j^a(x+i) u_i^a(x+j) u_j^a(x) \Big) . \qquad (8)$$

This is obviously the same as (8) since under the measure $\int_{-\infty}^{\infty} du\, \delta\, |u|$, the differentiation $\frac{1}{2}\partial^2/\partial u^2$ has the same effect upon u as U^{*2} has upon U in (7). This observation gives rise to the real field theory

$$Z = \prod_{x,i,a} \int_{-\infty}^{+\infty} du_i^a(x) \int_{-i\infty}^{i\infty} \frac{d\alpha_i^a(x)}{2\pi i} \prod_{x,i} \Big(1 + \frac{1}{2} \sum_a [\alpha_i^a(x)]^2 \Big) \exp\Big(- \sum_{x,i,a} \alpha_i^a(x) u_i^a(x) \Big)$$

$$\times \prod_{x,i<j} \Big(1 + \sum_a \beta_a u_i^a(x) u_j^a(x+i) u_i^a(x+j) u_j^a(x) \Big) . \qquad (9)$$

‡1 After performing the trivial manipulation $\Sigma_{x,\, i \neq j}\, \theta_i^a(x) [n_{ij}^a(x-j) + n_{ij}^a(x)] = \Sigma_{x,\, i \neq j}\, [\theta_i^a(x+j) + \theta_i^a(x)] n_{ij}^a(x)$ $= \Sigma_{x,\, i<j}\, [\theta_i^a(x) + \theta_j^a(x+i) + \theta_i^a(x+j) + \theta_j^a(x)] n_{ij}^a(x)$.

In this form, the theory has manifest $(Z_2)^m$ symmetry (reflection symmetry of each colored pair u_i^a, α_i^a). Due to the strong truncation effect of the factors $\Pi_{x,i}\,[1+\frac{1}{2}\,\Sigma_a\,\alpha_i^a(x)^2]$, however, the full symmetry could be larger than this.

For $m = 1$, the partition function (9) differs from the Z_2 lattice gauge theory by having in the measure of integration, $\frac{1}{2}\,\Sigma_{s=-1,1}\,e^{\alpha s} = \mathrm{ch}\,\alpha$ replaced by $1+\frac{1}{2}\alpha(x)^2$ (and, certainly, β_{Z_2} by β) [‡2]. This difference is negligible for small β where the bubble gas is dilute. It becomes quantitatively important only very close to the transition temperature such that we can estimate $\beta^c \sim \mathrm{th}\,\beta^c_{Z_2}$.

In three dimensions, the Z_2 gauge theory is dually equivalent to the Ising model (the surfaces are the domain walls) with $\exp(-2\beta_{Z_2}) = \mathrm{th}\,\beta_{\mathrm{Ising}}$, i.e. $\mathrm{sh}\,2\beta_{Z_2} = 1/\mathrm{sh}\,2\beta_{\mathrm{Ising}}$. Using $\beta^c_{\mathrm{Ising}} = 0.2217$, gives $\beta^c_{Z_2} = 0.761$ and the estimate for one-color self-avoiding random surfaces $\beta^c \sim 0.642$. In four dimensions, there is self-duality and $\beta^c_{Z_2} = \beta_{\mathrm{Ising}} = 0.44$, giving $\beta^c \sim 0.41$.

It is useful to realize that the formulation of the field theory (7) is no way unique. We can always introduce an additional redundant phase variable $W_i(x) = \exp[\mathrm{i}\gamma_i(x)]$, multiply each $U_i^a(x)$ by $W_i(x)$, and integrate over all $\gamma_i(x) \in (-\pi, \pi)$. This brings the integrand to the form

$$\prod_{x,i<j}\left(1+\sum_a \beta_a W_\square U_\square^a\right), \tag{10}$$

where the subscript □ stands for the product of elements around each plaquette, for example

$$U_\square^a = U_i^a(x)U_j^a(x+i)U_i^a(x+j)U_j^a(x). \tag{11}$$

The transformed measure of integration is

$$\prod_{x,i,a}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_i^a(x)}{2\pi}\prod_{x,i}\int_{-\pi}^{\pi}\frac{\mathrm{d}\gamma_i(x)}{2\pi}\prod_{x,i}\left(1+\sum_a [U_i^{a*}(x)]^2[W_i^*(x)]^2\right). \tag{12}$$

The $U_i^a(x)$ integrals have the effect of knitting together the plaquettes of the integrand for equal color. The new redundant $W_i(x)$ integrals do once more the same thing *irrespective* of color. Since each link is either occupied by $U_i^a(x)$ *and* $W_i(x)$ or empty, the measure of integration can just as well be taken as

$$\prod_{x,i}\left(1+\sum_a [U_i^{a*}(x)]^2\right)\prod_{x,i}\{1+[W_i^*(x)]^2\}, \tag{13}$$

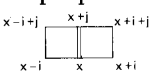
without changing the result (the mixed products cannot become active). In this way we arrive at the following general representation

$$Z=\prod_{x,i,a}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_i^a(x)}{2\pi}\prod_{x,i}\int_{-\pi}^{\pi}\frac{\mathrm{d}\gamma_i(x)}{2\pi}\prod_{x,i}\left(1+\sum_a [U_i^{a*}(x)]^2\right)\prod_{x,i}\left(1+[W_i^*(x)]^2\right)\prod_{x,i<j,a}\left(1+\sum_a \beta_a U_\square^a W_\square\right). \tag{14}$$

It is instructive to follow the way in which the integrations build up the different possible random surfaces. By doing the W_i integrations we obtain the constraint $C^1[n_{ij}]$, forming all possible bubbles of a single color [see (4)], and the partition function reads

$$Z=\sum_{\{n_{ij}(x)=0,1\}} C^1[n_{ij}]\prod_{x,i,a}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_i^a(x)}{2\pi}\prod_{x,i}\left(1+\sum_a [U_i^{a*}(x)]^2\right)\prod_{x,i<j}\left(\sum_a \beta_a U_\square^a\right)^{n_{ij}(x)}. \tag{15}$$

The integrals over $U_i^a(x)$ produce the different colors of the bubbles. We can see this by considering first two adjacent plaquettes



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[‡2] This can be seen directly by executing the α integrations with ch α which force all $u(x)$ to become +1 or −1 only.

They receive contributions from

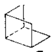
$$\prod_a \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_j(x)}{2\pi} \sum_a [U_j^{a*}(x)]^2 \sum_a \beta_a U_i^a(x-i) U_j^a(x) U_i^a(x-i+j) U_j^a(x-i) \sum_a \beta_a U_i^a(x) U_j^a(x+i) U_i^a(x+j) U_j^a(x) .$$

The integral eliminates the two $U_j^a(x)$ on the common link x, j and leaves

$$\sum_a \beta_a^2 U_i^a(x-i) U_i^a(x) U_j^a(x+i) U_i^a(x+j) U_i^a(x-i+j) U_j^a(x-i) .$$

For larger surfaces of area A, this becomes

$$\sum_a \beta_a^A \prod_B U_i^a(x) , \tag{16}$$

with the product covering all boundary links collectively denoted by B. The same statement holds for a surface with corners since an elementary corner  has a contribution from the twelve $U_i^a(x)$ of the three plaquettes, the common six which are eliminated by the $U_i^a(x)$ integrations, giving a factor $\Sigma_a \beta_a^3 \Pi_B U_i^a$ with the product running again over the boundary links. In general, whenever two open surface pieces meet along several boundary links, the integrals over $U_i^a(x)$ ensure the equality of their color such that there is the multiplication rule

$$\sum_a \beta_a^{A_1} \prod_{B_1} U_i^a \sum_a \beta_a^{A_2} \prod_{B_2} U_i^a \to \sum_a \beta_a^{A_1+A_2} \prod_{B_1 \cup B_2 - B_1 \cap B_2} U_i^a . \tag{17}$$

For closed surfaces, this leaves $\Sigma_a \beta_a^A$. Notice that the result is independent of the genus of the surface. For, if there is a handle, it may be obtained by rolling up a tube from an open surface. The integrations at the junctures merely ensure equality of the colors and give no new β_a factors. Therefore, also a tube gives the contribution (16). This, in turn, may be attached to another surface. If the boundary links match, the result is $\Sigma_a \beta_a^A$. Thus, if we let b enumerate the individual bubbles of area $A(b)$ in the ensemble, we obtain the partition function

$$Z = \sum_{\{S\}} \prod_b \left(\sum_a \beta_a^{A(b)} \right) . \tag{18}$$

If all colors have equal surface energy, this simplifies to

$$Z = \sum_{\{S\}} m^{n[S]} \beta^A , \tag{19}$$

where $n[S]$ is the number of bubbles and A their total area.

We have gone through this discussion in detail, since it illustrates an important property of representations (14), (15): Once the ensemble of self-avoiding bubbles is formed by the W_i integrations, there is ample freedom of choosing different statistical weights for the bubbles. Up to now, we have allowed only for different colors. There is, however, no problem in attributing different properties to the bubbles. Of particular practical interest will be the curvature energies mentioned in the beginning which will be discussed in detail elsewhere. In this note we shall confine ourselves to illustrating this freedom by replacing the integrals over $U_\square^a$ in (15) by

$$\prod_{x,i} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_i(x)}{2\pi} \prod_{x,i<j} [\beta U_i(x) U_j(x+i) U_i^*(x+j) U_i^*(x) + \text{c.c.}]^{n_{ij}(x)} . \tag{20}$$

This replacement is particularly interesting since it produces a theory possessing manifest U(1) *local* gauge invariance, just as electrodynamics [‡3]: For each x, i, we can multiply $U_i(x)$ by phases $U_i(x) \to \exp[-\mathrm{i}\varphi(x)]\, U_i(x) \times \exp[\mathrm{i}\varphi(x+i)]$. This transformation can be absorbed into the angles θ_i by adding the lattice gradient

[‡3] Quite similar theories have recently begun [5] playing an important role in the theory of elementary particles [5].

$$\theta_i(x) \to \theta_i(x) + \nabla_i \varphi(x) , \tag{21}$$

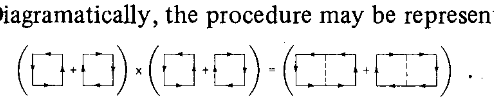
and has therefore no effect upon the integrations (20). Let us perform these first on two adjacent plaquettes. They are accompanied by two factors of the type (20)

$$\prod_{x,j} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_j(x)}{2\pi} [U_i(x-i)U_j(x)U_i^*(x-i+j)U_j^*(x-i) + \text{c.c.}][U_i(x)U_j(x+i)U_i^*(x+j)U_j^*(x) + \text{c.c.}] . \tag{22}$$

The common link x, j is associated with phases $U_j(x)U_j^*(x)$ and $U_j^*(x)U_j(x)$ whose integration leaves

$U_i(x-i)U_i(x)U_j(x+i)U_i^*(x+j)U_i^*(x-i+j)U_j^*(x-i) + \text{c.c.}$

Diagramatically, the procedure may be represented as follows



Each $U_i(x)$ amounts to a current flowing in from x along the link i, with complex conjugation reversing the direction. From this representation it is obvious that every closed surface with no handles can be covered twice by small current loops. Each plaquette contributes a factor β, and the total weight is $2\beta^A$ (A = area of bubble). By going through the same discussion as above we can convince ourselves that the result is unaffected by the presence of handles. Indeed, the situation is precisely the same as for surfaces of two colors, the role of the colors being played by the two orientations of the loops.

Thus we conclude that the gauge invariant theory

$$Z = \prod_{x,i} \int_{-\pi}^{\pi} \frac{\mathrm{d}\gamma_i(x)}{2\pi} \prod_{x,i} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_i(x)}{2\pi} \prod_{x,i} \{1 + [W_i^*(x)]^2\}$$

$$\times \prod_{x,i<j} \{1 + \beta[U_i(x)U_j(x+i)U_i^*(x+j)U_j^*(x) + \text{c.c.}]\, W_\square\} \tag{23}$$

represents an ensemble of self-avoiding random surfaces of two colors. We can bring this to yet another form by treating W as u in (8) and inserting dummy operators of the type

$$1 = \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\alpha}{2\pi\mathrm{i}} \exp[-\alpha(u - U)] :$$

$$Z = \prod_{x,i} \int_{-\infty}^{\infty} \mathrm{d}u_i(x)\, \mathrm{d}u_i^*(x) \prod_{x,i} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\alpha_i(x)\, \mathrm{d}\alpha_i^*(x)}{(2\pi\mathrm{i})^2} \prod_{x,i} \int_{-\infty}^{\infty} \mathrm{d}w_i(x)\delta(w_i(x)) \prod_{x,i} \left(1 + \frac{1}{2}\frac{\partial^2}{\partial w_i(x)^2}\right)$$

$$\times \exp\left(-\frac{1}{2}\sum_{x,i} [\alpha_i^*(x)u_i(x) + u_i^*(x)\alpha_i(x)]\right) \prod_{x,i} I_0(|\alpha_i(x)|)$$

$$\times \prod_{x,i<j} \{1 + \beta[u_i(x)u_j(x+i)u_i^*(x+j)u_j^*(x) + \text{c.c.}]\, w_\square\} . \tag{24}$$

We now replace $u_i \to u_i/w_i$, $\alpha_i \to \alpha_i w_i$ and perform the w_i integrals. This leaves only the first two terms of the Bessel function and we arrive at

$$Z = \prod_{x,i} \int_{-\infty}^{\infty} \mathrm{d}u_i(x)\, \mathrm{d}u_i^*(x) \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}\alpha_i(x)\, \mathrm{d}\alpha_i^*(x)}{(2\pi \mathrm{i})^2} \exp\left(-\frac{1}{2} \sum_{x,i} [\alpha_i^*(x) u_i(x) + \alpha_i(x) u_i^*(x)]\right)$$

$$\times \prod_{x,i} [1 + \tfrac{1}{2} |\alpha_i(x)|^2] \prod_{x,i<j} \{1 + \beta[u_i(x) u_j(x+i) u_i^*(x+j) u_j^*(x) + \mathrm{c.c.}]\} . \qquad (25)$$

This is an important result. It shows that the self-avoiding surfaces of two colors are a subset of graphs of the U(1) lattice gauge theory whose field theory [‡4] would be the same as (25), except with $(1 + \frac{1}{2}|\alpha_i|^2)$ replaced by $I_0(|\alpha_i|)$ $\{1 + \beta[u_i(x) \ldots u_j^+(x) + \mathrm{c.c.}]\}$ by the exponential $\exp\{\beta[u_i(x) \ldots u_j^+(x) + \mathrm{c.c.}]\}$. The U(1) lattice gauge theory is equivalent to a Coulomb gas of magnetic monopoles [7]. It has Debye screening for *all* β and no phase transition except at $\beta = \infty$. There is permanent confinement of electric charges. This leads us to conclude that two different colors prevent also self-avoiding random surfaces from having a phase transition. Their number does increase at a certain temperature, but without forming infinite bubble areas.

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[‡4] For a review see ref. [6].

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