TWO-LOOP EFFECTIVE ACTION OF O(N) SPIN MODELS IN $1/D$ EXPANSION

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We calculate the two-loop effective action of O(N) spin models on the lattice in a $1/D$ expansion to order $1/D^2$. The resulting free energy depends on $\beta = 1/T$ and the order parameter Φ. It matches the high and low temperature regimes and is quite reliable close to the phase transition where it has a simple Landau expansion.

While recent progress in understanding the numerical properties of lattice theories has been remarkable, due to Monte Carlo computer calculations, the development of analytic methods leaves much to be desired. In order to compare Monte Carlo data with theory, the main tools are still strong-coupling expansions [1] for high temperatures and mean field approximations plus loop corrections for low temperatures [1–4].

The separate treatment of the two temperature regimes is rather unpleasant for two reasons:

First, the loop corrections to the mean field solution have unphysical cusps at the mean field transition point β_c^{MF} (see fig. 2).

Second, the order parameter is in strong disagreement with what it should be. In mean field approximation, it vanishes at β_c^{MF} and this usually lies considerably below its proper value β_c. Loop corrections are extremely inefficient in correcting this discrepancy. That is why, for many transitions of many-body systems, the agreement of calculated and experimental order parameters is very bad. It is therefore desirable to find an approach in which β_c^{MF} does not play such a special role but in which all corrections are oriented from the beginning on the physical transition point β_c. Such an approach is available, in principle, via the so-called "effective energy" (or "effective action" in the case of quantum systems). It depends explicitly on the order parameter Φ of the system. Minimization in Φ gives the free energy $-\beta f$. Phase transitions show up by Φ becoming non-zero. The effective energy can be used to study a system equally well on both sides of the critical point.

The associated theory has been known for a long time [5] [‡1]. The effective energy is calculated by forming the sum of all one-particle irreducible diagrams involving the Green's function of fluctuations around a fixed background field Φ. At first, it may look surprising that this formalism has not yet been applied to lattice theories. There is, however, a simple reason: The calculation of more than one-loop corrections in terms of lattice Green's functions becomes very difficult. The one-loop correction itself, on the other hand, is an unsatisfactory approximation.

With the need of calculating at least two-loop diagrams we are forced to some approximation. Such an approximation is provided by a systematic $1/D$ expansion of all loop diagrams. This has the additional advantage that the parameter $1/D$ efficiently organizes the diagrams: The leading order of an M-loop diagram is at least $1/D^M$.

[‡1] For a recent review see ref. [6], we shall use the same notation as there.

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In the present note, we shall limit ourselves to corrections of order $1/D^2$. This removes all diagrams with more than two loops and leads to simple algebraic expressions.

In order to be specific we apply our ideas directly to O(N) spin models. The results are amazingly good for the free energy as well as for the critical temperature for $D \geqslant 3$ (see fig. 1 and table 1). The order parameter appears to be reliable for $D \geqslant 4$ (see fig. 3).

The partition function of the O(N) model in D space dimensions is

$$Z = \exp(-\beta L f) \equiv \prod_{x,a} \int \mathrm{d}\hat{S}_a \det\left(\beta \sum_x \sum_{i=1}^{D} S_a(x) S_a(x+i)\right), \tag{1}$$

where x runs over the L sites of a simple cubic lattice, i through the D oriented links, and $a = 1, \ldots, N$ is the internal O(N) index. The measure of integration covers the surface of the N-dimensional unit sphere $S_a^2 = 1$. By introducing an auxiliary field $\varphi_a(x)$, the unit vectors S_a can be turned into an N-component field fluctuating from $-\infty$ to ∞ with a partition function [‡2]

$$Z = \prod_{x,a} \int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi_a(x)}{(4\pi\beta D)^{1/2}} \exp\{-\beta E[\varphi]\} = \prod_{x,a} \int_{-\infty}^{\infty} \frac{\mathrm{d}\varphi_a(x)}{(4\pi\beta D)^{1/2}} \exp\left(\sum_x \left[-(4\beta D)^{-1}\varphi_a^2 + W(\rho)\right]\right). \tag{2}$$

Here $W(\rho)$ is given in terms of associated Bessel functions as

$$W(\rho) = \log\left[\Gamma(N/2)(\rho/2)^{1-N/2} I_{N/2-1}(\rho)\right],$$

where ρ is short for $\rho \equiv [\hat{\varphi}| = (\hat{\varphi}_a\hat{\varphi}_a)^{1/2}$ and $\hat{\varphi}_a$ is defined as $\hat{\varphi}_a(x) = \sum_y \mathscr{D}^{1/2}(x,y)\varphi_a(y)$. The operator $\mathscr{D}(x,y) = (1 + \overline{\nabla}\nabla/2D)(x,y)$ is $1/2D$ times the hopping operator

$$\hat{H}(x,y) \equiv 2D + \overline{\nabla}\nabla = \sum_i \nabla_{\pm i} = \sum_i (\delta_{x,y+i} + \delta_{x,y-i}).$$

According to the rules of the game, the lowest approximation to the effective energy $\Gamma[\Phi]$ is simply the energy $E[\Phi]$ calculated in the background field $\varphi = \Phi$:

$$-\beta\Gamma[\Phi] = -\beta E[\Phi] = \sum_x \left[-(4\beta D)^{-1}\Phi_a(x)^2 + W(|\hat{\Phi}(x)|)\right]. \tag{3}$$

In order to find the loop corrections we have to form the propagator in the background field

$$\begin{aligned} G_\Phi^{-1}(x,y)_{ab} &= |\beta\delta^2 E(\varphi_a)/\delta\varphi_a(x)\delta\varphi_b(y)|_{\varphi_a=\Phi_a} \\ &\equiv \frac{1}{2\beta D}\delta_{ab}\delta_{xy} - \sum_z \mathscr{D}^{1/2}(x,z)\sigma_{ab}(z)\mathscr{D}^{1/2}(z,y) \\ &\equiv \frac{1}{2\beta D}\delta_{ab}\delta_{x,y} - \sum_z \mathscr{D}^{1/2}(x,z)\left[\sigma_T(z)P_{ab}^{T}(z) + \sigma_L(z)P_{ab}^{L}(z)\right]\mathscr{D}^{1/2}(z,y), \end{aligned} \tag{4}$$

with

$$\begin{aligned} &\sigma_T(z) = W'(\rho(z))/\rho(z), \quad P_{ab}^{T}(z) \equiv \left(\delta_{ab} - \hat{\Phi}_a(z)\hat{\Phi}_b(z)/\rho(z)^2\right), \\ &\sigma_L(z) = W''(\rho(z)), \quad P_{ab}^{L}(z) \equiv \hat{\Phi}_a(z)\hat{\Phi}_b(z)/\rho(z)^2, \end{aligned} \tag{5}$$

[‡2] This method is well-known. In our notation it appears in detail in ref. [7], or ref. [8].

and calculate $-\beta\Gamma^{\text{loops}}[\Phi]$ as the sum over all one-particle irreducible vacuum diagrams involving this propagator and the vertices $\delta^n W(|\hat{\varphi}|)/\delta\varphi_{a_1}(x_1)\ldots\delta\varphi_{a_n}(x_n)|_{\varphi_a=\Phi_a}$.

For the purpose of obtaining only the free energy and the order parameter of the system we can confine our attention to the effective energy per site at constant order parameter, $L^{-1}\Gamma[\boldsymbol{\Phi}]|_{\Phi=\text{const}}$ called the "*effective potential*" $V(\boldsymbol{\Phi})$. This simplifies the calculation. At a constant Φ, the operation $\mathscr{D}^{1/2}$ can be moved to the right in (4) and G_Φ^{-1} takes the form

$$G_\Phi^{-1}(x,y)|_{\boldsymbol{\Phi}=\text{const}} = G_\Phi^{-1}(x-y)|_{\boldsymbol{\Phi}=\text{const}} = (1/2\beta D)\,\delta_{ab}\delta_{x,y} - (\sigma_{ab}/2D)\cdot\hat{H}(x,y).$$

We shall expand all diagrams in powers of $1/D$ by keeping $b \equiv \beta D \equiv$ constant and $\boldsymbol{\Phi}^2 = \rho^2 =$ const. Because of the derivative coupling in $W(|\hat{\varphi}|)$, the differentiation with respect to $\varphi_a(x)$, when forming vertices, produces factors $\mathscr{D}^{1/2}$ which may be absorbed into the propagator G_Φ, forming

$$\hat{G}_\Phi(x,y)_{ab}\,|_{\boldsymbol{\Phi}=\text{const}} = (b/D)\hat{H}\left[1-(b/D)\sigma\hat{H}\right]^{-1} = \frac{b}{D}\sum_{n=0}^{\infty}(b\sigma/D)^n_{ab}\hat{H}^{n+1}(x,y)$$

$$= \frac{b}{D}\sum_{n=0}^{\infty}\left\{\left[(b/D)\sigma^{\mathrm{T}}\right]^n P^{\mathrm{T}}_{ab} + \left[(b/D)\sigma^{\mathrm{L}}\right]^n P^{\mathrm{L}}_{ab}\right\}\hat{H}^{n+1}(x,y). \tag{6}$$

The factor $1/D$ in front shows that the M-loop diagrams produce corrections at least of the order $1/D^M$, such that, with our restriction to $1/D^2$ corrections, we can indeed ignore all but the one- and two-loop diagrams.

The one-loop diagram is

$$-\beta V^{\text{1-loop}} = -(1/2L)\mathrm{Tr}\,\log\left(2bG_\Phi^{-1}\right) = \frac{1}{2L}\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}\left((b/D)\sigma\hat{H}\right)^n$$

$$= \tfrac{1}{4}\mathrm{tr}_N\left((b/D)\sigma\right)^2\cdot 2D + \tfrac{1}{8}\mathrm{tr}_N\left((b/D)\sigma\right)^4\cdot 12D^2 + \mathrm{O}\left(\frac{1}{D^3}\right), \tag{7}$$

where we have used $\mathrm{tr}_x\hat{H}^{2M+1} = 0$, $L^{-1}\,\mathrm{tr}_x\hat{H}^2 = 2D$, $L^{-1}\,\mathrm{tr}_x\hat{H}^4 = 6D(2D-1),\ldots$.

Performing the remaining traces in the $\mathrm{O}(N)$ space gives

$$-\beta V^{\text{1-loop}} = (b^2/2D)\left[(N-1)\sigma_{\mathrm{T}}^2 - \sigma_{\mathrm{L}}^2\right] + (3b^4/2D^2)\left[(N-1)\sigma_{\mathrm{T}}^4 + \sigma_{\mathrm{L}}^4\right], \tag{8}$$

where we have used $\mathrm{tr}_N\,P^{\mathrm{T}} = N-1$, $\mathrm{tr}_N\,P^{\mathrm{L}} = 1$. There are two two-loop diagrams, one gives

$$-\beta V^{\text{2-loop}}_{\circ\circ} = (1/4!)\Big[W_1(\rho)\big(\delta_{a_1a_2}\delta_{a_3a_4} + 2\text{ perm.}\big) + W_2(\rho)\big(\Phi_{a_1}\Phi_{a_2}\delta_{a_3a_4} + 5\text{ perm.}\big) + W_3(\rho)\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\Phi_{a_4}\Big]$$
$$\times\left[\hat{G}_\Phi(\mathbf{0})_{a_1a_2}\hat{G}_\Phi(\mathbf{0})_{a_3a_4} + 2\text{ perm.}\right], \tag{9}$$

with

$$W_1(\rho) = \rho^{-1}\mathrm{d}\sigma_{\mathrm{T}}/\mathrm{d}\rho,\quad W_2(\rho) = \rho^{-1}\mathrm{d}W_1(\rho)/\mathrm{d}\rho,\quad W_3(\rho) = \rho^{-1}\mathrm{d}W_2(\rho)/\mathrm{d}\rho. \tag{10}$$

This can be worked out to

$$-\beta V^{\text{2-loop}}_{\circ\circ} = (1/4!)\Big\{W_1\left[3(\mathrm{tr}_N\hat{G}(\mathbf{0}))^2 + 6\mathrm{tr}_N\hat{G}(\mathbf{0})^2\right] + W_2\left[6\mathrm{tr}_N G^1(\mathbf{0})\cdot\overline{\hat{G}(\mathbf{0})} + 12\overline{\hat{G}(\mathbf{0})^2}\right] + W_3 3\left(\overline{\hat{G}(\mathbf{0})}\right)^2\Big\}, \tag{11}$$

where

$$\overline{\hat{G}_\Phi(x)} = \Phi_a \hat{G}_\Phi(x)_{ab}\Phi_b, \quad \overline{\hat{G}_\Phi^2(x)} = \Phi_a \hat{G}_\Phi(x)_{ab}\hat{G}_\Phi(x)_{bc}\Phi_c, \quad \dots .$$

From (6) we see that the relevant part of the 1/D expansion of $\hat{G}_\Phi$ is

$$\hat{G}_\Phi(x)_{ab} = (b^2/D^2)(\sigma_T P_{ab}^T + \sigma_L P_{ab}^L)\hat{H}^2(x, \mathbf{0}) + O(1/D^2)$$

such that

$$\overline{\hat{G}_\Phi(\mathbf{0})} = 2(b^2/D)\sigma^L\rho^2 + O(1/D^2), \quad \overline{\hat{G}_\Phi(\mathbf{0})^2} = 4(b^4/D^2)\sigma^{L^2}\rho^2 + O(1/D^3). \tag{12}$$

Inserting this into (11) leads to

$$-\beta V_{\circ\circ}^{\text{2-loop}} = (b^4/2D^2)\left\{W_1\left[(W'/\rho)^2(N^2-1) + 3W''^2 + 2(W'/\rho)W''(N-1)\right]\right.$$
$$\left. + W_2\left[2W'W''\rho(N-1) + 6W''^2\rho^2\right] + W_3 W''^2\rho^4\right\} + O(1/D^3). \tag{13}$$

The other two-loop diagram gives

$$-\beta V_{\ominus}^{\text{2-loop}} = \tfrac{1}{2}(3!)^{-2}\sum_x\left[W_1\left(S_{a_1a_2}\Phi_{a_3} + \delta_{a_1a_3}\Phi_{a_2} + \delta_{a_2a_3}\Phi_{a_1}\right) + W_2\Phi_{a_1}\Phi_{a_2}\Phi_{a_3}\right]$$
$$\times\left[W_1\left(\delta_{a_4a_5}\Phi_{a_6} + \delta_{a_4a_6}\Phi_{a_5} + \delta_{a_5a_6}\Phi_{a_4}\right) + W_2\Phi_{a_4}\Phi_{a_5}\Phi_{a_6}\right]$$
$$\times\left[\hat{G}_\Phi(x)_{a_1a_4}\hat{G}_\Phi(x)_{a_2a_5}\hat{G}_\Phi(x)_{a_3a_6} + 5\text{ perm.}\right]$$
$$= \frac{1}{2}\frac{1}{(3!)^2}\sum_x\left\{W_1^2\left[18\,\mathrm{tr}_N\left(G_\Phi^1(x)\right)^2\overline{\hat{G}_\Phi(x)} + 36\overline{\hat{G}_\Phi(x)^3}\right]\right.$$
$$\left. + 36W_1W_2\overline{\hat{G}_\Phi(x)^2}\,\overline{G_\Phi(x)} + 6W_2^2\left(\overline{G_\Phi^1(x)}\right)^3\right\}. \tag{14}$$

Using (6) we see that the leading $1/D$ terms are

$$\mathrm{tr}_N\hat{G}_\Phi(x)^2 = N(b^2/D^2)\hat{H}(x,\mathbf{0}), \quad \overline{\hat{G}_\Phi(x)} = (b/D)\rho^2\hat{H}(x,\mathbf{0}), \quad \overline{\hat{G}_\Phi(x)^3} = (b^3/D^3)\rho^4\hat{H}(x,\mathbf{0}),$$
$$\overline{\hat{G}_\Phi(x)^2}\cdot\overline{\hat{G}_\Phi(x)} = (b^3/D^3)\rho^4\hat{H}(x,\mathbf{0}), \quad \left(\overline{\hat{G}_\Phi(x)}\right)^3 = (b^3/D^3)\rho^6\hat{H}(x,y), \tag{15}$$

where we have used the fact that the matrix elements of $\hat{H}(x,y)$ are only zero or 1 such that they satisfy the identity $(\hat{H}(x,y))^M = \hat{H}(x,y)$. The sum $\Sigma_x\hat{H}(x,\mathbf{0})$ is equal to $2D$ and (14) becomes

$$-\beta V_{\ominus} = (b^3/6D^2)\left[3W_1^2(N+2)\rho^2 + 6W_1W_2\rho^4 + W_2^2\rho^6\right] + O(1/D^3). \tag{16}$$

Adding up the three contributions (8), (13), (16) gives the effective potential as follows

$$-\beta V(\boldsymbol{\Phi}) = -(1/4b)\boldsymbol{\Phi}^2 + W(|\boldsymbol{\Phi}|) + (b^2/2D)\left[(N-1)\sigma_T^2 + W''^2\right] + (b^3/6D^2)\left[3(N-1)\sigma_T'^2 + W'''^2\right]$$
$$+ (b^4/2D^2)\left\{(N-1)\left\{3\sigma_T^4 - (4/\rho)\sigma_T\sigma_T'W'' - \left[(N+1)/\rho\right]\sigma_T^2\sigma_T' + (2/\rho)\sigma_T W''W'''\right\}\right.$$
$$\left. + 3W''^4 + W^{(4)}W''^2\right\}. \tag{17}$$

For small b, the minimum lies at $\Phi = 0$ and is simply

$$-\beta V(\mathbf{0})/N = (b/N)^2/2D + (b/N)^4/2D^2 + O(1/D^3). \tag{18}$$

This is a universal curve in b/N which, for $N=1$, agrees with the $1/D$ approximation to the usual high temperature expansion by Fisher and Gaunt [9].

For large b, the minimum is given by the gap equation:

$$-\Phi_a^{-1}\partial\beta V(\Phi)/\partial\Phi_a = 0 = g_{-1}b^{-1} + g_0 + g_2 b^2 + g_3 b^3 + g_4 b^4, \tag{19}$$

where

$$\begin{aligned}
&g_{-1} = -1/2, \quad g_0 = \sigma_T = W'(\rho)/\rho, \quad g_2 = (1/D\rho)\left[(N-1)\sigma_T\sigma_T' + W''W'''\right],\\
&g_3 = (1/3D^2\rho)\left[W'''W^{(4)} + (3/\rho)(N-1)\sigma_T' W''' - (6/\rho)(N-1)\sigma_T'^2\right],\\
&g_4 = (1/2D^2\rho)\left[12(N-1)\sigma_T'\sigma_T^3 + 12W''^3W''' + \tilde{g}_4\right],\\
&\tilde{g}_4 = (N-1)\{(2/\rho^3)(N+11)\sigma_T W''^2 - \left[(7N+23)/\rho^3\right]\sigma_T^2 W'' - (12/\rho^2)\sigma_T W''W'''\\
&\qquad + \left[(N+5)/\rho^2\right]\sigma_T^2 W'' + \left[5(N+1)/\rho^3\right]\sigma_T^3 - (4/\rho^3)W''^3 + (2/\rho^2)W''^2W''' + (2/\rho)\sigma_T W'''^2\\
&\qquad + (2/\rho)\sigma_T W''W^{(4)}\} + W''^2W^{(5)} + 2W''W'''W^{(4)}.
\end{aligned} \tag{20}$$

The critical value for which Φ turns non-zero is found by solving this equation at $\Phi=0$, where

$$g_{-1} = -1/2, \quad g_0 = 1/N, \quad g_2 = -2/DN^3, \quad g_3 = 4/D^2N^4(N+2), \quad g_4 = 0. \tag{21}$$

The solutions b_c/N are shown in table 1. They exist for $N=1$ and $N=2$ starting from $D=3$, and for $N=3$ and higher starting from $D=4$. The second entry in table 1 shows our expression in all reliable powers of $1/D$, i.e.,

$$b_c/N = \tfrac{1}{2}\left[1 - 1/2D - (N+1)/2D^2(N+2)\right]^{-1} + O(1/D^3), \tag{22}$$

which agrees with ref. [9] up to this order, and with refs. [1,2]. Notice that our solutions given by eqs. (19), (21) are more accurate than those. In fact, they are very close to the values of Gerber and Fisher [9], who extended eq. (22) up to order $1/D^5$,

$$\begin{aligned}
\frac{b_c}{N} = \frac{1}{2}\Bigg[&1 - \frac{1}{2D} - \frac{N+1}{2D^2(N+2)} - \frac{1}{8D^3}\left(3 + \frac{4N}{N+2}\right)\\
&- \frac{1}{16D^4}\left(16 + \frac{(21N+32)N}{(N+2)^2} - \frac{2N^2}{(N+2)(N+4)}\right)\\
&- \frac{1}{32D^5}\left(102 + \frac{(129N^2+422N+340)N}{(N+2)^3} - \frac{16N^2}{(N+2)(N+4)}\right)\Bigg]^{-1} + O(D^{-6}),
\end{aligned} \tag{23}$$

which are given in the third entry in table 1. Thus we see that an expansion of the effective energy up to the second explicit order in $1/D$ gives, after minimization in Φ, a critical temperature which is much more accurate than a direct systematic expansion of b_c^{-1} up to $1/D^2$.

The free energy is plotted for $D=3, 4, 5$ and $N=1, 2, 3, \infty$ in fig. 1. In fig. 2 the free energy for $D=3$, $N=1, 2$ is compared with those of the usual strong coupling expansion and of the mean field plus one-loop correction. The order parameter ρ is plotted in fig. 3. For $D=3$ it has an unphysical non-monotonous transition. Obviously $D=3$ is too small a value to apply our $1/D$ expansion. For $D \geqslant 4$, however, the effective energy leads to a bona fide second order transition and the order parameters should be reliable.

The poor quality of the $D=3$ solution is not specific to our approach but is a consequence of the asymptotic nature of the $1/D$ expansion. This can be seen directly [9] when using the formula for b_c/N expanded in $1/D$ for a spherical model ($N=\infty$). It is known up to $O(D^{-13})$ [9], and for $D=3$, gives an absurd negative value, while for $D=4$ it gives $b_c/N=0.6081$ as compared to the exact value 0.6197.

One possibility [10] of curing the $D=3$ disease is to replace the $1/D$ expansion with the hopping operator expansion, which is convergent for $D>2$. In it, the lattice Green's function is expanded in powers of hopping operator $\hat{H}$ (see eqs. (6), (7));

$$(-\overline{\nabla}\nabla+m^2)^{-1}(x,x)=1/s+2D/s^3+6D(2D-1)/s^5+O(\mathrm{Tr}\,\hat{H}^6/s^7), \tag{24}$$

where $s=2D+m^2$ and $m^2=O(D)$. In the previous $1/D$ expansion the s^{-5} term was $12D^2/s^5$.

If we do the corresponding change in our calculation of the effective potential (17), it gives rise to an extra factor $(1-1/2D)$ multiplying to the terms $b^4(2D^2)^{-1}[3(N-1)\sigma_T^4+3(W'')^4]$ on the right-hand side. Hence $-\beta V(0)/N$ of eq. (18) aquires the additional term $-3(b/N)^4/4D^3$. This factor $(1-1/2D)$ also influences the first two terms of g_4 in eq. (20), and so g_4 in eq. (21) reads $g_4=6/D^3N^5$.

It is gratifying to see that this simple replacement does remove the unphysical backtracking in the $D=3$ order parameter. The "noses" in the $D=3$ curves in fig. 3 ($N=1,2$) are removed in favor of a monotonously decreasing order parameter which has a proper second order phase transition. There is, however, a disadvantage: The intercept of the order parameter is pushed to the left. Therefore b_c/N comes out somewhat too low: $b_c/N=0.6138$ for $(D,N)=(3,1)$ and 0.6242 for (3, 2).

Finally we point out that in the neighborhood of the transition we can extract from $V(\Phi)$ a simple Landau expansion,

$$-\beta V(\Phi)/N=\beta V(0)/N-\tfrac{1}{2}m^2\rho^2-(g/4!)\rho^4+O(\rho^6), \tag{25}$$

Table 1
Our critical values b_c/N ($\equiv\beta_c D/N$) (first row, taken from eqs. (19), (21)) as compared with the different approximations, eq. (22) (second row), eq. (23) (third row), and with the exact results (in the brackets).

N	D = 3	4	5	6
1	0.6724	0.5948	0.5676	0.5529
	0.6279	0.5854	0.5639	0.5510
	0.6737	0.5979	0.5689	0.5535
	(0.665)	(0.600)	(0.570)	(0.554)
2	0.7125	0.5998	0.5698	0.5541
	0.6316	0.5872	0.5650	0.5517
	0.6852	0.6016	0.5708	0.5546
	(0.68)			
3	–	0.6030	0.5711	0.5548
	0.6338	0.5882	0.5656	0.5521
	0.6725	0.6040	0.5719	0.5553
	(0.692)			
4	–	0.6053	0.5720	0.5553
	0.6353	0.5890	0.5660	0.5524
	0.6975	0.6055	0.5727	0.5557
∞	–	0.6180	0.5767	0.5579
	0.6429	0.5926	0.5682	0.5538
	0.7239	0.6135	0.5765	0.5579
	(0.7582)	(0.6197)	(0.5782)	(0.558)

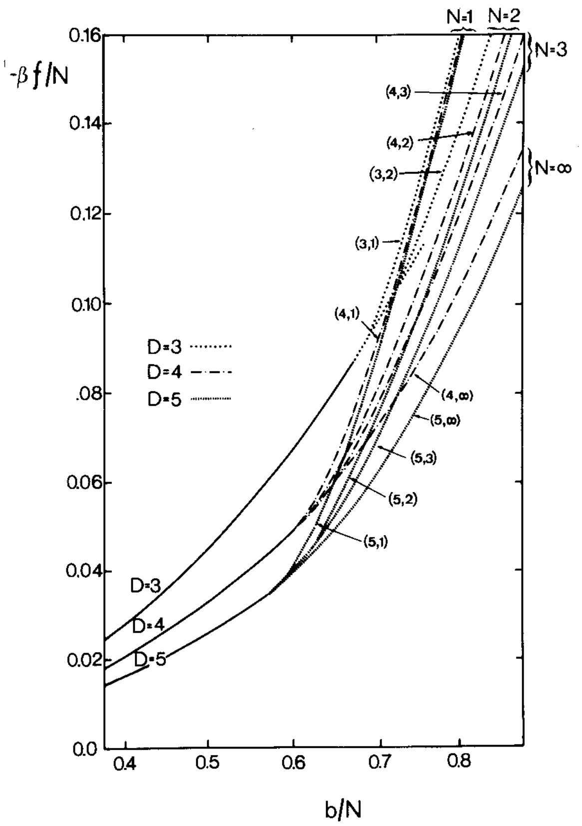

Fig. 1. The free energy of O(N) spin models as a function of β. Each curve is labelled by (D, N). For $\beta < \beta_c$, the curves are independent of N. For $\beta > \beta_c$, $-\beta f/N$ grows slower for higher N. Incidentally, the curves in the limit of $N = \infty$ with b/N and ρ/N kept fixed approach the free energy of the spherical model (not shown in the figure).

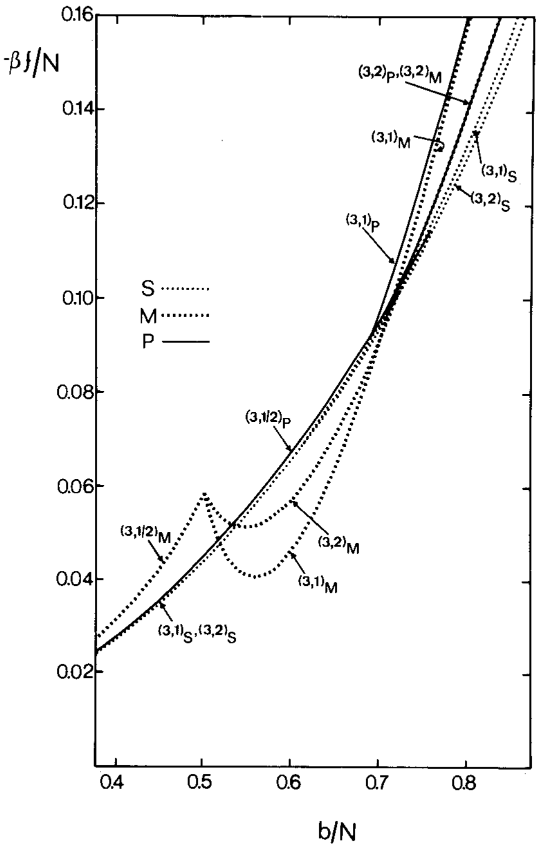

Fig. 2. The $D = 3$ curves for $-\beta f/N$, labelled by $(3, N)_P$, are compatible with the usual strong coupling expansion up to β^{12} [labelled by $(3, N)_S$] and the mean field plus one-loop correction [labelled by $(3, N)_M$].

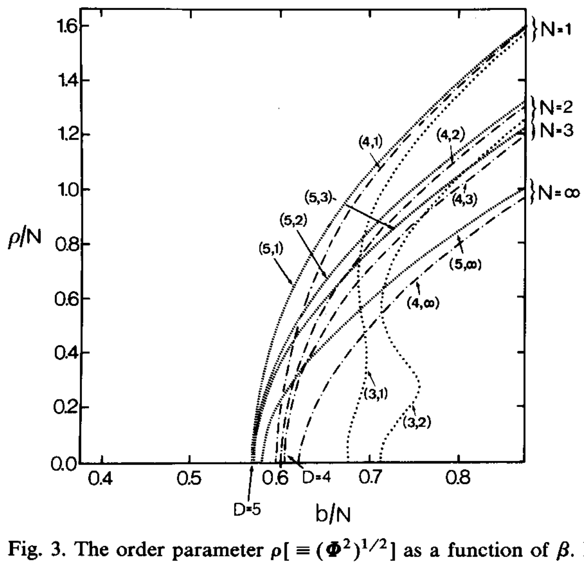

Fig. 3. The order parameter $\rho[\equiv \langle\Phi^2\rangle^{1/2}]$ as a function of β. Each curve is labelled by (D, N). Also here, $N = \infty$ curves approach those of the spherical model (not shown here).

where

$$m^2 = -\gamma(b - b_c),$$

$$\gamma = \frac{1}{2b_c^2} - \frac{4b_c}{DN^3} + \frac{12b_c^2}{D^2N^4(N+2)} + \frac{24b_c^3}{D^3N^5},$$

$$\frac{g}{4!} = \frac{1}{4N^2(N+2)} - \frac{b_c^2}{2D}\frac{5N+16}{N^4(N+2)^2} + \frac{b_c^3}{D^2}\frac{16}{N^5(N+2)^2} + \frac{b_c^4}{D^2}\frac{7N^2+30N+44}{N^6(N+2)^3} + \frac{b_c^4}{D^3}\frac{3(7N+32)}{2N^6(N+2)^2}. \quad (26)$$

The $O(D^{-3})$ terms in γ and g above represent the additional contributions of the hopping operator expansion, and will disappear if the simple $1/D$ expansion is employed.

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