Transition Form Factors in the H Atom

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Transition form factors between arbitrary excited states of the H atom have been evaluated in closed highly symmetric form within the framework of the noncompact dynamical group $O(4,2)$.

The purpose of this paper is to present the exact form of the charge form factors in the H atom for transitions between any two excited states. The motivations for this work are: (1) These arbitrary transition form factors have, to our knowledge, not been given in the literature; (2) to show the power of the new simple algebraic methods using the representations of noncompact groups; and (3) for possible adaptation of the results to the dynamics of strongly interacting particles.

The form factors in question, denoted by $\mathcal{F}_{\ell m \rightarrow \ell' m'} \left( q^2 \right)$, are the vertex amplitudes shown in Fig. 1 as a function of the momentum transfer $q^2 = \ell^2 + \ell'^2 - 2k^2 \cos \theta$. They govern the inelastic scattering of the H atom by other charged particles or atoms if single photon exchange is dominant, and are measured by such scattering experiments. The transition form factors from the ground state $|100\rangle$ to an arbitrary state $|nlm\rangle$ were first calculated by Massey and Mohr by Schrödinger theory. To our knowledge, these are the only form factors known explicitly. We present here an evaluation of arbitrary form factors solely within the conformal group $O(4,2)$. The method does not make any reference to spatial wave functions.

It has been shown recently that the dipole transitions in the H atom can be described in a simple manner by using the dynamical group $O(4,2)$, the conformal group. Nambu has investigated relativistic infinite-component wave equations for H-like systems and has indicated the calculation of form factors. Later, further properties of the H atom within the group $O(4,2)$ were investigated by Fronsdal and the present authors. Fronsdal gave also the form of the Galilei boost transformations on the group $O(4,2)$, and evaluated the form factor of the ground state.

We summarize briefly the $O(4,2)$ description of the H atom. Let $L_{ab} = -L_{ba}; a, b = 1, 2, \ldots, 6$, be the 16 generators of $O(4,2)$. The subgroup $O(4)$ generated by $L_{ab}; a, b = 1, 2, 3, 4$, describes the degeneracy of the states of a given energy; the subgroup $O(4,1)$—dynamical group in the rest frame—describes all bound states $|nlm\rangle$, and, finally, the remaining generators $L_{56}$ are associated with dipole transitions, and $L_{56}$ with the quantum number $n$.

The vector form factors are given by

$$\mathcal{F}_{\ell m \rightarrow \ell' m'} = \langle n' l' m' | n l m, k \rangle,$$

where $|nlm,k\rangle$ is the Galilei-boosted state of momentum $k$, i.e., $e^{ikx} \psi_{nlm}(x)$. The generators $M_i$ of the Galilei transformations $\exp(i k \cdot M)$ are given by

$$M_i = (L_{15} - L_{14}),$$

provided we introduce the new states

$$|\tilde{n}lm\rangle = -e^{-i\alpha L_{15}} |nlm\rangle,$$

and a current operator $\Gamma_n = (L_{56} - L_{14}, L_{16})$. Then the charge form factors can be written as (for a booster in the 3 direction)

$$\mathcal{F}_{\ell m \rightarrow \ell' m'} = \langle \tilde{n}'l'm' | \Gamma_0 e^{-i k M} | n l m \rangle.$$

Using Eqs. (2) and (3) and the commutation relations of $O(4,2)$ we can bring Eq. (4) to the form

$$\mathcal{F}_{\ell m \rightarrow \ell' m'} = \frac{1}{n} \langle n' l' m' | \Gamma_0 | n l m \rangle \times \langle n' l' m' | e^{-i\alpha n \cdot L e^{-i k (L_{15} - L_{14})}} | n l m \rangle,$$

where $R$ is the rotation matrix $e^{-i\alpha n \cdot L}$.
where
\[ \tanh \frac{1}{2} \theta_{n', n} = \frac{n - n'}{n + n'}; \quad \theta_{n', n} = \ln \left( \frac{n}{n'} \right). \]

In the evaluation of Eq. (5) we notice that \( L_{44} = K_3, \) \( L_{33} = -K_2, \) \( L_{43} = K_1 \) generate an \( O(2,1) \) subalgebra (transition group) of \( K. \) The second matrix element in Eq. (5) is that of a finite transformation of \( O(2,1) \) that we express in terms of the Euler operators \( \varphi, \psi, \chi: \)
\[ e^{-i\varphi K_3} e^{-i\psi K_2} e^{-i\chi K_1} = e^{-i\varphi} e^{-i\psi K_2} e^{-i\chi K_1} e^{-i\psi K_2} e^{-i\chi K_1}, \]
and obtain
\[ \sinh \frac{1}{2} \chi = \frac{1}{(4n' n)^{1/2}} \left[ (n - n')^2 + i n^2 n^2 \right]^{1/2}, \]
\[ \sin \varphi = \frac{(n^2 - n'^2) + i n^2 n^2}{\left[ (n - n')^2 + i n^2 n^2 \right]^{1/2}}. \]

\( \sin \psi \) is exactly like \( \sin \varphi \) with \( n \) and \( n' \) interchanged. Next we express the operators \( K \) in terms of the generators corresponding to parabolic coordinates
\[ N^i_+, N^i_-, \quad N^i = \pm N^i_+, \ i = 1, 2 \]
where
\[ N^i_{n_1 n_2 n_3} = [n_1 + (m + 1)/2] n_1 n_2 n_3; \]
\[ N^i_{\pm} = n_1 \pm 1, n_2, n_3 \]
as follows:
\[ K_3 = N^3_1 - N^3_2; \quad K_1 = \frac{1}{2i} (N^1_+ + N^1_- - N^1 - N^-_2); \]
\[ K_2 = -\frac{i}{2} (N^2_+ + N^2_- + N^-_1 - N^-_2). \]

It is therefore easy to evaluate the matrix elements of Eq. (6) in parabolic coordinates:
\[ \langle n'_1 n'_2 n'_3 | e^{-i\varphi K_3} e^{-i\psi K_2} e^{-i\chi K_1} | n_1 n_2 n_3 \rangle \]
\[ = e^{-i\left( (n'_1 - n'_2) \varphi + (n'_2 - n'_3) \psi + (n'_3 - n_1) \chi \right)} \langle n'_1 n'_2 n'_3 | n_1 n_2 n_3 \rangle \]
\[ \times e^{-i\left( (n'_1 - n_2) (n'_3 + n'_1) - (n_2 + n'_3) (n_1 + n'_3) \right)} \langle n_1 n_2 n_3 | n_1 n_2 n_3 \rangle, \]
where the last matrix element is the product of two finite \( O(2,1) \) transformations:
\[ G_{n'_1 n'_2 n'_3 n_1 n_2 n_3} = V_{n'_1 + (m + 1)/2, n'_2 + (m + 1)/2}^{(m + 1)/2} (x) \times V_{n_1 n_2}^{(m + 1)/2} (x). \]

The \( V \) function for \( n' > n_1 \) is given by
\[ V_{n'_1 + (m + 1)/2, n'_2 + (m + 1)/2}^{(m + 1)/2} (x) = \theta_{n'_1, n_1} \left( \cosh \frac{x}{2} \right)^{-(n'_1 + n'_2 - n_1 - n_2)} \exp \left( -i \sinh \frac{x}{2} x' \right), \]
\[ \theta_{n'_1, n_1} = \frac{1}{(n'_1 + n_1)} \left[ n'_1 (n'_1 + m) + 1 \right]^{1/2} \]
\[ \left[ n_1 (n_1 + m) + 1 \right]^{1/2}, \]
for \( n'_1 < n_1, \) use \( V_{n'_1 + (m + 1)/2, n'_2 + (m + 1)/2}^{(m + 1)/2} (x) \), which occurs universally in all form-factor calculations (scalar or vector) and in the approximate evaluation of scattering amplitudes.\(^6\)

Similarly, the first matrix element in Eq. (5) is easily calculated in parabolic coordinates, because
\[ L_{44} = \frac{1}{2} (N^1_+ + N^1_- + N^2_+ + N^2_-); \quad L_{45} = N. \]

It remains then to change the basis \( |n_1 n_2 n_3 \rangle \) into \( |nlm \rangle. \) Because this change of basis is connected with the reduction of \( O(4) \) into \( O(3) \times O(3), \) we have immediately in terms of the \( 3j \) symbols
\[ \langle nlm | n_1 n_2 n_3 \rangle = (-1)^m (2l + 1)^{1/2} \]
\[ \times \left( \begin{array}{ccc} n - 1 & 1 & l \\ m - n_1 + n_2 & 0 & 0 \\ m + n_1 - n_2 & 0 & 0 \end{array} \right). \]

Consequently, collecting all the terms the final result is
\[ \mathcal{F}_{n', n}^{lm} = (2l + 1)^{1/2} \sum_{n} \left( \begin{array}{ccc} n - 1 & 1 & l \\ m - n_1 + n_2 & 0 & 0 \\ m + n_1 - n_2 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} n - 1 & 1 & l \\ m - n_1 + n_2 & 0 & 0 \\ m + n_1 - n_2 & 0 & 0 \end{array} \right) \times \left[ n' e^{-i\left( (n'_1 - n'_2) \varphi + (n'_2 - n'_3) \psi + (n'_3 - n_1) \chi \right)} \right] \left[ (n'_1 + m - 1) \right]^{1/2} \left[ (n'_2 + m - 1) \right]^{1/2} \left[ (n'_3 + m - 1) \right]^{1/2} \exp \left( -i \sinh \frac{x}{2} x' \right), \]
\[ \text{for } (1)^{n - n} = -1 \]
\[ \text{for } (1)^{n - n} = +1. \]

\(^6\) A. O. Barut and H. Kleinert, Phys. Rev. 156, 1546 (1967).
\(^7\) A. O. Barut and H. Kleinert, Phys. Rev. Letters 18, 754 (1967).
The form factor $\bar{\sigma}$ has a singularity where the $\cosh^{1/2}X$ term in Eq. (11) vanishes. From Eq. (7), this occurs at

$$t = t_1 = -\frac{(n'+n)^3}{n^2 n'^2} = 2(\sqrt{B'} + \sqrt{B})^2$$

(17)

and coincides exactly with the anomalous threshold singularity obtained from the triangular diagram shown in Fig. 2 ($B =$ binding energy).

The final result, Eq. (16), reduces in the special case to the Massey and Mohr result which now has been written in a highly symmetric form.