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Transition Form Factors in the H Atom*

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Transition form factors between arbitrary excited states of the H atom have been evaluated in closed highly symmetric form within the framework of the noncompact dynamical group $O(4,2)$.

THE purpose of this paper is to present the exact form of the charge form factors in the H atom for transitions between any two excited states. The motivations for this work are: (1) These arbitrary transition form factors have, to our knowledge, not been given in the literature; (2) to show the power of the new simple algebraic methods using the representations of noncompact groups; and (3) for possible adaptation of the results to the dynamics of strongly interacting particles.

The form factors in question, denoted by $\mathfrak{F}_{n'l'm',nlm}{}^{\mu}\times(q^2)$, are the vertex amplitudes shown in Fig. 1 as a function of the momentum transfer $q^2=t=k^2+k'^2-2kk'\cos\theta$. They govern the inelastic scattering of the H atom by other charged particles or atoms if single photon exchange is dominant, and are measured by such scattering experiments. The transition form factors from the ground state $|100\rangle$ to an arbitrary state $|nlm\rangle$ were first calculated by Massey and Mohr[1] by Schrödinger theory. To our knowledge, these are the only form factors known explicitly. We present here an evaluation of arbitrary form factors solely within the conformal group $O(4,2)$. The method does not make any reference to spatial wave functions.

It has been shown recently that the dipole transitions in the H atom can be described in a simple manner by using the dynamical group $O(4,2)$, the conformal group.[2] Nambu[3] has investigated relativistic infinite-component wave equations for H-like systems and has indicated the calculation of form factors. Later, further properties of the H atom within the group $O(4,2)$ were investigated by Fronsdal[4] and the present authors.[5] Fronsdal gave also the form of the Galilei booster transformations on the group $O(4,2)$, and evaluated the form factor of the ground state.

We summarize briefly the $O(4,2)$ description of the H atom. Let $L_{ab}=-L_{ba}$; $a,b=1,2,\cdots 6$, be the 16 generators of $O(4,2)$. The subgroup $O(4)$ generated by L_{ab}; $a,b=1,2,3,4$, describes the degeneracy of the states of a given energy; the subgroup $O(4,1)$—dynamical group in the rest frame—describes *all* bound states $|nlm\rangle$, and, finally, the remaining generators L_{i6} are associated with dipole transitions, and L_{56} with the quantum number n.

The vector form factors are given by

$$\mathfrak{F}_{n'l'm',nlm}=\langle n'l'm'|nlm,k\rangle\,, \tag{1}$$

where $|nlm,k\rangle$ is the Galilei-boosted state of momentum

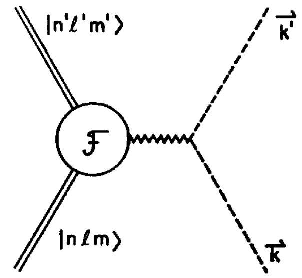

FIG. 1. The transition form factor $\mathfrak{F}$ is measured in an inelastic collision with another system of momentum k when one-photon exchange is dominant.

k, i.e., $e^{ikx}\psi_{nlm}(x)$. The generators M_i of the Galilei transformations $\exp(i\mathbf{k}\cdot\mathbf{M})$ are given by[4]

$$M_i=(L_{i5}-L_{i4})\,, \tag{2}$$

provided we introduce the new states[5]

$$|\bar{n}lm\rangle=\frac{1}{n}e^{-i\theta_n L_{45}}|nlm\rangle,\quad \theta_n=\ln n\,, \tag{3}$$

and a current operator $\Gamma_\mu=(L_{56}-L_{46},\,L_{i6})$. Then the charge form factors can be written as (for a booster in the 3 direction)

$$\mathfrak{F}_{n',n}{}^{lm}=\langle\bar{n}'lm|\Gamma_0 e^{-ikM_3}|\bar{n}lm\rangle\,. \tag{4}$$

Using Eqs. (2) and (3) and the commutation relations of $O(4,2)$ we can bring Eq. (4) to the form

$$\mathfrak{F}_{n',n}{}^{lm}=\frac{1}{n}\langle n'lm|\Gamma_0|n''lm\rangle \times\langle n''lm|e^{-i\vartheta_{n'n}L_{45}}e^{-ink(L_{34}-L_{35})}|nlm\rangle\,, \tag{5}$$

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[1] H. S. W. Massey and C. B. O. Mohr, Proc. Roy. Soc. (London) A132, 605 (1931). See also H. S. W. Massey, in *Handbuch der Physik*, edited by S. Flügge (Springer Verlag, Berlin, 1956), Vol. 34.

[2] A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1541 (1967). The relevance of the group $O(4,2)\sim SU(2,2)$ to the H atom, beyond the minimal dynamical group $O(4,1)$, was also noticed by I. A. Malkin and V. I. Man'ko, JETP Pis'ma v Redaktsiyn **2**, 230 (1966) [English transl.: JETP Letters **3**, 146 (1966)], but these authors did not consider dipole operators.

[3] Y. Nambu, Progr. Theoret. Phys. (Kyoto) Suppl. **37**, 368 (1966).

[4] C. Fronsdal, Phys. Rev. **156**, 1665 (1967).

[5] A. O. Barut and H. Kleinert, Phys. Rev. **157**, 1180 (1967).

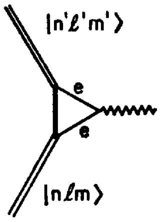

FIG. 2. The triangular diagram which gives the anomalous threshold obtained in Eq. (17).

where

$$\tanh\tfrac{1}{2}\vartheta_{n'n}=\frac{n-n'}{n+n'};\quad \vartheta_{n'n}=\ln(n/n').$$

In the evaluation of Eq. (5) we notice that $L_{34}=K_3$, $L_{35}=-K_2$, $L_{45}=K_1$ generate an $O(2,1)$ subalgebra (transition group[5] K). The second matrix element in Eq. (5) is that of a finite transformation of $O(2,1)$ that we express in terms of the Euler angles φ, ψ, χ:

$$e^{-i\vartheta_{n'n}K_1}e^{-ikn(K_3+K_2)}=e^{-i\varphi K_3/2}e^{-i\chi K_2/2}e^{-i\psi K_3/2},\qquad(6)$$

and obtain

$$\sinh\tfrac{1}{2}\chi=\frac{1}{(4n'n)^{1/2}}[(n-n')^2+tn^2n'^2]^{1/2},$$
$$\sin\varphi=\frac{(n'^2-n^2)+tn'^2n^2}{\{[(n'-n)^2+tn'^2n^2][(n'+n)^2+tn'^2n^2]\}^{1/2}};\qquad(7)$$

$\sin\psi$ is exactly like $\sin\varphi$ with n and n' interchanged. Next we express the operators K in terms of the generators corresponding to parabolic coordinates[2]

$$[N_i^+,N_i^-]=-2N_i^3,\quad [N_i^3,N_i^\pm]=\pm N_i^\pm,\quad i=1,2\qquad(8)$$

where

$$N_i^3|n_1n_2m\rangle=[n_i+(m+1)/2]|n_1n_2m\rangle;$$
$$N_1^\pm|n_1n_2m\rangle=-[(n_1+\tfrac{1}{2}\pm\tfrac{1}{2})(n_1+m+\tfrac{1}{2}\pm\tfrac{1}{2})]^{1/2}|n_1\pm1,n_2m\rangle\qquad(9)$$

as follows:

$$K_3=N_1^3-N_2^3;\quad K_1=\frac{1}{2i}(N_1^++N_2^+-N_1^--N_2^-);$$
$$K_2=-\tfrac{1}{2}(N_1^+-N_2^++N_1^--N_2^-).\qquad(10)$$

It is therefore easy to evaluate the matrix elements of Eq. (6) in parabolic coordinates:

$$\langle n_1''n_2''m|e^{-i\varphi K_3}e^{-i\chi K_2}e^{-i\psi K_3}|n_1n_2m\rangle=e^{-i(n_1''-n_2'')\varphi}e^{-i(n_1-n_2)\psi}\langle n_1''n_2''m|\times e^{-i(\chi/2)(N_1^++N_1^-)}e^{-i(\chi/2)(N_2^++N_2^-)}|n_1n_2m\rangle,\qquad(11)$$

where the last matrix element is the product of two finite $O(2,1)$ transformations:

$$G_{n_1''n_2''n_1n_2}{}^m\equiv V_{n_1''+(m+1)/2,n_1+(m+1)/2}{}^{(m+1)/2}(\chi)\times V_{n_2''+(m+1)/2,n_2+(m+1)/2}{}^{(m+1)/2}(\chi).\qquad(12)$$

The V function for $n_1'>n_1$ is given by

$$V_{n_1'+(m+1)/2,n_1+(m+1)/2}{}^{(m+1)/2}(\chi)=\theta_{n_1'n_1}(\cosh\tfrac{1}{2}\chi)^{-(n_1'+n_1+m+1)}(-i\sinh\tfrac{1}{2}\chi)^{n_1'-n_1}\times F(-n_1,-n_1-m,1+n_1'-n_1,-\sinh^2(\tfrac{1}{2}\chi)),\qquad(13)$$

$$\theta_{n_1'n_1}=\frac{1}{(n_1'-n_1)!}\left[\frac{n_1'!(n_1'+m)!}{n_1!(n_1+m)!}\right]^{1/2}$$

[for $n_1'<n_1$, use $V_{n_1+(m+1)/2,n_1'+(m+1)/2}{}^{(m+1)/2}(\chi)$], and occurs universally in all form-factor calculations (scalar or vector) and in the approximate evaluation of scattering amplitudes.[5–7]

Similarly, the first matrix element in Eq. (5) is easily calculated in parabolic coordinates, because

$$L_{46}=\tfrac{1}{2}(N_1^++N_1^-+N_2^++N_2^-);\quad L_{56}=N.\qquad(14)$$

It remains then to change the basis $|n_1n_2m\rangle$ into $|nlm\rangle$. Because this change of basis is connected with the reduction of $O(4)$ into $O(3)\times O(3)$, we have immediately in terms of the $3j$ symbols

$$\langle nlm|n_1n_2m\rangle=(-1)^m(2l+1)^{1/2}\times\begin{pmatrix}(n-1)/2 & (n-1)/2 & l\\ (m-n_1+n_2)/2 & (m+n_1-n_2)/2 & -m\end{pmatrix}.\qquad(15)$$

Consequently, collecting all the terms the final result is

$$\mathfrak{F}_{n'n}{}^{lm}=\frac{(2l+1)}{n}\sum_{n_1'n_2',n_1n_2}\begin{pmatrix}(n'-1)/2 & (n'-1)/2 & l\\ (m-n_1'+n_2')/2 & (m+n_1'-n_2')/2 & -m\end{pmatrix}\begin{pmatrix}(n-1)/2 & (n-1)/2 & l\\ (m-n_1+n_2)/2 & (m+n_1-n_2)/2 & -m\end{pmatrix}$$
$$\times\{n'e^{-i[(n_1'-n_2')\varphi+(n_1-n_2)\psi]}G_{n_1'n_2'n_1n_2}{}^m+[(n_1'+1)(n_1'+m+1)]^{1/2}h^+{}_{n_1'-n_2',n_1-n_2}{}^{n',n}G_{n_1'+1,n_2'n_1n_2}{}^m$$
$$+[n_1'(n_1'+m)]^{1/2}h^-{}_{n_1'-n_2',n_1-n_2}{}^{n',n}G_{n_1'-1,n_2'n_1n_2}{}^m\},\qquad(16)$$

where

$$h^\pm{}_{s',s}{}^{n',n}=\cos[(s'\pm1)\varphi+s\psi]\quad\text{for}\quad(-1)^{n'-n}=-1$$
$$=-i\sin[(s'\pm1)\varphi+s\psi]\quad\text{for}\quad(-1)^{n'-n}=+1.$$

[6] A. O. Barut and H. Kleinert, Phys. Rev. **156**, 1546 (1967).
[7] A. O. Barut and H. Kleinert, Phys. Rev. Letters **18**, 754(1967).

The form factor $\mathfrak{F}$ has a singularity where the $\cosh\frac{1}{2}\chi$ term in Eq. (11) vanishes. From Eq. (7), this occurs at

$$t=t_1=-\frac{(n'+n)^2}{n^2n'^2}=2(\sqrt{B'}+\sqrt{B})^2 \tag{17}$$

and coincides exactly with the anomalous threshold singularity obtained from the triangular diagram shown in Fig. 2 (B=binding energy).

The final result, Eq. (16), reduces in the special case to the Massey and Mohr result[1] which now has been written in a highly symmetric form.