ARE GLUONS COMPOSITE? A NEW LATTICE GAUGE MODEL WITH AN EXACT U(∞) SOLUTION

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The search for an exact $N \to \infty$ solution of U(N) gauge theories has led us to a new lattice model in which the gauge field dynamics is generated by four fundamental tensor fields describing subgluons. Our model has the same $\beta \to 0$ limit as Wilson's and the same $\beta \to \infty$ limit as far as the soft weak field excitations are concerned. It allows for the introduction of colorless Hartree like collective fields and has a simple $N \to \infty$ solution.

The existence of an exact $N \to \infty$ solution of O(N) spin models has inspired hopes that also U(N) lattice gauge theories might possess such a limiting solution [1,2]. So far, attempts in this direction have failed, except for D = 2, where the gluons have no proper transverse degrees of freedom [3,4].

We would like to report a solution to the problem based on a new lattice gauge model in which gluon dynamics is generated by "bubble sums" of fundamental subgluon fields. Our model has the same $\beta \to 0$ (strong coupling) limit as Wilson's. The $\beta \to \infty$ (weak coupling) limit is the same only as far as the soft weak-field gluons are concerned, as we shall see later. The main advantage of our model is that it permits the introduction of colorless Hartree like collective fields for all U(N) [‡1].

The partition function of our "composite gluon model" can be written, just as Wilson's [7], in the form

$$Z = \prod_{x,\mu} \int \mathrm{d}U_\mu(x) \prod_{x,\mu<\nu} z(U_{\mu\nu}(x)) , \tag{1}$$

where the x are the D lattice sites, the $U_\mu(x)$ are U(N) matrices living on the oriented links μ, the $\mathrm{d}U_\mu(x)$ are the invariant group integrals, and the $U_{\mu\nu}(x)$ the plaquette objects

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\mu) U_\mu^+(x+\nu) U_\nu^+(x) . \tag{2}$$

The difference of our model with Wilson's lies in the fugacity factor which in his case is

$$z(U_{\mu\nu}) = \exp[\tfrac{1}{2}\beta \, \mathrm{tr}_N(U_{\mu\nu} + U_{\mu\nu}^+ - 2)] , \tag{3}$$

while ours is chosen to be the result of fluctuations of four complex antisymmetric "tensor" fields $\phi_{\mu\nu}^{(1)}, \phi_{\mu\nu}^{(2)}, \phi_{\mu\nu}^{(3)}, \phi_{\mu\nu}^{(4)}$, which transform according to the fundamental representation of the group U(N),

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[‡1] In the special case of SU(2) and SU(3), Rühl [5] has succeeded in introducing colorless fields, although he could not find an explicit action, as we have done. See also ref. [6].

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$$z(U_{\mu\nu}) = [(\lambda^2-2)^2-4]^N \prod_{\substack{r=1\\ \mu<\nu}}^{4} \int d\phi_{\mu\nu}^{(r)} d\phi_{\mu\nu}^{(r)} + \pi^{-1}$$

$$\times \exp\Big(-\lambda \sum_{\mu<\nu} \phi_{\mu\nu}^{(1)+}(x)\phi_{\mu\nu}^{(1)}(x) + \phi_{\mu\nu}^{(2)+}(x+\mu)\phi_{\mu\nu}^{(2)}(k+\mu)$$

$$+ \phi_{\mu\nu}^{(3)+}(x+\mu+\nu)\phi_{\mu\nu}^{(3)}(x+\mu+\nu) + \phi_{\mu\nu}^{(4)+}(x+\nu)\phi_{\mu\nu}^{(4)}(x+\nu)]$$

$$+ \sum_{\mu<\nu} [\phi_{\mu\nu}^{(1)+}(x)U_\mu(x)\phi_{\mu\nu}^{(2)}(x+\mu) + \phi_{\mu\nu}^{(2)+}(x+\mu)U_\nu(x+\mu)\phi_{\mu\nu}^{(3)}(x+\mu+\nu)$$

$$+ \phi_{\mu\nu}^{(3)+}(x+\mu+\nu)U_\mu^+(\sigma+\nu)\phi_{\mu\nu}^{(4)}(x+\nu) + \phi^{(4)+}(x+\nu)U_\nu^+(x)\phi_{\mu\nu}^{(1)}(x) + \text{h.c.}]\Big) . \quad (4)$$

Geometrically speaking, the fields $\phi_{\mu\nu}^{(r)}(x)$ may be imagined as living on the corners of the four plaquettes with given $\mu<\nu$ having a common site X (see fig. 1). The arguments of the four fields in the fugacity factor (4) are shifted in such a way that they are associated with the four inside corners of the upper right plaquette in fig. 1 (marked bv ○).

After four quadratic completions, the integrals can be performed with the result

$$z(U_{\mu\nu}) = (1-2\kappa)^N \det(1-\kappa(U_{\mu\nu}+U_{\mu\nu}^+))^{-1} , \quad (5)$$

where κ is short for

$$\kappa \equiv [(\lambda^2-2)^2-2]^{-1} . \quad (6)$$

Contact with Wilson's action is established by identifying

$$\kappa \equiv \tfrac{1}{2}\beta/(1+\beta) , \quad (7)$$

such that for small $\beta, z \sim 1 + \frac{1}{2}\beta\, \mathrm{tr}_N(U_{\mu\nu}+U_{\mu\nu}^+-2)$ and for large $\beta, z \sim \exp\{-\mathrm{tr}_N \log[1-\frac{1}{2}\beta(U_{\mu\nu}+U_{\mu\nu}^+-2)]\}$. The first limit is the same as Wilson's. In the second limit $U_\mu = \exp(iagA_\mu)$ is squeezed close to the unit matrix such that

$$1-U_{\mu\nu} \approx \tfrac{1}{2}a^4g^2F_{\mu\nu}^2 \quad (8)$$

is of the order $1/\beta$, where a is the lattice spacing and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + ig[A_\mu, A_\nu]$. We can therefore write

$$\prod_{x,\mu<\nu} z(U_{\mu\nu}(x)) \underset{\beta\to\infty}{\approx} \exp\Big(-\sum_{x,\mu<\nu} \mathrm{tr}_N \log[1+\tfrac{1}{2}\beta a^4g^2F_{\mu\nu}^2(x)]\Big) . \quad (9)$$

Setting $\beta = g^{-2}a^{D-4}$ and $\Sigma_x = a^{-D}\int d^Dx$, the formal continuum limit $a\to 0$ at fixed $F_{\mu\nu}$ looks the same as Wilson's

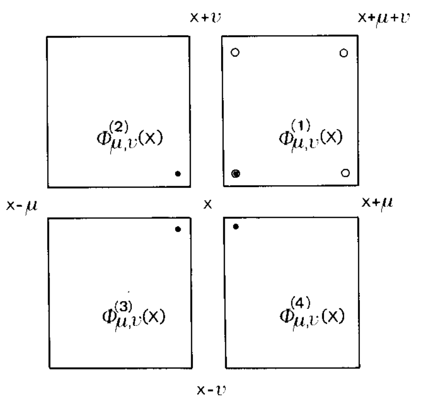

Fig. 1. The fundamental subgluon fields living on the inside corners of the four plaquettes adjacent to the point X (for a given plane $\mu<\nu$). The fugacity factor (4) contains the shifted fields at the positions marked by ○.

$$\prod_{x,\mu<\nu} z(U_{\mu\nu}(x)) \sim \exp\Big(-\sum_{x,\mu<\nu} \tfrac{1}{2}\beta a^4 g^2 F_{\mu\nu}^2\Big) \to \exp\Big(-\frac{1}{2}\sum_{\mu<\nu}\int \mathrm{d}^D x F_{\mu\nu}^2\Big). \quad (10)$$

For a given a, however, the partition functions are different, even for very large β, since only the small weak field fluctuations have the same Boltzmann weight.

We now show that our model possesses a colorless collective field representation. For this we integrate out the $\phi_{\mu\nu}^{(r)}$ fields in Z and find

$$Z = [(\lambda^2-2)^2-4]^{NN_\mathrm{p}} \prod_{\substack{x,r=1\\ \mu<\nu}}^{4} \int \mathrm{d}\phi_{\mu\nu}^{(r)+}(x)\, \mathrm{d}\phi_{\mu\nu}^{(r)}(x) \exp\Big(-\frac{\lambda}{2}\sum_{\substack{x,r\\ \mu<\nu}} \phi_{\mu\nu}^{(r)+}\phi_{\mu\nu}^{(r)}\Big) \exp\Big(N\sum_{x,\mu} w_N(J_\mu J_\mu^+)\Big), \quad (11)$$

where $N_\mathrm{p} = \frac{1}{2}D(D-1)L^D$ and

$$J_\mu{}^{\beta\alpha}(x) \equiv \frac{1}{N}\Big(\sum_{\nu<\mu}(\phi_{\mu\nu}^{(3)\beta}(x+\mu)\phi_{\mu\nu}^{(2)\alpha}(x)^* + \phi_{\mu\nu}^{(4)\beta}(x+\mu)\phi_{\mu\nu}^{(1)\alpha}(x)^*)$$
$$+\sum_{\nu<\mu}(\phi_{\mu\nu}^{(3)\beta}(x+\mu)\phi_{\mu\nu}^{(2)\alpha}(x)^* + \phi_{\mu\nu}^{(4)\beta}(x+\mu)\phi_{\mu\nu}^{(1)\alpha}(x)^*)\Big) \quad (12)$$

and $Nw_N(JJ^+)$ is the usual U(N) integral

$$\exp[Nw_N(JJ^+)] \equiv \int \mathrm{d}U(x) \exp[N\,\mathrm{tr}_N(U^{\alpha\beta}J^{\beta\alpha} + \text{h.c.})]\ . \quad (13)$$

The partition function (11) represents gluon dynamics entirely in terms of fundamental subgluon fields $\phi_{\mu\nu}^{(r)}(x)$. In the literature one can find two alternative expressions for the function $w_N(JJ^+)$. One involves the eigenvalues of the matrix JJ^+ [8]. Defining $z_\alpha \equiv 2N\sqrt{x_\alpha}$ and the matrices $M_{\alpha\beta}^{(1)} \equiv z_\alpha{}^{\beta-1} I_{\beta-1}(z_\alpha)$, $M_{\alpha\beta}^{(2)} \equiv (z_\alpha{}^2)^{\beta-1}$, Brower et al. give the formula

$$\exp[Nw(JJ^+)] = 2^{N(N-1)/2}\prod_{k=0}^{N-1} k!\,\frac{\det M^{(1)}}{\det M^{(2)}}\,. \quad (14)$$

The other expression by Bars [9] consists of a linear combination of traces of powers, $\mathrm{tr}_N(JJ^+)^n$, called "moments".

We are now ready to introduce our collective Hartree like fields. For brevity, let us denote the four fields $\phi_{\mu\nu}^{(1)}, ..., \phi_{\mu\nu}^{(4)}$ by $\phi_{\mu\nu}, \phi_{\mu,-\nu}, \phi_{-\mu,\nu}, \phi_{-\mu,-\nu}$, respectively, and introduce the $2D$ indices m, n running from $-D$ to D. Then $J_\mu{}^{\beta\alpha}(x)$ becomes simply $J_\mu{}^{\beta\alpha}(x) = N^{-1}\sum_{n\neq\pm\mu}\phi_{-\mu,n}{}^\beta(x+\mu)\phi_{\mu,n}{}^\alpha(x)^*$. If we now form the first moment $\mathrm{tr}_N(J_\mu J_\mu^+)$ we see that this can be regrouped as

$$\frac{1}{N^2}\sum_{n\neq\pm\mu}\sum_{l\neq\pm\mu}\phi_{\mu,n}{}^\alpha(x)^*\phi_{\mu,l}{}^\alpha(x)\phi_{-\mu,l}{}^\beta(x+\mu)^*\phi_{-\mu,n}{}^\beta(x+\mu) = \mathrm{tr}_{2D}(\alpha^\mu(x)\alpha^{-\mu}(x+\mu))\,,$$

where the $\alpha_{nl}{}^\mu(x)$ are the colorless $2D \times 2D$ matrices

$$\alpha_{nl}{}^\mu(x) \equiv N^{-1}\phi_{\mu,n}^+(x)\,\phi_{\mu,l}(x)\,, \quad (15)$$

and tr_{2D} refers to this matrix space. The regrouping of fields is analogous to the Fierz transformation of the four-fermion interactions which leads to the introduction of meson fields in quark theories [10]. The "dimensional transmutation" from color to link indices occurs in all higher moments

$$\mathrm{tr}_N(J_\mu(x)J_\mu^+(x))^n = \mathrm{tr}_{2D}(\alpha^\mu(x)\alpha^{-\mu}(x+\mu))^n\ . \quad (16)$$

Thus, using the expression of Bars, we know immediately w_N as a function of $\alpha^\mu(x)\alpha^{-\mu}(x+\mu)$:

$$w_N(J_\mu J_\mu^+) = \widetilde{w}_{2D}\,(\alpha^\mu(x)\alpha^{-\mu}(x+\mu))\ . \quad (17)$$

For $N > 2D$ there is an even more convenient way of calculating $\tilde{w}_{2D}$. If we define the $2D \times 2D$ matrix

$$\tilde{J}_\mu(x) \equiv (\alpha^\mu(x)\alpha^{-\mu}(x+\mu))^{1/2}\,, \tag{18}$$

we can write

$$\mathrm{tr}_N(J_\mu J_\mu^+)^n \equiv \mathrm{tr}_{2D}(\tilde{J}_\mu \tilde{J}_\mu^+)^n\,, \tag{19}$$

and hence

$$w_N(J_\mu J_\mu^+) = w_N(\tilde{J}_\mu \tilde{J}_\mu^+)\,, \tag{20}$$

where the $2D \times 2D$ matrix $\tilde{J}_\mu$ in (20) is trivially extended to an $N \times N$ matrix by adding zeros. In the limit $N \to \infty$, we now use formula (14) and calculate

$$w_N(J_\mu J_\mu^+) = \tilde{w}_{2D}(\alpha^\mu(x)\alpha^{-\mu}(x+\mu))$$

$$\xrightarrow[N\to\infty]{} \mathrm{tr}_{2D}[\;+4\alpha^\mu(x)\alpha^{-\mu}(x+\mu)]^{1/2} - 2D - \mathrm{tr}_{2D}\log\{\tfrac{1}{2}\mathbf{1} + \tfrac{1}{2}[\mathbf{1} + 4\alpha^\mu(x)\alpha^{-\mu}(x+\mu)]^{1/2}\}\,, \tag{21}$$

having replaced $J_\mu \tilde{J}_\mu$ by $\alpha^\mu(x)\alpha^{-\mu}(x+\mu)$.

Let us now come to our first goal, namely that of expressing the partition function in terms of a guage invariant collective field. This can be come by inserting the following identity

$$\mathbf{1} = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathcal{D}\sigma \int_{-\infty}^{\infty} \mathcal{D}\alpha \exp\Big(\tfrac{1}{4}N \sum_{\substack{x,l,m,n\\ m,n,\neq \pm l}} [\sigma_{mn}^{l*}(\alpha_{mn}^l - N^{-1}\phi_{lm}^+\phi_{ln}) + \mathrm{c.c.}]\Big)\,, \tag{22}$$

into Z, which ensures the relation (15). The measure for $\alpha_{mn}^l(x) = \alpha_{nm}^{l*}(x)$ and $\sigma_{mn}^l(x) = \sigma_{nm}^{l*}(x)$ is

$$\mathcal{D}\sigma\mathcal{D}\alpha = \prod_{x,l}\Bigg(\prod_{\substack{m\\ m\neq\pm l}} \frac{\mathrm{d}\sigma_{mm}^l\,\mathrm{d}\alpha_{mm}^l}{4\pi\mathrm{i}/N} \prod_{\substack{m<n\\ m\neq -n}} \frac{\mathrm{d}\sigma_{mn}^l\,\mathrm{d}\sigma_{mn}^{l*}\,\mathrm{d}\alpha_{mn}^l\,\mathrm{d}\alpha_{mn}^{l*}}{(2\pi\mathrm{i}/N)^2}\Bigg).$$

Integrating out the ϕ_{lm} fields leads to

$$Z = [(\lambda^2-2)^2-4]^{NN_\mathrm{p}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathcal{D}\sigma \int_{-\infty}^{\infty}\mathcal{D}\alpha \exp\Bigg\{N\Bigg[\sum_{x,\mu}\tilde{w}_{2D}(\alpha^\mu(x)\alpha^{-\mu}(x+\mu))$$

$$+\frac{1}{2}\Bigg(\sum_{\substack{x,l,m<n\\ m,n\neq\pm l}}(\sigma_{mn}{}^{l*}(x)\alpha_{mn}{}^l(x) + \mathrm{c.c.}) + \sum_{\substack{x,l,m\\ m\neq\pm l}}\sigma_{mn}{}^l(x)\alpha_{mn}{}^l(x)\Bigg) - \sum_x \mathrm{tr}_{2D}\log G^{-1}(\sigma(x))\Bigg]\Bigg\}\,, \tag{23}$$

where

$$\exp[-N\,\mathrm{tr}_{2D}\log G^{-1}(\sigma)]$$

$$\equiv \prod_{\substack{m<n\\ m\neq -n}} \int \mathrm{d}\phi_{mn}(x)\,\mathrm{d}\phi_{mn}^+(x)\pi^{-1}\exp\Bigg(-\lambda\sum_{\substack{m<n\\ m\neq -n}}\phi_{mn}^+(x)\phi_{mn}(x) - \tfrac{1}{2}\sum_{\substack{l,m,n\\ m,n\neq\pm l}}\sigma_{mn}{}^l\phi_{lm}^+(x)\phi_{lm}(x)\Bigg). \tag{24}$$

For $N\to\infty$, the collective fields $\sigma_{mn}{}^l$, $\alpha_{mn}{}^l$ are squeezed into the energy minimum ‡2. For symmetry reasons, this

‡2 Observe that the number of integration variables in (23) is $\sim D^3$ while the exponent grows as $\sim ND^2$.

may be assumed to have three order parameters depending on whether the two plaquettes ml, nl with common link l are orthogonal to each other $(\sigma_\perp, \alpha_\perp)$ parallel to each other $(\sigma_\parallel, \alpha_\parallel)$, or folded on top of each other (σ_0, α_0). A straightforward but tedious calculation produces a mean-field energy density

$$\begin{aligned}-\beta f/N = {} & \tfrac{1}{2}D(D-1)\log[(\lambda^2-2)^2-4] + D(D-2)([1+4(\alpha_0+\alpha_\parallel-2\alpha_\perp)^2]^{1/2} - 1 - \log\{\tfrac{1}{2} \\ & + \tfrac{1}{2}[1+4(\alpha_0+\alpha_\parallel-2\alpha_\perp)^2]^{1/2}\}) + D(D-1)([1+4(\alpha_0-\alpha_\parallel)^2]^{1/2} - 1 - \log\{\tfrac{1}{2}+\tfrac{1}{2}[1+4(\alpha_0-\alpha_\parallel)^2]^{1/2}\}) \\ & + D[\{1+4[\alpha_0+\alpha_\parallel+2(D-2)\alpha_\perp]^2\}^{1/2} - 1 - \log(\tfrac{1}{2}+\tfrac{1}{2}\{1+4[\alpha_0+\alpha_\parallel+2(D-2)\alpha_\perp]^2\}^{1/2})] \\ & + 2D(D-1)(\sigma_0\alpha_0+\sigma_\parallel\alpha_\parallel) + 4D(D-1)(D-2)\sigma_\perp\alpha_\perp - D(D-2)\log(\lambda+\sigma_0-\sigma_\perp) \\ & - \tfrac{1}{2}D(D-3)\log(\lambda+\sigma_0+\sigma_\parallel-2\sigma_\perp) - \tfrac{1}{2}D(D-1)\log(\lambda+\sigma_0-\sigma_\parallel) - D\log[\lambda+\sigma_0+(D-2)\sigma_\perp] \\ & - (D-1)\log[\lambda+\sigma_0+\sigma_\parallel+(D-4)\sigma_\perp] - \log[\lambda+\sigma_0+\sigma_\parallel+2(D-2)\sigma_+] \,. \qquad (25)\end{aligned}$$

Its minimum is given by

$$\sigma_0 = -4/[\lambda+(\lambda^2-4)^{1/2}] \,, \quad \sigma_\parallel = \sigma_\perp = 0 \,, \quad \alpha_0 = (\lambda^2-4)^{-1/2} \,, \quad \alpha_\parallel = \alpha_\perp = 0 \,, \qquad (26)$$

and has a free energy

$$-\beta f/N = \tfrac{1}{2}D(D-1)\log[(\lambda^2-2)^2-4] - 2D(D-1)\log[\tfrac{1}{2}\lambda+\tfrac{1}{2}(\lambda^2-4)^{1/2}] \,. \qquad (27)$$

The vanishing of the $\parallel$, $\perp$ order parameters implies the absence of non-trivial surfaces in the $N \to \infty$ limit of our model. Therefore, the solution (27) has the same λ dependence as what would have been found by applying Gross and Witten's method to our model in the two-dimensional case (apart from the factor $\frac{1}{2}D(D-1)$ which they would be unable to obtain).

Notice that, contrary to Wilson's, our model has no phase transition in the limit $N \to \infty$ (caused in this model by the infinite group space [3]). The reason for this can be found by looking at the high temperature series which consists of surfaces associated with the different group representations in the character expansion of $z(U_{\mu\nu})$ [11]. In our model there are the Young tableaux $(n, 0, 0, ..., 0)$ with dimension $d_n = \binom{N+n-1}{n} \to N^n/n!$. A non-self-intersecting surface of area A and genus g contribute with a weight $z(\beta)^{nA} d_n^{2-g} d_n^{-A}$ where $z(\beta) \in [0, 1)$. It is the last factor (absent in Wilson's action) which suppresses all non-trivial surfaces.

The simple collective field properties of our model could make it a useful tool in developing further analytic approaches to gauge theories.

The analogy of the coupling of the subgluon fields in (4) with that of quarks may be a sign that these fields are more than just a convenient mathematical tool for studying the $N \to \infty$ limit. This is also suggested by the fact that out procedure is precisely the color analogue of Gell-Mann's way of decomposing flavor octet objects into more fundamental flavor triplet substructures. More speculatively, our subgluons could even be the Bose partners of quarks in an as yet unknown supersymmetric theory of fundamental particles (i.e. quarks could be *subgluinos*).

The details of our calculations will be published separately.

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