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Disorder Field Theory of the Ensemble of Random Loops without Spikes.

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Summary. – We prove that a grand-canonical ensemble of random loops without spikes (*i.e.* without immediate backtrackers) obey a *free* disorder field theory with a mass parameter, on a simple cubic lattice,

$$m^2 = \exp[\varepsilon/T] - 2D + (2D-1)\exp[-\varepsilon/T]\,,$$

where ε is the energy per link and D the spatial dimension. Thus the lines proliferate at a temperature

$$T_c = \varepsilon/\log(2D-1)$$

as one might naively expect.

A free ensemble of unoriented random loops (1) is known to be described by the free scalar field theory (2)

$$Z = \prod_{\boldsymbol{x}} \int \frac{d\varphi(\boldsymbol{x})}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{x}'} \phi(\boldsymbol{x}) G^{-1}(\boldsymbol{x},\boldsymbol{x}')\phi(\boldsymbol{x}')\right], \tag{1}$$

where

$$G^{-1}(\boldsymbol{x},\boldsymbol{x}') = \delta_{\boldsymbol{x},\boldsymbol{x}'} - zH_{\boldsymbol{x},\boldsymbol{x}'} \tag{2}$$

is the lattice Green's function, $H_{\boldsymbol{x},\boldsymbol{x}'}$ the hopping matrix, and z the fugacity of the

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(1) The mathematic of random walks is described in M. SPITZER: *Random Walks* (Springer, Berlin, 1970); M. BARBER and B. W. NINHAM: *Random Walks and Restricted Walks* (Gordon and Breach, New York, N. Y., 1970).

(2) The random loop content in a $\phi(x)$ field theory was first discussed extensively by K. SYMANZIK: in *Euclidean Quantum Field Theory* , edited by R. JOST (Academic Press, New York, N. Y., 1969). For mathematical aspects see D. C. BRYDGES, J. FRÖHLICH and T. SPENCER: *Commun. Math. Phys.*, **80**, 892 (1983). Disorder fields are extensively used in H. KLEINERT: *Gauge Theory of Stresses and Defects* (Gordon and Breach, New York, N. Y., 1985).

loop elements. In terms of the energy ε per element,

$$z \equiv \exp\left[-\varepsilon/T\right]. \tag{3}$$

On a simple cubic lattice,

$$H_{\boldsymbol{x},\boldsymbol{x}'} = \sum_{\pm i=1}^{D} \delta_{\boldsymbol{x},\boldsymbol{x}'+\boldsymbol{i}} = 2D + \sum_{i=1}^{D} \bar{\nabla}_i \nabla_i\,, \tag{4}$$

where $i = 1, \ldots, D$ describes the D oriented links pointing to the next neighbours and $\nabla_i \varphi(x) \equiv \varphi(\boldsymbol{x}+\boldsymbol{i}) - \varphi(\boldsymbol{x})$, $\bar{\nabla}_i \varphi(\boldsymbol{x}) \equiv \varphi(\boldsymbol{x}) - \varphi(\boldsymbol{x}-\boldsymbol{i})$.

Hence the mass of the $\phi(\boldsymbol{x})$ field is given by

$$m^2 = \frac{1}{z} - 2D \tag{5}$$

and turns negative if the temperature becomes larger than

$$T_c = \varepsilon/\log 2D\,. \tag{6}$$

Since the exponent of (1) is quadratic in $\phi(\boldsymbol{x})$, we can integrate out the $\phi(\boldsymbol{x})$ field and find

$$Z = \exp\left[Z_1\right], \tag{7}$$

where

$$Z_1 = -\tfrac{1}{2}\,\mathrm{tr}\log G^{-1} \tag{8}$$

is the one-loop partition function.

The purpose of this note is to find a similar field-theoretic formulation for an ensemble of random loops which are not allowed to have spikes, *i.e.* to back-track on a link they just have passed (*).

In order to restrict certain movements in an ensemble of random walks [3], it is useful to re-express the partition function in terms of complex link fields $\psi_i(\boldsymbol{x})$. They are introduced via the trivial identity

$$\prod_{\boldsymbol{x},i} \int \frac{\mathrm{d}\psi_i(\boldsymbol{x})\,\mathrm{d}\psi_i^*(\boldsymbol{x})}{\pi} \exp\left[-\sum_{\boldsymbol{x},i} \left(\psi_i^*(\boldsymbol{x}) - \sqrt{z}\,\phi(\boldsymbol{x})\right)\left(\psi_i(\boldsymbol{x}) - \sqrt{z}\,\phi(\boldsymbol{x}+\boldsymbol{i})\right)\right], \tag{9}$$

where $\int \mathrm{d}\psi_i \mathrm{d}\psi_i^*$ means $\int_{-\infty}^{\infty} \mathrm{d}\,\mathrm{Re}\,\psi \int_{-\infty}^{\infty} \mathrm{d}\,\mathrm{Im}\,\psi$. Thus we rewrite

$$Z = \prod_{\boldsymbol{x}} \int \frac{\mathrm{d}\varphi(\boldsymbol{x})}{\sqrt{2\pi}} \prod_{\boldsymbol{x},i} \int \frac{\mathrm{d}\psi_i(\boldsymbol{x})\,\mathrm{d}\psi_i^*(\boldsymbol{x})}{\pi} \cdot \\ \cdot \exp\left[-\frac{1}{2}\sum_{\boldsymbol{x}} \phi^2(\boldsymbol{x}) + \sqrt{z}\sum_{\boldsymbol{x},i} \phi(\boldsymbol{x})\left(\psi_i(\boldsymbol{x}) + \psi_i^*(\boldsymbol{x}-\boldsymbol{i})\right)\right] \exp\left[-\sum_{\boldsymbol{x},i} \psi_i^*(\boldsymbol{x})\,\psi_i(\boldsymbol{x})\right]. \tag{10}$$

(*) Individual lines of this type are well understood [3].

[3] H. N. Y. TEMPERLEY: *Phys. Rev.*, **103**, 1 (1956); J. GILLIS: in *Proc. Cambridge Philos. Soc.*, **51**, 639 (1956); C. DOMB and M. E. FISHER: in *Proc. Cambridge Philos. Soc.*, **54**, 48 (1958). For a recent Monte Carlo study of individual lines see B. BERG and D. FOERSTER: *Phis: Lett. B*, **106**, 323 (1981).

Performing the $\phi(\boldsymbol{x})$ integration gives

$$(11)\qquad Z=\prod_{\boldsymbol{x},i}\int\frac{\mathrm{d}\psi_i(\boldsymbol{x})\,\mathrm{d}\psi_i^*(\boldsymbol{x})}{\pi}\exp\Big[-\sum_{\boldsymbol{x},i}\psi_i^*(\boldsymbol{x})\psi_i(\boldsymbol{x})\Big]\exp\Big[\frac{z}{2}\sum_{\boldsymbol{x}}\Big(\sum_i\big(\psi_i(\boldsymbol{x})+\psi_i^*(\boldsymbol{x}-\boldsymbol{i})\big)\Big)^2\Big].$$

Defining $\psi_{-i}(\boldsymbol{x})\equiv\psi_i(\boldsymbol{x}-\boldsymbol{i})$ for $i=1,\dots,D$, we work out the second exponential as follows:

$$(12)\qquad \exp\Big[\frac{z}{2}\sum_{\boldsymbol{x},i}[\psi_i(\boldsymbol{x})^2+\psi_i^*(\boldsymbol{x})^2+2\psi_i(\boldsymbol{x})\psi_i^*(\boldsymbol{x})]+$$
$$+z\sum_{\boldsymbol{x},i<j}[\psi_i(\boldsymbol{x})\psi^i(\boldsymbol{x})+\psi_{-i}^*(\boldsymbol{x})\psi_{-j}^*(\boldsymbol{x})]+z\sum_{\boldsymbol{x},i\neq j}\psi_i(\boldsymbol{x})\psi_{-j}^*(\boldsymbol{x})\Big].$$

Expanding this in a power series gives

$$(13)\qquad \prod_{\boldsymbol{x},\mu\geqslant\nu}\Big(\sum_{m(\boldsymbol{x},\mu,\nu)=0,1,2,\dots}\frac{z^{m(\boldsymbol{x},\mu,\nu)}}{m(\boldsymbol{x},\mu,\nu)!}\Big)\prod_{\boldsymbol{x},i}[(\tfrac{1}{2})^{m(\boldsymbol{x},i,i)}(\tfrac{1}{2})^{m(\boldsymbol{x},-i,-i)}]\cdot$$
$$\cdot\prod_{\boldsymbol{x},i}[\psi_i(\boldsymbol{x})^{2m(\boldsymbol{x},i,i)}\psi_{-i}^*(\boldsymbol{x})^{2m(\boldsymbol{x},-i,-i)}\psi_i(\boldsymbol{x})^{m(\boldsymbol{x},i,-i)}\psi_{-i}^*(\boldsymbol{x})^{m(\boldsymbol{x},-i,i)}]\cdot$$
$$\cdot\prod_{\boldsymbol{x},i>j}[\psi_i(\boldsymbol{x})^{m(\boldsymbol{x},i,j)}\psi_j(\boldsymbol{x})^{m(\boldsymbol{x},i,j)}\psi_{-i}^{*m(\boldsymbol{x},-i,-j)}\psi_{-j}^*(\boldsymbol{x})^{m(\boldsymbol{x},-i,-j)}]\prod_{\boldsymbol{x},i\neq j}[\psi_i(\boldsymbol{x})^{m(\boldsymbol{x},i,-j)}\psi_{-j}^*(\boldsymbol{x})^{m(\boldsymbol{x},i,-j)}].$$

Here $m(\boldsymbol{x},i,j)$, $m(\boldsymbol{x},i,-j)$, $m(\boldsymbol{x},-i,j)$, $m(\boldsymbol{x},-i,-j)$ can be interpreted as occupation numbers of pairs of links emerging from the point $\boldsymbol{x}$ (see fig. 1).

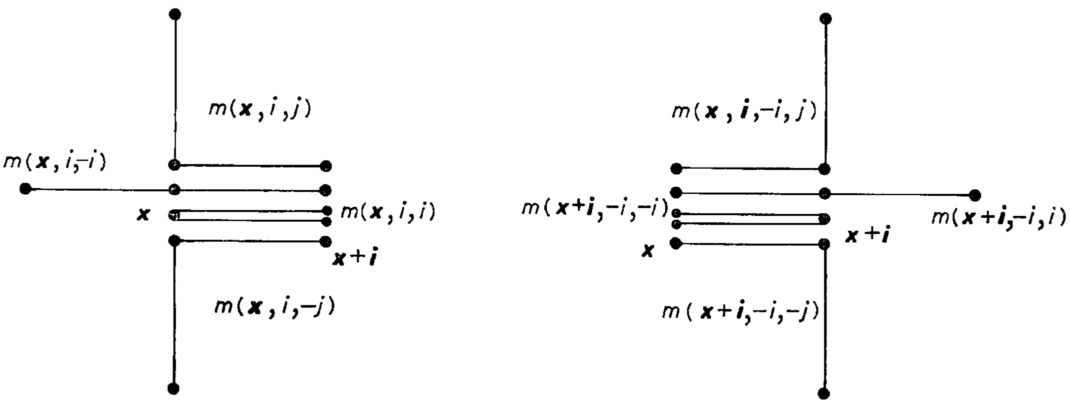

Fig. 1. – Illustration of the hook occupation numbers.

For brevity, we have used the symbol $m(\boldsymbol{x},\mu,\nu)$ with $\mu=\pm i$, $\nu=\pm i$ running through oriented and oppositely oriented links. Geometrically, a configuration of $m(\boldsymbol{x},\mu,\nu)$ with $\mu\geqslant\nu$ may be pictured as an ensemble of *hooks* with corners situated at the point $\boldsymbol{x}$ (or the midpoint, for the stretched hook i,i). Each field $\psi_i(\boldsymbol{x})$ and $\psi_i^*(\boldsymbol{x})$ occurs with total power

$$(14)\qquad n_i(\boldsymbol{x})=2m(\boldsymbol{x},i,i)+m(\boldsymbol{x},i,-i)+\sum_{j\neq i}\big(m(\boldsymbol{x},i,j)+m(\boldsymbol{x},i,-j)\big),$$

$$(15)\qquad n_i^*(\boldsymbol{x})=2m(\boldsymbol{x}+\boldsymbol{i},-i,-i)+m(\boldsymbol{x}+\boldsymbol{i},i,-i)+$$
$$+\sum_{j\neq i}\big(m(\boldsymbol{x}+\boldsymbol{i},-i,-j)+m(\boldsymbol{x}+\boldsymbol{i},-i,j)\big),$$

respectively. It counts the different ways in which a link is occupied by hooks, as illustrated in fig. 1.

By rewriting the partition function in the form

$$(16)\qquad Z=\prod_{x,\mu\geqslant\nu}\left(\sum_{m(x,\mu,\nu)=0,1,\dots}\frac{z^{m(x,\mu,\nu)}}{m(x,\mu\ \nu)}\left(\frac{1}{2}\right)^{\sum\limits_{x,i}(m(x,i,i)+m(x,-i,-i))}\right)\cdot \\ \cdot\prod_{x,i}\left[\int\frac{\mathrm{d}\psi\,\mathrm{d}\psi^*}{\pi}\exp\left[-\psi^*\psi\right]\psi^{n_i(x)}\psi^{*n_i^*(x)}\right],$$

we see that the integrals over $\mathrm{d}\psi\,\mathrm{d}\psi^*$ play the role of knitting together hooks if their numbers of like elements on each link matches, i.e. if $n_i(x)=n_i^*(x)$.

Among all these hooks, there are also the immediate back-trackers or spikes. These are counted by $m(x,i,i)$ or $m(x,-i,-i)$. They are accompanied by a factor

$$\left(\frac{1}{2}\right)^{\sum\limits_{x,i}(m(x,i,i)+m(x,-i,-i))},$$

since the branches of the spikes are indistinguishable.

If we want to construct a disorder field theory for random loops in which these spikes are forbidden, we simply have to omit in (16) the sums over $m(x,i,i)$, $m(x,-i,-i)$. Going back to the exponential (12) we see that this is achieved by omitting the terms $\psi_i^2(x)+\psi_{-2}^{*2}(x)$ in (12). The partition function of random loops without spikes is, therefore,

$$(17)\qquad Z_{\text{no spikes}}=\prod_x\left[\int\frac{\mathrm{d}\varphi(x)}{\sqrt{2\pi}}\right]\prod_{x,i}\left[\int\frac{\mathrm{d}\psi_i(x)\,\mathrm{d}\psi_i^*(x)}{\pi}\right]\exp\left[-\frac{1}{2}\sum_{x,i}\phi^2(x)-\sum_x\psi_i^*(x)\psi_i(x)\right]\cdot \\ \cdot\exp\left[\sqrt{z}\sum_{x,i}\phi(x)\big(\psi_i(x)+\psi_i^*(x-i)\big)-\frac{z}{2}\sum_{x,i}\left(\psi_i(x)^2+\psi_i^*(x-i)^2\right)\right].$$

Using translational invariance, the exponent can be rewritten as

$$(18)\qquad \frac{1}{2}\sum_x\phi^2(x)+\frac{\sqrt{z}}{2}\sum_{x,i}\big(\psi_i(x)+\psi_i^*(x)\big)\big(\phi(x)+\phi(x+i)\big)+ \\ +\frac{\sqrt{z}}{2}\sum_{x,i}\big(\psi_i(x)-\psi_i^*(x)\big)\big(\phi(x)-\phi(x+i)\big)-\sum_{x,i}\psi_i^*(x)\psi_i(x)-\frac{z}{2}\sum_{x,i}\left(\psi_i(x)^2+\psi_i^*(x)^2\right).$$

Thus, if ψ_i^1, ψ_i^2 denote the real and imaginary parts of ψ_i, we have

$$(19)\qquad \tfrac{1}{2}\sum\phi^2(x)+\sqrt{z}\sum_x\psi_i^1(x)\big(\phi(x)+\phi(x+i)\big)+i\sqrt{z}\sum_x\psi_i^2(x)\big(\phi(x)-\phi(x+i)\big)- \\ -\sum_x\left[(1+z)\psi_i^1(x)^2+(1-z)\psi_i^2(x)^2\right].$$

After a quadratic completion, the fields $\psi_i^{1,2}(x)$ can be integrated out and we obtain

$$(20)\qquad Z_{\text{no spikes}}=A\prod_x\int\frac{\mathrm{d}\phi(x)}{\sqrt{2\pi}}\exp\left[S[\phi]\right],$$

where

$$A = \sqrt{1 - z^2}^{-(D-1)N} \sqrt{1 + z^2(2D-1)}^{-N}$$

and

$$S[\phi] = -\tfrac{1}{2} \sum_{x,x'} \phi(x) \tilde{G}^{-1}(x, x') \phi(x') \tag{21}$$

is the field entropy with

$$\tilde{G}^{-1}(x, x') \equiv \delta_{x,x'} - \tilde{z} H_{x,x'} \equiv \delta_{x,x'} - \frac{z}{1 + (2D-1) z^2} H_{x,x'} , \tag{22}$$

being the inverse propagator.

Thus, apart from a wave function renormalization A, the random loops without spikes follow again a free disorder field theory with a modified fugacity

$$\tilde{z} = \frac{z}{1 + (2D-1) z^2} . \tag{23}$$

The mass of this field is

$$\tilde{m}^2 = \frac{1}{\tilde{z}} - 2D = \frac{1}{z} - 2D + (2D-1) z . \tag{24}$$

This shows that such an ensemble undergoes a phase transition if the temperature exceeds

$$\tilde{T}_c = \varepsilon / \log \frac{1}{z_c} = \varepsilon / \log (2D - 1) . \tag{25}$$

This value was to be expected on naive grounds, since $\log (2D - 1)$ is the entropy of a single step in a random walk which cannot back-track right away.

Notice that this result implies the partition function of the ensemble *without spikes* (4) to exponentiate as

$$Z_{\text{no spikes}} = A \exp [\tilde{Z}_1] ,$$

where $\tilde{Z}_1$ is the single random walk *with spikes*, but with the modified fugacity $\tilde{z}$.

(4) For field theories with the stronger restriction of complete self-avoidance see T. HOFSÄSS and H. KLEINERT: *Phys. Lett. A*, **105**, 60, 463 (1984).