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Gauge theory of time-dependent stresses and defects: quantum defect dynamics

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Abstract. We construct the two-dimensional quantum field theory which governs an idealised ensemble of dislocations and disclinations, including their higher gradient elastic interactions. The action contains four gauge fields of phonons A_i, C_{ij} and H, D_i, coupled minimally to two complex Higgs fields φ_i, φ, which are the disorder fields of dislocations and disclinations, respectively. Because of the close analogy with the quantum field theory of electrons and photons, called quantum electrodynamics (QED), the new theory of defects and phonons may be named quantum defect dynamics (QDD).

The interaction of defects in a crystal is described by field equations which bear a close resemblance to the Maxwell–Lorentz theory of electrons [1]‡. The electromagnetic fields B_i and E_i correspond to the stress tensor σ_{ij} and the momentum density p_i, respectively. The local coupling of the gauge field A_μ with the conserved electron current J_μ corresponds to the coupling of the stress gauge field with the conserved defect tensor.

The principal difference between the two systems lies in the fact that electrons describe world lines in 4-space with currents

$$J_\mu = \int_L \mathrm{d}s\, \mathrm{d}x'_\mu/\mathrm{d}s\, \delta^{(4)}(x - x'(s))$$

while defects form lines in real space, such that in four dimensions they form world sheets. In two-dimensional systems, however, the analogy between the two systems is very close: defects are point-like and form world lines in spacetime just as electrons. It is the purpose of this paper to exploit this analogy and perform, in the defect system, the same steps which lead from the Maxwell–Lorentz theory to quantum electrodynamics. The result will be a simple field theoretic action which governs the quantum phenomena of defects and phonons. It may be called quantum defect dynamics (QDD). The defects appearing in this theory are *idealised* objects. They can freely glide and climb. Processes which impede the motion of real defects [2] must be included separately. A similar treatment has been given before to vortices in films of superfluid helium (quantum vortex dynamics) [3] which the reader may find useful to read before studying the present more complicated defect problem.

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‡ See also [1a].

The starting point is the quantum mechanical partition function† of stress fluctuations in the presence of plastic distortions

$$Z=\int \mathscr{D}u(x)\exp\left(\frac{\mathrm{i}}{\hbar}\mathscr{A}\right) \tag{1}$$

with the action

$$\mathscr{A}=\int \mathrm{d}^3x[\tfrac{1}{2}(\partial_0 u_i-\beta_{0i}{}^{\mathrm{P}})^2+\tfrac{1}{2}\theta(\partial_0\omega-\kappa_0{}^{\mathrm{P}})^2-\tfrac{1}{4}(\partial_i u_j+\partial_j u_i-\beta_{ij}{}^{\mathrm{P}}-\beta_{ji}{}^{\mathrm{P}})^2$$
$$-\tfrac{1}{2}\lambda(\partial_i u_i{}^{\mathrm{P}}-\beta_{ii}{}^{\mathrm{P}})^2-2\mu l^2(\partial_i\omega-\kappa_i{}^{\mathrm{P}})^2]. \tag{2}$$

We have used natural units in which the transverse sound velocity $C_t=(\mu/\rho)^{1/2}$ and the shear modulus μ are both equal to one‡. The constant l^2 controls higher gradient elasticity (we have omitted gradients of the strain tensor since they produce no interesting new qualitative structures). The gradients of ω are necessary in order to acquire sensitivity to disclinations [4]§. The plastic quantities $\beta_{\mu i}{}^{\mathrm{P}}$, $\chi_\mu{}^{\mathrm{P}}$ are given by

$$\begin{aligned}
&\beta_{ij}{}^{\mathrm{P}}(\boldsymbol{x})=\delta_i(S)(b_j-\Omega\varepsilon_{jr}x_r)\\
&\kappa_i{}^{\mathrm{P}}(\boldsymbol{x})=\varepsilon_{kl}\partial_i\beta_{kl}{}^{\mathrm{P}}+\phi_i{}^{\mathrm{P}} \qquad \phi_i{}^{\mathrm{P}}=\delta_i(S)\Omega\\
&\beta_{0i}{}^{\mathrm{P}}(\boldsymbol{x})=-v_k\delta_k(S)(b_i-\Omega\varepsilon_{ir}x_r)\\
&\kappa_0{}^{\mathrm{P}}(\boldsymbol{x})=\varepsilon_{kl}\partial_0\beta_{kl}{}^{\mathrm{P}}+\phi_0{}^{\mathrm{P}} \qquad \phi_0{}^{\mathrm{P}}=-v_k\delta_k(S)\Omega
\end{aligned} \tag{3}$$

where b_i are the Burgers' vectors, Ω is the Frank scalar, and S is the *time-dependent* Volterra cutting surface, which in two dimensions is really a line and v_k the velocity with which S moves through space. The δ function $\delta_i(S)$ is singular on S and points along the normal vector. Since S is a line, we may also write $\delta_i(S)$ as $-\varepsilon_{ij}\tilde{\delta}_j(S)$ where $\tilde{\delta}_j(S)=\int \mathrm{d}s\,\mathrm{d}x_j'/\mathrm{d}s\,\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}'(s,t))$. We shall keep the notation (3) because of its analogy with the three-dimensional situation.

The stresses and torque stresses are introduced by taking (1) to the canonical form [4]

$$Z=\int \mathscr{D}u_i(x)\int \mathscr{D}\sigma_{ij}(x)\int \mathscr{D}\omega(x)\int \mathscr{D}\tau_i(x)\int \mathscr{D}p_i\int \mathscr{D}\pi\exp\left(\frac{\mathrm{i}}{\hbar}\mathscr{A}_{\mathrm{can}}\right) \tag{4}$$

$$\mathscr{A}_{\mathrm{can}}=\int \mathrm{d}^3x\left\{\left[-\frac{1}{2}p_i^2-\frac{1}{2\theta}\pi^2+\frac{1}{4}\left(\overset{\mathrm{s}}{\sigma}{}_{ij}^2-\frac{\nu}{1+\nu}\overset{\mathrm{s}}{\sigma}{}_{ll}^2\right)+\frac{1}{8l^2}\tau_i^2\right]-\sigma_{ij}(\partial_i u_j-\varepsilon_{ij}\omega-\beta_{ij}{}^{\mathrm{P}})\right.$$
$$\left.-\tau_i(\partial_i\omega-\phi_i{}^{\mathrm{P}})+p_i(\partial_0 u_i-\beta_{0i}{}^{\mathrm{P}})+\pi(\partial_0\omega-\phi_0{}^{\mathrm{P}})\right\}$$
$$\equiv \mathscr{A}_0+\mathscr{A}_{\mathrm{int}} \tag{5}$$

where the elastic energy depends only on the symmetric part $\overset{\mathrm{s}}{\sigma}{}_{ij}$ of σ_{ij}. The integration over the antisymmetric part enforces the connection between ω and $\frac{1}{2}\varepsilon_{ij}\partial_i u_j$, modulo the plastic part $\frac{1}{2}\varepsilon_{ij}\beta_{ij}{}^{\mathrm{P}}$. Integrating out $u_i(x)$ and $\omega(x)$ produces the conservation laws

$$\partial_i\sigma_{ij}=\partial_0 p_j \qquad \partial_i\tau_i=\partial_0\pi-\varepsilon_{kl}\sigma_{kl}. \tag{6}$$

† If external currents are added, this object permits calculating all correlation functions. The sources are omitted for brevity.

‡ Notation: $x^0=t=$ time, $\boldsymbol{x}=(x^i)=(x^1,x^2)=$ space, $x=(x^\mu)=(x^0,x^1,x^2)$, $\mathrm{d}^3x=\mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2$, $\partial_\mu=\partial/\partial x^\mu$.

§ For static interaction energies of defects within higher gradient elasticity, in three dimensions, see [4a].

They can be fulfilled by introducing the phonon (or stress) gauge fields A_i, H, c_{ij}, d_{ij}:

$$\sigma_{ij}=\varepsilon_{ik}\partial_k A_j+\partial_0 c_{ij} \qquad p_j=\partial_i c_{ij}$$
$$\tau_i=\varepsilon_{ik}\partial_k H-A_i+\partial_0 d_i \qquad \pi=\partial_i d_i+\varepsilon_{ij}c_{ij}. \tag{7}$$

The gauge transformations which leave this decomposition invariant are somewhat degenerate, due to the reduced dimensionality of space†:

$$A_i\to A_i+\partial_i\xi+\partial_0\Lambda_i \qquad H\to H+\partial_0\xi \tag{8}$$
$$c_{ij}\to c_{ij}-\varepsilon_{ik}\partial_k\Lambda_j \qquad d_i\to d_i-\varepsilon_{ik}\partial_k\xi+\Lambda_i. \tag{9}$$

It is useful to introduce $A_{ij}=\varepsilon_{il}c_{lj}$, $H_i=\varepsilon_{il}d_l$, such that (9) becomes

$$A_{ij}\to A_{ij}+\partial_i\Lambda_j \qquad H_i\to H_i+\partial_i\xi+\varepsilon_{il}\Lambda_l. \tag{9'}$$

Inserting (7) into (5), the interaction with the defects can be brought to the form

$$\begin{aligned}\mathscr{A}_{\text{int}}=\int \mathrm{d}^3x[&A_i(\varepsilon_{kj}\partial_k\beta_{ji}-\phi_i^{\mathrm{P}})+H(\varepsilon_{kj}\partial_k\phi_j^{\mathrm{P}})\\&-A_{ij}\varepsilon_{il}(\partial_0\beta_{lj}^{\mathrm{P}}-\partial_l\beta_{0j}^{\mathrm{P}}+\varepsilon_{lj}\phi_0^{\mathrm{P}})-H_i\varepsilon_{il}(\partial_0\phi_l^{\mathrm{P}}-\partial_l\phi_0^{\mathrm{P}})]\\=\int \mathrm{d}^3x(&A_i\alpha_i+H\theta-A_{ij}J_{ij}-H_jS_j).\end{aligned}\tag{10}$$

The sources

$$\begin{aligned}&\alpha_i\equiv\varepsilon_{kj}\partial_k\beta_{ji}^{\mathrm{P}}-\phi_i^{\mathrm{P}} &&\theta\equiv\varepsilon_{kj}\partial_k\phi_j^{\mathrm{P}}\\&J_{ij}=\varepsilon_{il}(\partial_0\beta_{lj}^{\mathrm{P}}-\partial_l\beta_{0j}^{\mathrm{P}}+\varepsilon_{lj}\phi_0^{\mathrm{P}}) &&S_i=\varepsilon_{il}(\partial_0\phi_l^{\mathrm{P}}-\partial_l\phi_0^{\mathrm{P}})\end{aligned}\tag{11}$$

are identified with dislocation density, disclination density, and their respective currents. Inserting (3) we find explicitly

$$\begin{aligned}&\alpha_j(x)=\delta(L(t))(b_j-\Omega\varepsilon_{jr}x_r) &&\theta(x)=\delta(L(t))\Omega\\&J_{ij}(x)=-v_i\delta(L(t))(b_{ij}-\Omega\varepsilon_{jr}x_r) &&S_i(x)=-v_i\delta(L(t))\Omega\end{aligned}\tag{12}$$

where L is the boundary 'line' of the Volterra cutting surface S which, in two dimensions, consists of the two end points. The $\delta(L)$ function is positive on the one and negative on the other end point.

The densities and currents obviously satisfy the conservation laws

$$\partial_iJ_{ij}=\partial_0\alpha_j-\varepsilon_{ji}S_i \qquad \partial_iS_i=\partial_0\theta. \tag{13}$$

These are necessary to ensure gauge invariance under (8) and (9′).

Notice that the plastic quantities in (11) are gauge fields on their own‡. Defect gauge transformations correspond to changing the shape of the Volterra cutting surface. Indeed, under $S\to S'$ we find that $\delta_i(S')=\delta_i(S)-\partial_i\delta(V)$ where V is the volume (here area) over which the surface S has swept. From (3) we see that under such a change

$$\begin{aligned}&\beta_{ij}^{\mathrm{P}}\to\beta_{ij}^{\mathrm{P}}+\partial_iN_j-\varepsilon_{ij}M &&\phi_i^{\mathrm{P}}\to\phi_i^{\mathrm{P}}+\partial_iM\\&\beta_{0j}^{\mathrm{P}}\to\beta_{0j}^{\mathrm{P}}+\partial_0N_j &&\phi_0^{\mathrm{P}}\to\phi_0^{\mathrm{P}}+\partial_0M\end{aligned}\tag{14}$$

† For the full three-dimensional gauge transformations in the static case see [4].
‡ The double gauge properties of elasticity and plasticity are discussed in [5].

where $M=-\delta(V)\Omega$, $N_i=-\delta(V)(b_i-\Omega\varepsilon_{ir}x_r)$. These transformations obviously preserve (11). Separating out self-energies of the defects [6–8] we arrive at a partition function†

$$Z=\int \mathscr{D}A_i\mathscr{D}H\mathscr{D}A_{ij}\mathscr{D}H_i \overset{\text{phon}}{\Phi}[A_i, H, A_{ij}, H_i]\exp\left(\frac{\mathrm{i}}{\hbar}\mathscr{A}_0'\right)$$
$$\times\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x(A_i\alpha_i+H\theta-A_{ij}J_{ij}-H_iS_i)\right)$$
$$\times\exp\left[-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x\left(\frac{1}{2\varepsilon_1}(\alpha_i^2-J_{ij}^2)+\frac{1}{2\varepsilon_2}(\theta^2-S_i^2)\right)\right] \tag{15}$$

where $\mathscr{A}_0'$ is the square bracket of the action (5), expressed in terms of the gauge fields, but modified at short distances, such as to separate out the core energies in the last line. The symbol Φ^{phon} denotes a gauge fixing functional for the phonon gauge fields. The defect partition function (15) is the analogue of the Maxwell–Lorentz theory of the electron

$$Z=\int \mathscr{D}A_\mu \overset{\text{phot}}{\Phi}[A_\mu]\exp\left(\frac{1}{4\hbar}\int \mathrm{d}^4xF_{\mu\nu}^2\right)\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t(\varphi-A_i\dot{x}_i)\right)$$
$$\times\exp\left(-\frac{im}{\hbar}\int \mathrm{d}t(1-\dot{x}^2)^{1/2}\right). \tag{16}$$

In order to turn (15) into the desired quantum field theory of defects and phonons we have to remember how the quantum field theory of electrons and photons may be obtained from (16). All we have to do is sum in Z over all random orbits of electrons, with specific constraints, such as to respect Pauli's exclusion principle. In the present case of defects we may simply sum over all non-backtracking world lines of dislocations and disclinations in the (2+1)-dimensional spacetime. Explicitly, this is most easily done by remembering that in a proper crystal, the plastic quantities β_{ij}^{P}, ϕ_i^{P} are really discrete. For example, in a simple cubic lattice with lattice spacing $a=2\pi$ (say) one has $\beta_{ij}=2\pi n_{ij}$, where n_{ij} are all integer numbers [4–7]. They present the jumps in the position variable u_j across the links i, thus parametrising an ensemble of Volterra cutting surfaces S. Similarly, we discretise ϕ_j^{P} to parametrise the jumping surfaces of the rotation angle ω. Taking a similar lattice spacing also for the time variable (which is taken to be zero at the end) the surfaces undergo a hopping motion as a function of time.

We are therefore led to describing the ensemble of all fluctuating defects by performing, on the second and third exponential in (15), the sum over all these jumping numbers, $\Sigma_{\{n_{ij}n_{0i},m_i,m_0\}}$. Since these are integer valued gauge fields, the sum requires a gauge fixing functional $\Phi^{\text{def}}[n_{ij},n_{0i},m_i,m_0]$.

It is now straightforward to transform this sum into a disorder field theory of dislocations and disclinations. The technique for doing so has been developed before [4a, 7, 8] and is explained in detail in reference [7].

† For brevity, we have omitted another possible invariant $J_{ij}J_{ji}$.

Four steps are necessary. First we observe that on the lattice, the defect sum

$$Z_{\rm def} = \sum_{\{n_{ij}, n_{0i}, m_{ij}, m_0\}}^{\rm def} \Phi\, [n_{ij}, n_{0i}, m_{ij}, m_0]$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x\left(-\frac{1}{2\varepsilon_1}(\alpha_i^2 - J_{ij}^2) - \frac{1}{2\varepsilon_2}(\theta^2 - S_j^2)\right.\right.$$
$$\left.\left. + A_i\alpha_i + H\theta - A_{ij}J_{ij} - H_iS_i\right)\right] \tag{17}$$

can be transformed, via a simple manipulation [8, 9], into a dual form†

$$Z_{\rm def} = \sum_{\{\tilde n_{ij}, \tilde n_{0i}, \tilde m_i, \tilde m_0\}} \tilde\Phi[\tilde n_{ij}, \tilde n_{0i}, \tilde m_i, \tilde m_0] \prod_{x,i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\gamma_i(x)}{2\pi}\prod_x\int_{-\infty}^{\infty}\frac{\mathrm{d}\delta(x)}{2\pi}$$
$$\times \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x[\tfrac{1}{2}\varepsilon_1(\nabla_0\gamma_i - A_i - 2\pi\tilde n_{0i})^2 - \tfrac{1}{2}\varepsilon_1(\nabla_i\gamma_j - A_{ij} - 2\pi\tilde n_{ij})^2\right.$$
$$\left. + \tfrac{1}{2}\varepsilon_2(\nabla_0\delta - H - 2\pi\tilde m_0)^2 - \tfrac{1}{2}\varepsilon_2(\nabla_i\delta + \varepsilon_{ij}\gamma_j - H_i - 2\pi\tilde m_i)]\right) \tag{18}$$

where ∇_0, ∇_i are lattice gradients and the integrations over γ_i, δ ensure the defect conservation laws (12). Second we remove an integer valued field $\tilde N_j$, $\tilde M$ from γ_j, δ and restrict these angles to the interval $(-\pi, \pi)$ only. The removed gradients $\nabla_\mu \tilde N_i$, $\nabla_\mu \tilde M$ can be absorbed into the integer valued gauge fields making the sum over $\tilde n_{ij}\tilde n_{0i}\tilde m_i\tilde m_0$ in (18) *unrestricted* (i.e. we can drop $\tilde\Phi$).

Third we use the Villain approximation‡

$$\sum_{\tilde n_\mu}\exp(-\tfrac{1}{2}\beta(\nabla_\mu\gamma - 2\pi\tilde n_\mu)^2 \approx R_{V^{-1}}(\beta)\exp(\beta_{V^{-1}}(\beta)\cos\nabla_\mu\gamma)$$
$$= R_{V^{-1}}(\beta)\exp(\beta_{V^{-1}}(\beta)\,\mathrm{Re}\,U_a(x)U_a^+(x+\mu)) \tag{19}$$

to rewrite the exponents in (17) in a two-vector form where

$$U_a(x) \equiv (\cos\gamma(x), \sin\gamma(x)).$$

Finally, we use the identity

$$f(U, U^+) = \int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{d}u^+\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\alpha\,\mathrm{d}\alpha^+}{(2\pi\mathrm{i})^2}f(u, u^+)\exp[-\tfrac{1}{2}(\alpha^+(u-U)+\mathrm{cc})] \tag{20}$$

to rewrite the integral of (19) over γ as follows:

$$R_{V^{-1}}(\beta)\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{d}u^+\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\alpha\,\mathrm{d}\alpha^+}{(2\pi\mathrm{i})^2}$$
$$\times\exp[\beta_{V^{-1}}\,\mathrm{Re}\,u^+(x)u(x+\mu) - \tfrac{1}{2}(\alpha^+u+\mathrm{cc}) + \log I_0(|\alpha|)]. \tag{21}$$

† The numbers $\tilde n_{\mu j}$, $\tilde m_\mu$ are integer valued gauge fields which are *dual* to the defect fields $n_{\mu j}$, m_μ. They represent the vortex lines in the disorder fields of (24).

‡ With $R_{V^{-1}}(\beta) \equiv (I_0(\beta)_{V^{-1}}\sqrt{2\pi\beta})^{-1}$, $\beta \equiv -1/(2\log(I_1(\beta)_{V^{-1}}/I_0(\beta)_{V^{-1}}))$; I_0, I_1 = associated Bessel functions. For more details see [11].

Performing these four steps on (18) leads to the following disorder field theory of the two dislocation fields and the disclination field:

$$Z_{\rm def}\propto\prod_{x,i}\int_{-\infty}^{\infty}\mathrm{d}u_i\,\mathrm{d}u_i^+(x)\int_{-\infty}^{\infty}\mathrm{d}v\,\mathrm{d}v^+(x)\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\alpha_i\,\mathrm{d}\alpha_i^+(x)}{(2\pi\mathrm{i})^2}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\zeta\,\mathrm{d}\zeta^+(x)}{(2\pi\mathrm{i})^2}$$

$$\times\exp-\mathrm{i}\Bigg[\left(\frac{\varepsilon_1}{\hbar}\right)_{V^{-1}}\sum_{x,j}\mathrm{Re}\,u_j^+(x)u_j(x+0)\exp(-\mathrm{i}A_j(x))$$

$$+\left(\frac{\varepsilon_2}{\hbar}\right)_{V^{-1}}\sum_{x}\mathrm{Re}\,v^+(x)v(x+0)\exp(-\mathrm{i}H(x))$$

$$-\left(\frac{\varepsilon_1}{\hbar}\right)_{V^{-1}}\sum_{x,j,i}\mathrm{Re}\,u_j^+(x)u_j(x+i)\exp(-\mathrm{i}A_{ij}(x))$$

$$-\left(\frac{\varepsilon_2}{\hbar}\right)_{V^{-1}}\sum_{x}\mathrm{Re}[v^+(x)v(x+1)\exp(-\mathrm{i}H_1(x))u_2(x)$$

$$+v^+(x)v(x+2)\exp(-\mathrm{i}H_2(x))u_1^*(x)]$$

$$-\tfrac{1}{2}\sum_{x}(\alpha_i^+u_i+\zeta^+v+\mathrm{cc})-\sum_{x,i}\log I_0(|\alpha_i|)-\sum_{x}\log I_0(|\zeta|)\Bigg]\qquad(22)$$

where we have dropped trivial overall constants. It is useful to define combinations like

$$\begin{aligned}D_i^{A_{ij}}u_j(x)&\equiv u_j(x+i)\exp(-\mathrm{i}A_{ij}(x))-u_j(k)\\ \bar{D}_i^{A_{ij}}u_j(x)&\equiv u_j(x)-u_j(k-i)\exp[+\mathrm{i}A_{ij}(x-i)]\end{aligned}\qquad(23)$$

as covariant lattice derivatives. Then it is easy to perform the following manipulation:

$$\begin{aligned}&\sum_x\mathrm{Re}\,u_j^+(x)u_j(x+i)\exp(-\mathrm{i}A_{ij}(x))\\ &\quad=\tfrac{1}{2}\sum_{2}\{u_j^+(x)[u_j(x+i)\exp(-\mathrm{i}A_{ij}(x))-u_j(x)]+u_j^+u_j+\mathrm{cc}\}\\ &\quad=\tfrac{1}{2}\sum_{x}[u_j^+(x)D_i^{A_{ij}}u_j(x)+(D_i^{A_{ij}}u_j)^+u_j(x)+2u_j^+u_j]\\ &\quad=\tfrac{1}{2}\sum_{x}[u_j^+(x)(D_j^{A_{ij}}-\bar{D}_i^{A_{ij}})u_j(x)+2u_j^+u_j]\\ &\quad=\sum_{x}u_j^+(x)(1+\bar{D}_i^{A_{ij}}D_i^{A_{ij}}/2)u_j(x).\end{aligned}\qquad(24)$$

Similarly we define†

$$\begin{aligned}D_i^{H_i,u}v(x)&\equiv v(x+i)\exp(-\mathrm{i}H_i(x))u(x)-v(x)\\ \bar{D}_i^{H_i,u}v(x)&=v(x)-v(x-i)\exp[\mathrm{i}H_i(x-i)]u(x-i).\end{aligned}\qquad(25)$$

† This satisfies $(D-\bar{D})v(x)=\bar{D}Dv+[|u|^2(x-i)-1]v$, in contrast to $D^{A_{ij}}$ of (24) where the second term is absent.

Then the first two terms in the exponent of (21) become simply

$$\left(\frac{\varepsilon_1}{\hbar}\right)_{V^{-1}} \sum_{x,j} \operatorname{Re} u_j^+(x)\left(1+\frac{\bar{D}_0^{A_j} D_0^{A_j}}{2}\right) u_j(x)+\left(\frac{\varepsilon_2}{\hbar}\right)_{V^{-1}} \sum_{x} \operatorname{Re} v(x)\left(1+\frac{\bar{D}_0^{H} D_0^{H}}{2}\right) v(x)$$
$$-\left(\frac{\varepsilon_1}{\hbar}\right)_{V^{-1}} \sum_{x,j} \operatorname{Re} u_j^+(x)\left(2+\tfrac{1}{2}\sum_i \bar{D}_i^{A_{ij}} D_i^{A_{ij}}\right) u_1(x)$$
$$-\left(\frac{\varepsilon_2}{\hbar}\right)_{V^{-1}} \sum_{x} \operatorname{Re} v^+(x)\left(2+\frac{D_1^{H_1 u_2}-\bar{D}_1^{H_1 u_2}}{2}+\frac{D_2^{H_2 u_1^*}-\bar{D}_2^{H_2 u_1^*}}{2}\right) v(x). \tag{26}$$

For smooth field configurations and small gauge fields, the covariant derivatives $D_i^{A_{ij}}, \bar{D}_i^{A_{ij}}, \ldots$ become simply $(\partial_i - \mathrm{i}A_i)$. The last covariant derivatives in (26) which account for the coupling between dislocations and disclinations via the first defect current conservation law (13) always have a non-trivial form. Even if $v(x)$ is smooth and H_i are small it reduces merely to

$$D_i^{H_i u_j} v(x) \rightarrow [u_j(x)(\partial_i - \mathrm{i}H_i) + u_j - 1] v(x) + \tfrac{1}{2} u_j(x)[(\partial_i - \mathrm{i}H_i)^2 + \mathrm{i}\partial_i H_i] v(x) + \ldots$$
$$\bar{D}_i^{H_i u_j} v(x) \rightarrow [u_j^*(k-i)(\partial_i - \mathrm{i}H_i) + 1 - u_j^*(x-i)] v(x)$$
$$-\tfrac{1}{2} u_j^*(x-i)[(\partial_i - \mathrm{i}H_i)^2 - \mathrm{i}\partial_i H_i] v(x) + \ldots. \tag{27}$$

In the cold phase, in which the expectation of dislocations is zero and $\langle u \rangle = 0$, disclinations have a vanishing next-neighbour coupling in (26). In the molten phase, however, where $u \rightarrow 1$, they move like ordinary particles with the usual gradient term $(\partial_i - \mathrm{i}H_i)^2$.

If the system has a second-order melting transition, which in two dimensions could be possible, due to quantum fluctuations, we can perform, close to T_c, a Landau expansion and obtain† the following partition function

$$Z = \int \mathscr{D}u_j \mathscr{D}u_j^+ \mathscr{D}v \mathscr{D}v^+ \mathscr{D}A_i \mathscr{D}H \mathscr{D}A_{ij} \mathscr{D}H_i \overset{\text{phon}}{\Phi} \exp \frac{\mathrm{i}}{\hbar} \mathscr{A}$$

with

$$\mathscr{A} = \int \mathrm{d}^3 x \bigg[-\tfrac{1}{2} p_i^2 - \frac{1}{2\theta} \pi^2 + \tfrac{1}{4}\left(\overset{s}{\sigma}_{ij}{}^2 - \frac{v}{1+v} \overset{s}{\sigma}_l{}^2\right) + \frac{1}{\gamma l^2} \tau_i^2$$
$$+ \hbar\left(\frac{\varepsilon_1}{\hbar}\right)_{V^{-1}} [\tfrac{1}{2}|(\partial_0 - \mathrm{i}A_j) u_j|^2 - \tfrac{1}{2}|(\partial_i - \mathrm{i}A_{ij}) u_j|^2]$$
$$+ \hbar\left(\frac{\varepsilon_2}{\hbar}\right)_{V^{-1}} \Big(\tfrac{1}{2}|(\partial_0 - \mathrm{i}H) v|^2 - (u_2 + u_2^*)[\tfrac{1}{2}|(\partial_1 - \mathrm{i}H_1) v|^2 + \tfrac{1}{2} v^+ v]$$
$$-(u_1 + u_1^*)[\tfrac{1}{2}|(\partial_2 - \mathrm{i}H_2) v|^2 + \tfrac{1}{2} v^+ v]$$
$$-\mathrm{i}(u_2 - u_2^*) \frac{1}{2\mathrm{i}} v^+ (\overleftrightarrow{\partial_1 - \mathrm{i}H_1}) v - \mathrm{i}(u_1^* - u_1) \frac{1}{2\mathrm{i}} v^+ (\overleftrightarrow{\partial_2 - \mathrm{i}H_2}) v \Big)$$
$$-\tfrac{1}{2} m_u^2 \sum_i u_j^2 - \tfrac{1}{4} q_u \sum_i u_j^4 - \tfrac{1}{2} m_v^2 v^2 - \tfrac{1}{2} q_v v^4 - \ldots \bigg]. \tag{28}$$

† After integrating out α_i and ζ in (22).

This partition function governs the quantum phenomena of defect systems in the continuum limit. The quantum statistical mechanics can be studied in the usual way by considering time to be an imaginary quantity with periodic boundary conditions on all fields in the interval $\tau = -\mathrm{i}t \in (0, 1/T)$.

It must be realised that, except for extreme quantum crystals, the melting transition is really of first order and the correlation lengths of the disorder fields never grow to infinity. This precludes one from taking a proper continuum limit and the disorder field theory must be used in its full lattice formulation (22). The limit (28) is, however, structurally interesting since it establishes the close correspondence of QDD with QED which for 'scalar electrons' would read

$$Z = \int \mathscr{D}A_\mu \overset{\text{phot}}{\Phi}[A_\mu] \exp\left(\frac{\mathrm{i}}{\hbar} \int \mathrm{d}^4x \, \tfrac{1}{4} F_{\mu\nu}{}^2\right)$$

$$\times \int \mathscr{D}\varphi \mathscr{D}\varphi^+ \exp\left(\frac{\mathrm{i}}{\hbar} \int \mathrm{d}^4x [\tfrac{1}{2}|(\partial_\mu - \mathrm{i}A_\mu)\varphi|^2 - \tfrac{1}{2}m^2|\varphi|^2 - \tfrac{1}{4}q|\varphi|^4]. \qquad (29)$$

The new quantum field theory will hopefully be useful for understanding the dynamic plastic properties of crystal as well as the defect mediated melting transition.

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