Renormalization of Charge in Villain Lattice Gauge Theory

H. Kleinert[a]
Department of Physics, University of California, San Diego, La Jolla, California 92093

and

W. Miller
Institut für Theorie der Elementarteilchen, 1 Berlin 33, Germany
(Received 3 September 1985)

We calculate the renormalization of charge in the Villain version of the U(1) lattice gauge theory due to monopole loops with their Biot-Savart-type long-range interactions, up to tenth order. The agreement with recent Monte Carlo data is very good except in the extreme vicinity of the critical point.

PACS numbers: 11.15.Ha, 11.10.Gh

Recently, the Villain form of the U(1) lattice gauge theory (LGT) in four dimensions has been investigated via Monte Carlo methods with great accuracy.[1,2] One of the reasons[3] is the dual relationship of this model with the Abelian Higgs model,[4] which is a prototype model for spontaneous symmetry breakdown in superconductors and unification schemes of weak and electromagnetic interactions. The U(1) LGT contains photons plus defects which are closed loops representing world lines of magnetic monopoles. If the stiffness parameter β is decreased, small loops are excited and renormalize the effective electric charge $e=1/\sqrt{\beta}$. At a certain critical stiffness, the entropy overwhelms the energy, the monopole loops become infinitely long, and the system goes into a phase in which the magnetic charges $m=2\pi\sqrt{\beta}$ are screened. This is the phase in which electric charges are confined.

The renormalization of charge due to magnetic monopoles is very similar to the reduction of stiffness, in the Villain form of the two-dimensional XY model, due to vortices which form a Coulomb gas in a plane. In the latter case, there exists a powerful renormalization-group approach which permits calculation of the renormalization up to the critical point.[5] In the U(1) LGT, there have been two attempts at explanation of the renormalization,[6] neither of them, however, being able to find a scaling equation valid near the critical point.

The purpose of this note is to present a systematic $\sum_{\text{loops}} e^{-\beta\epsilon}$ expansion of the charge renormalization into monopole loops of increasing length: 4, 6, 8, and 10. We count the loops explicitly, calculate their magnetic interaction energy ϵ on the lattice, and sum up their contributions to the charge renormalization (all by hand). The counting process is quite tedious because of the large number of different graphs at the level of tenth order, most of them having different magnetic interaction energies. Their contribution to the renormalization of the stiffness is given *exactly* by the formula

$$\beta_R = 1 - \tfrac{1}{6}\pi^2\beta\sum_{\mathbf{x},\mu}\mathbf{x}^2\langle l_\mu(\mathbf{x})\,l_\mu(0)\rangle\,. \tag{1}$$

The integer numbers on the four links μ, $l_\mu(\mathbf{x})$, satisfy $\overline{\nabla}_\mu l_\mu(\mathbf{x})\equiv l_\mu(\mathbf{x})-l_\mu(\mathbf{x}-\mu)=0$, parametrizing the closed monopole loops on the cubic lattice. The expectation values are taken with respect to the loop partition function

$$Z=\sum_{\{l_\mu(\mathbf{x})\}}\delta_{\overline{\nabla}_\mu l_\mu(\mathbf{x}),0}\exp\left[-\frac{\beta}{2}4\pi^2\sum_{\mathbf{x},\mu}l_\mu(\mathbf{x})v(\mathbf{x}-\mathbf{x}')\,l_\mu(\mathbf{x}')\right], \tag{2}$$

where

$$v(\mathbf{x})=\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}e^{i\mathbf{k}\cdot\mathbf{x}}\frac{1}{K_\mu\overline{K}_\mu} \tag{3}$$

(with $K_\mu\equiv e^{ik_\mu}-1$) is the Biot-Savart-type magnetic interaction on the lattice, which shows that $m=2\pi\sqrt{\beta}$ is the magnetic charge in accordance with Dirac's relation $me=2\pi$.

The proof of formula (1) is quite straightforward. The U(1) Villain form is defined by the path integral

$$Z=\left[\prod_{\mathbf{x},\mu}\int dA_\mu(\mathbf{x})\right]\sum_{\{n_{\mu\nu}(\mathbf{x})\}}\exp\left[-\frac{\beta}{2}\sum_{\mathbf{x},\mu\nu}(\nabla_\mu A_\nu-\nabla_\nu A_\mu-2\pi n_{\mu\nu})^2\right] \tag{4}$$

$[\nabla_\mu A_\nu(\mathbf{x}) \equiv A_\nu(\mathbf{x}+\mu) - A_\nu(\mathbf{x})]$, which shows that $e = 1/\sqrt{\beta}$ plays the role of the electric charge. The integer numbers $n_{\mu\nu}(\mathbf{x})$ generate the closed loops $l_\mu(\mathbf{x})$ of magnetic monopoles [explicitly, $l_\mu(\mathbf{x}) = \epsilon_{\mu\nu\lambda\kappa}\nabla_\nu n_{\lambda\kappa}(\mathbf{x}+\mu)$].

The partition function (4) can, after a quadratic completion and an application of Poisson's sum formula $\sum_l e^{2\pi i l A} = \sum_n \delta(A-n)$, be brought to the form

$$Z = \prod_{\mathbf{x},\mu}\int d\tilde{A}_\mu(x)\prod_{\mathbf{x}}\delta(\nabla_\mu\tilde{A}_\mu)\sum_{\{l_\mu(x)\}}\delta_{\overline{\nabla}_\mu l_\mu(\mathbf{x}),0}\exp\left\{-\frac{1}{2\beta}\sum_{\mathbf{x},\mu<\nu}\tilde{F}^2_{\mu\nu}(\mathbf{x}) + 2\pi i\sum_{\mathbf{x},\mu} l_\mu(\mathbf{x})\tilde{A}_\mu(\mathbf{x})\right\}, \tag{5}$$

where $\tilde{A}_\mu$ is the dual (magnetic) photon and $\tilde{F}_{\mu\nu} = \nabla_\mu\tilde{A}_\nu - \nabla_\nu\tilde{A}_\mu$ is the dual field strength $\frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}(\nabla_\lambda A_\kappa - \nabla_\kappa A_\lambda)$. Integrating out the $\tilde{A}_\mu$ field gives directly the loop partition function (2).

The renormalized charge is defined by the correlation

$$\tfrac{1}{6}\sum_{\mu,\nu}\langle\tilde{F}^2_{\mu\nu}(\mathbf{x})\rangle \equiv \beta_R = e_R^{-2}. \tag{6}$$

It can be recovered from (5) by addition of a source term $i\sum_{x,\mu<\nu}\tilde{F}_{\mu\nu}j_{\mu\nu}$ and differentiation of $\ln Z$ twice with respect to $j_{\mu\nu}$. Since $j_{\mu\nu}$ enters into (2) via the replacement $l_\mu \to l_\mu + (2\pi)^{-1}\nabla_\nu j_{\mu\nu}$, we find directly that

$$\langle\tilde{F}_{\mu\nu}\tilde{F}_{\lambda\kappa}\rangle = -\frac{\delta^2\ln Z}{\delta j_{\mu\nu}\delta j_{\lambda\kappa}} = \beta\frac{1}{\overline{\nabla}\nabla}\{[\overline{\nabla}_\nu\nabla_\kappa\delta_{\mu\lambda} - (\mu\nu)] - [\lambda\kappa]\}$$
$$+4\pi^2\beta^2(\{\langle[\overline{\nabla}_\mu\nabla_\kappa(\overline{\nabla}\nabla)^{-2}l_\mu]l_\lambda - (\mu\nu)\rangle\} - [\lambda\kappa]). \tag{7}$$

Contracting this with $\frac{1}{6}\delta_{\mu\lambda}\delta_{\mu\kappa}$ gives

$$\beta_R = \beta + \tfrac{8}{6}\pi^2\sum_{\mathbf{x},\mu}\langle[(\overline{\nabla}\nabla)^{-1}l_\mu(\mathbf{x})]l_\mu(\mathbf{x})\rangle. \tag{8}$$

The correction term can be written in Fourier space as

$$-\frac{8\pi^2}{6}\lim_{\mathbf{k}\to 0}\frac{1}{\mathbf{k}^2}\langle l_\mu(\mathbf{k})l_\mu(\mathbf{k})\rangle. \tag{9}$$

Since for small $\mathbf{k}$, $\langle l_\mu(\mathbf{k})l_\mu(\mathbf{k})\rangle \equiv \tilde{f}(k)$ starts out with $\mathbf{k}^2\tilde{f}(0)$, we may replace the operation $\lim_{\mathbf{k}\to\infty}k^{-2}$ by $-\frac{1}{8}\sum_{\mathbf{x}}\mathbf{x}^2 f(\mathbf{x})$ and arrive at the exact formula (1).

The curves resulting from the explicit evaluation of formula (1) up to loops of length 10 are shown in Figs. 1 and 2 and compared with the Monte Carlo data of DeGrand and Toussaint[1] and Jersàk *et al.*[2] While agreement with the earlier data is excellent, the immediate neighborhood of the critical point measured in Ref. 2 is not so well reproduced, as was to be expected. It is here where a renormalization-group approach will

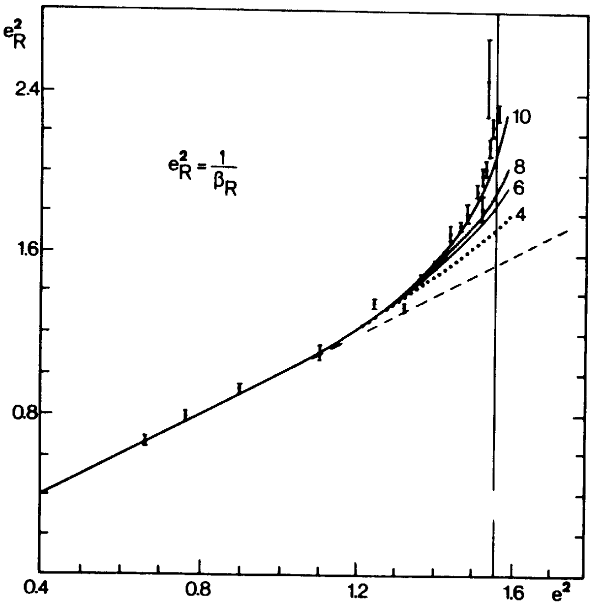

FIG. 1. Renormalized charge e_R as a function of the bare charge e as compared with the Monte Carlo data of Ref. 1. The numbers 4, 6, 8, and 10 indicate the sizes of free-monopole loops included in formula (1).

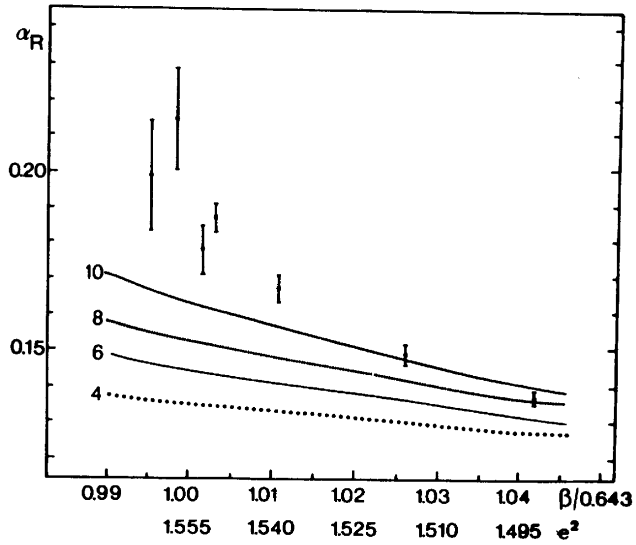

FIG. 2. The renormalized fine-structure constant $\alpha_R = e_R^2/(4\pi)$ in the immediate vicinity of the critical point in comparison with the most recent Monte Carlo data of Ref. 2. The curve labels 4, 6, 8, and 10 mean the same as in Fig. 1.

have to be developed.

One of us is (H.K.) is grateful to Professor N. Kroll for his hospitality at the University of Calfornia at San Diego, and to Professor J. Kuti and Professor D. Toussaint for useful discussions. H.K. also acknowledges partial support by the Deutsche Forschungsgemeinschaft under Grant No. Kl 256 and the Department of Energy under Contract No. DEAT-03-81ER40029.

---

[a]On sabbatical leave from Institut für Theorie der Elementarteilchen, Freie Universität Berlin, Arnimallee 14, 1000 Berlin 33, Germany.

[1]T. A. DeGrand and D. Toussaint, Phys. Rev. D **24**, 466 (1981).

[2]J. Jersàk, T. Neuhaus, and P. M. Zerwas, Phys. Lett. **133B**, 103 (1983); H. G. Evans, J. Jersàk, T. Neuhaus, and P. M. Zerwas, Nucl. Phys. **B251** [FS13], 279 (1985).

[3]H. Kleinert, Phys. Lett. **128B**, 69 (1983).

[4]For the standard derivation in our notation see, for example, H. Kleinert, in *Gauge Interactions,* edited by A. Zichichi (Plenum, New York, 1984).

[5]J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973); J. M. Kosterlitz, J. Phys. C **7**, 1046 (1974). For similar softening effects in crystals see H. Kleinert, *Guage Theory of Stresses and Defects* (Gordon and Breach, New York, 1986).

[6]An approximate version of this equation for small renormalization was given by J. Cardy, Nucl. Phys. **B170**, 369 (1980), and studied by J. M. Luck, Nucl. Phys. **B210** [FS6], 111 (1982), using different methods.