PATH INTEGRALS AND THE $N\to\infty$ SOLUTION OF U(N) LATTICE GAUGE THEORIES

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Abstract

If Wilson's action of U(N) lattice gauge theory is modified slightly, the partition function can be calculated exactly in the limit $N\to\infty$.

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Path integrals are most useful if one succeeds, by a change of integration variables, to find a field representation in which fluctuations are small. If it is possible to go to the limit of zero fluctuations, they lead to exact solutions. There are three important cases where this happens:

1) $\hbar\to 0$, where the original field ceases to fluctuate and the classical field becomes exact.
2) $D\to\infty$, in lattice models, where the gradient terms are suppressed and the mean field becomes exact.
3) $N\to\infty$, in O(N) spin models, where an O(N) invariant collective field ("Hartree-like") can be introduced, which solves the problem exactly.

It has been hoped for a long time that gauge theories with a continuum partition function à la Yang-Mills

$$Z_{YM} = \int DA_\mu \, e^{-1/(4g^2)\int d^4x F_{\mu\nu}^2} \qquad (1)$$

or a lattice partition function

$$Z_W = \int DU_\mu \, e^{(\beta/2)\mathrm{tr}_N(U_{\mu\nu}+U_{\mu\nu}^+-2)} \qquad (2)$$

à la Wilson, where

$$U_{\mu\nu} = U_\mu(x)U_\nu(x+\mu)U_\mu^+(x+\nu)U_\nu^+(x) \qquad (3)$$

are NxN unitary matrices, may also have an exact solution for $N\to\infty$. This might help in understanding the physical case N=3. Unfortunately, all attempts in this direction have so far failed. My collaborators, Hofsäss, Matsui and I, have succeeded in constructing a new lattice gauge model ("HKM model") which does allow for an exact $N\to\infty$ solution and has a chance of giving a similar continuum physics as Wilson's at least for a certain range of energies. The HKM model has the characteristic feature that its gluon interactions are generated by the exchange of two U(N) subgluon fields. These have the same confinement properties as fundamental gluons. At shorter distances there is asymptotic freedom. But at extremely short distances there is a new feature: The asymptotic freedom potential loses its singularity as r invades into the subgluon wave function.

Instead of Wilson's integrand

$$e^{(\beta/2)\mathrm{tr}_N(U_{\mu\nu}+U_{\mu\nu}^{+}-2)} \qquad (4)$$

we use

$$e^{-\mathrm{tr}_N \log\{1-[\beta/(2(1+\beta))U_{\mu\nu}+U_{\mu\nu}^{+})]\}} \qquad (5)$$

In certain limits, this gives no change. The actions are certainly the same for small β. For small field fluctuations $U_\mu \approx 1+iagA_\mu$, our integrand amounts to replacing the Boltzmann peak in (4)

$\sim e^{-(\beta/2)a^4 g^2 F_{\mu\nu}^{2}}$ by a Lorentzian peak $\sim \frac{1}{1+(\beta/2)a^4 g^2 F_{\mu\nu}^{2}}$ in (5).

It is easy to see that this gives the same weight if, at fixed field strength, the lattice spacing goes to zero. But at fixed lattice spacing, larger field strengths receive quite different weights.

The advantage of our integrand is that it can be rewritten in terms of four complex scallar fields $\phi_\square^{(i)}$ on each plaquette μν (the "subgluon fields")

$$\int d\phi_\square^{(1)} d\phi_\square^{(2)} d\phi_\square^{(3)} d\phi_\square^{(4)}\ e^{-\lambda \sum_{i=1}^{4} \phi_\square^{(2)} \phi_\square^{(i)} + [\phi_\square^{(1)} U_1 \phi_\square^{(2)} + (23) + (34) + (41)]} \qquad (6)$$

The four fields are U(N) vectors and may be imagined as living on the four corners indicated by the small circles in Fig. 1. For brevity, we have denoted the four factors associated with the lines around the plaquette in (3) by U_1, U_2, U_3, U_4.

The expression (6) permits integrating out the U fields with a result which depends only on color neutral, i.e., gauge invariant, collective fields

$$\alpha_{\square\square'} = \phi_{\square}^{+}\phi_{\square'} \tag{7}$$

where the subscript $\square\square'$ indicates that the plaquettes $\square$ and $\square'$ of the subgluon fields have to posses a common link. By a further change of variables it is possible to re-express the partition function entirely in terms of these fields $\alpha_{\square\square'}$, plus an additional Lagrange multiplier field $\sigma_{\square\square'}$ which ensures the identity (7). In this way we have obtained a partition function of the form

$$Z = \int D\alpha_{\square\square'}\, D\sigma_{\square\square'}\, e^{-NA[\alpha_{\square\square'},\, \sigma_{\square\square'}]} \tag{8}$$

in which the number of colors N appears only as an overall factor in front of the action.

For $N\to\infty$, the collective fields no longer fluctuate and the exact solution of the model is found by minimizing the action. For technical details of this approach you are referred to the original papers.[1]

REFERENCES

1. T. Hofsäss, H. Kleinert, T. Matsui, Phys. Lett. B156, 96 (1985) and University of California, San Diego, preprint, November 1985.

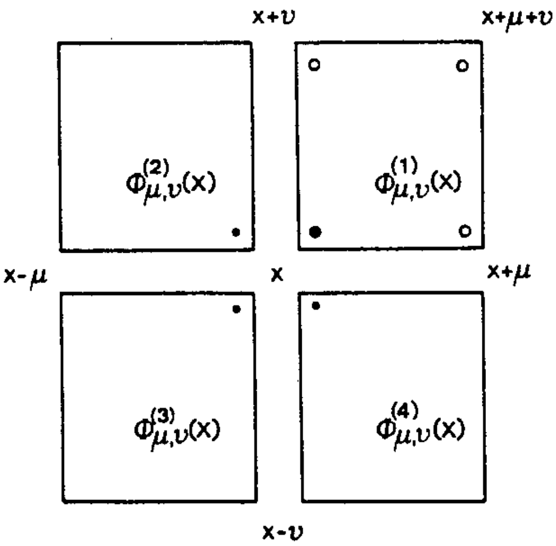

Figure 1