Path integral for second-derivative Lagrangian $L=(\kappa/2)\ddot{x}^2+(m/2)\dot{x}^2+(k/2)x^2-j(\tau)x(\tau)$

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(Received 19 February 1986; accepted for publication 6 August 1986)

For the above second-derivative Euclidean Lagrangian, the quantum statistical probability distribution $(x_b v_b \tau_b | x_a v_a \tau_a)$ that an orbit $x(\tau)$ with initial position x_a and velocity $\dot{x}_a = v_a$ arrives at a final point x_b and velocity $\dot{x}_b = v_b$ is calculated.

I. INTRODUCTION

The behavior of many physical systems cannot be understood without allowing for higher-gradient terms in the field energy. In relativistic quantum field theory such terms have not enjoyed much popularity, due to notorious difficulties with positivity of either the energy or of the metric of the quantum mechanical Hilbert space.[1] In statistical mechanics, however, such terms are ubiquitous and impossible to avoid. Some examples follow.

(1) Polymers on an intermediate distance scale are stiff objects and their energy requires the inclusion of a bending energy which involves the square of the second derivative, $\ddot{x}^2(s)$, where s is the length parameter of the space curve.[2]

(2) The walls of many living cells are free of tension and undergo fluctuations controlled mainly by second-gradient curvature energy.[3] This makes the fluctuations so large that they can be seen in an ordinary light microscope, as first observed on human red blood cells in 1890.[4] These giant fluctuations are crucial to prevent the cells from sticking to each other, in spite of their attractive van der Waals forces.[5]

(3) The formation of microemulsions cannot take place without the ampliphilic soap layer between water and oil losing its surface tension.[6]

(4) The strings of color electric flux lines, which bind quarks and antiquarks, can lose their tension in a phase transition, in which case they are controlled completely by second-gradient elasticity.[7]

(5) Finally, the cosmos at an early stage of evolution may not have been controlled by the Einstein action, but by the Weyl action which involves the square of the curvatures and thus contains the square of two derivatives of the metric. The geophysically observed deviations from Newton's law, when masses come closer to each other than ≈ 200 m, could be a signal for such terms (the sign is correct).[8]

II. THE PATH INTEGRAL

In all these physical situations, the prototype of the fluctuation problem to be solved is the path integral

$$(x_b v_b \tau_b | x_a v_a \tau_a) = \int \mathscr{D}x(\tau) \exp\left(- \int_{\tau_a}^{\tau_b} d\tau\, L(\tau) \right),$$
$$x(\tau_a) = x_a, \quad \dot{x}(\tau_a) = v_a, \qquad (1)$$
$$x(\tau_b) = x_b, \quad \dot{x}(\tau_b) = v_b,$$

with the Euclidean Lagrangian ($\cdot \equiv d/d\tau$)

$$L(\tau) = \frac{\kappa}{2}\ddot{x}^2(\tau) + \frac{m}{2}\dot{x}^2(\tau) + \frac{k}{2}x^2(\tau) - j(\tau)x(\tau) . \quad (2)$$

In order to make all integrals convergent we have rotated the time variable t to imaginary values $t = -i\tau$.

After rescaling the variables τ to $\tau = \kappa^{-1/3}\tau_{\rm old}$ and introducing the frequencies ω_1, ω_2 via

$$\omega_1^2 + \omega_2^2 = (m/2)\kappa^{-1/3}, \quad \omega_1^2\,\omega_2^2 = k\kappa^{1/3}, \qquad (3)$$

we are confronted with the probability distribution

$$(x_b v_b \tau_b | x_a v_a \tau_a) = \int \mathscr{D}x(\tau) \exp\left(- \int_{\tau_a}^{\tau_b} d\tau\, L(\tau) \right),$$
$$x(\tau_a) = x_a, \quad \dot{x}(\tau_a) = v_a, \qquad (4)$$
$$x(\tau_b) = x_b, \quad \dot{x}(\tau_b) = v_b,$$

where L is now the Euclidean Lagrangian

$$L = \tfrac{1}{2}[\ddot{x}^2 + (\omega_1^2 + \omega_2^2)\dot{x}^2 + \omega_1^2\,\omega_2^2 x^2] - j(\tau)x(\tau) \qquad (5)$$

with an appropriately rescaled current ($j = \kappa^{1/3} j_{\rm old}$). This can be separated into a pure surface term

$$L_{\rm sf} = \frac{d}{dt}\Lambda = \frac{d}{dt}\left[\frac{1}{2}(\dot{x}\ddot{x} - x\dddot{x}) + (\omega_1^2 + \omega_2^2)x\dot{x}\right], \qquad (6a)$$

plus a source term

$$L_{\rm source} = -\int_{\tau_a}^{\tau_b} d\tau\, j(\tau)x(\tau), \qquad (6b)$$

plus a term

$$L_0 = \tfrac{1}{2}x(\ddddot{x} - (\omega_1^2 + \omega_2^2)\ddot{x} + \omega_1^2\,\omega_2^2 x), \qquad (6c)$$

which vanishes for solutions of the free field equation

$$(\partial_\tau^2 - \omega_1^2)(\partial_\tau^2 - \omega_2^2)x(\tau) = 0 . \qquad (7)$$

These correspond to two independent harmonic oscillators and have the general form

$$x_{\rm cl}(\tau) = A\cosh\omega_1(\tau - \tau_a) + B\sinh\omega_1(\tau - \tau_a) + C\cosh\omega_2(\tau - \tau_a) + D\sinh\omega_2(\tau - \tau_a) . \quad (8)$$

(The two oscillators can be exhibited by introducing the two auxiliary variables $q_1 = \ddot{x} - \omega_1^2 x$, $q_2 = \ddot{x} - \omega_2^2 x$ and noting that $L_3 = [q_1(\partial_\tau^2 - \omega_2^2)q_1 - q_2(\partial_\tau^2 - \omega_1^2)q_2]/(\omega_1^2 - \omega_2^2)$. The negative sign in front of the second term leads to the difficulties with a quantum mechanical formulation due to an indefinite Hamiltonian.) The proper measure of integration in the path integral (4) is found via the canonical formalism. For a higher-gradient Lagrangian (3) we can follow the method of Ostrogradski,[9] according to which $\dot{x}(\tau) \equiv v(\tau)$ may be considered as an *independent* degree of freedom replacing the Lagrangian (5) by the equivalent one

$$\widetilde{L} = \tfrac{1}{2}(\dot{v}^2 + (\omega_1^2 + \omega_2^2)v^2 + \omega_1^2\,\omega_2^2 x^2) - ip(\dot{x} - v) - jx, \qquad (5')$$

where the Lagrangian multiplier $p(\tau)$ ensures the correct relation between v and $\dot{x}$. The canonical momenta are $p = i(\partial L/\partial \dot{x})$ and $p_v = i(\partial \tilde{L}_E/\partial \dot{v}) = i\dot{v}$ such that the Hamiltonian is

$$H(p,x,p_v,v,\tau) = ip\dot{x} + ip_v\dot{v} + \tilde{L}$$
$$= \tfrac{1}{2}(p_v^2 + (\omega_1^2 + \omega_2^2)v^2 + \omega_1^2\omega_2^2 x^2)ip_v - j(\tau)x\,, \qquad (9)$$

with the Hamiltonian equations of motion

$$i\dot{p} = -\frac{\partial H}{\partial x} = -\omega_1^2\omega_2^2 x + j(\tau)\,,$$
$$i\dot{p}_v = -\frac{\partial H}{\partial v} = -(\omega_1^2 + \omega_2^2)v - ip\,,$$
$$i\dot{v} = \frac{\partial H}{\partial p_v} = p_v, \quad i\dot{x} = \frac{\partial H}{\partial p} = iv\,. \qquad (10)$$

It is now straightforward to specify the measure of the path integral. In phase space it has the form

$$(x_b v_b \tau_b | x_a v_a \tau_a)$$
$$= \int \mathscr{D}x\,\mathscr{D}v \int \frac{\mathscr{D}p}{2\pi} \int \frac{\mathscr{D}p_v}{2\pi}$$
$$\times \exp\left(\int_{\tau_a}^{\tau_v} d\tau (ip\dot{x} + ip_v\dot{v} - H(p,x,p_v,v,\tau))\right), \qquad (11)$$

where $\int\mathscr{D}x$ means, as usual, the product of all

$$\prod_{n=1}^{N} \int_{-\infty}^{\infty} dx_n$$

over the time sliced positions

$$x_n \equiv x(\tau_n), \quad \tau_n \equiv \tau_a + \epsilon n\,,$$

where $\epsilon = (\tau_b - \tau_a)/(N+1)$ and $x_a = x(\tau_0)$, $x_b = x(\tau_{N+1})$ are held fixed, and $\int\mathscr{D}p/2\pi$ is the product of integrals

$$\prod_{n=1}^{N+1} \int_{-\infty}^{\infty} \frac{dp_n}{2\pi}$$

involving all $N+1$ momenta that appear in the canonical term

$$\int_{\tau_a}^{\tau_b} d\tau\, ip\dot{x} = \sum_{n=1}^{N+1} ip_n(x_n - x_{n-1})\,.$$

The same rule applies to the conjugate variable pair v and p_v, which are split into $v_1,\ldots,v_{N+1}$ and $p_{v_1},\ldots,p_{v_{N+1}}$ with $v_1 = v_a$, $v_{N+1} = v_b$ held fixed and the canonical integral measure is

$$\prod_{n=2}^{N} \int_{-\infty}^{\infty} dv_n \prod_{n=2}^{N+1} \int_{-\infty}^{\infty} \frac{dp_{v_n}}{2\pi}\,. \qquad (12)$$

By construction, the amplitude (11) and hence also (4) satisfies the Schrödinger equation

$$(H(-i\partial_x, x, -i\partial_v, v, \tau) + \partial_\tau)(xv\tau|x'v'\tau')$$
$$= (-\tfrac{1}{2}\partial_v^2 + (\omega_1^2 + \omega_2^2)v^2 + \omega_1^2\omega_2^2 x^2 + v\partial_x - j(\tau)x + \partial_\tau)(xv\tau|x'v'\tau')$$
$$= \delta(x-x')\delta(v-v')\delta(\tau-\tau')\,. \qquad (13)$$

Using (11), it is now straightforward to obtain the measure of integration for the pure x space path integral (3) as follows: Integrating out the variables p_2 and p_N gives a product of δ functions

$$\prod_{n=2}^{N} \delta(x_n - x_{n-1} - \epsilon v_n)\,, \qquad (14)$$

which can be used to eliminate the integrals over $v_2,\ldots,v_N$, thereby producing a factor $1/\epsilon^{N-1}$. The integrals over $pv_2,\ldots,pv_{N+1}$ produce a further factor $(1/\sqrt{2\pi\epsilon})N$. Thus the measure of the path integral (3) can be written as follows:

$$\int \mathscr{D}x(\tau) = \epsilon \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} \frac{dx_n}{\sqrt{2\pi\epsilon}\,\epsilon}\right] \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \int_{-\infty}^{\infty} \frac{dp_{N+1}}{2\pi}$$
$$\times \exp[ip_{N+1}(x_{N+1} - x_N - \epsilon v_{N+1}) + ip_1(x_1 - x_0 - \epsilon v_1)]\,. \qquad (15)$$

We may now also integrate out the remaining two momentum integrals, thereby eliminating the integrations over dx_1 and dx_0. For the calculations to come it will, however, be convenient to leave the measure in this form.

Due to the quadratic form of the energy, the integration over the spatial variables can most easily be done following the same procedure as developed for lowest-gradient quadratic energy in Feynman and Hibbs.[10] We expand all orbits around a fixed classical trajectory $x_{\rm cl}(\tau)$, which connects the initial point x_a at velocity v_a with the final point x_b at velocity v_b, and write $x(\tau) = x_{\rm cl}(\tau) + \delta x(\tau)$. The fluctuations $\delta x(\tau)$ then have the property that

$$\delta x(\tau_a) = \delta x(\tau_b) = 0\,,$$
$$\delta v(\tau_a) = \delta v(\tau_v) = 0\,. \qquad (16)$$

Inserting this into the action $\mathscr{A} = \int_{\tau_a}^{\tau_b} d\tau\, L(\tau)$, there is a classical contribution coming entirely from the surface term (6a),

$$\mathscr{A}_{\rm cl,sf} = \tfrac{1}{2}[v_b\ddot{x}_b - x_b\dddot{x}_b + \tfrac{1}{2}(\omega_1^2 + \omega_2^2)x_b v_b - (ab)]|_{x = x_{\rm cl}(\tau)}\,, \qquad (17)$$

plus a contribution from the source term (6b)

$$\mathscr{A}_{\rm cl,source} = -\int_{\tau_a}^{\tau_b} d\tau\, j(\tau)x_{\rm cl}(\tau)\,, \qquad (18)$$

plus a fluctuation piece,

$$\mathscr{A}_{\rm fl} = \int_{\tau_a}^{\tau_b} d\tau\left\{\frac{1}{2}[(\delta\ddot{x})^2 + (\omega_1^2 + \omega_2^2)(\delta\dot{x})^2 + (\delta x)^2] - j(\tau)\delta x(\tau)\right\}. \qquad (19)$$

Hence we may write

$$(x_b v_b \tau_b | x_a v_a \tau_a)$$
$$= e^{-\mathscr{A}_{\rm cl,sf} - \mathscr{A}_{\rm cl,source}} \int \mathscr{D}\,\delta x(\tau)$$
$$\times \exp\left(\int_{\tau_a}^{\tau_b}\left\{\frac{1}{2}[(\delta\ddot{x})^2 + (\omega_1^2 + \omega_2^2)(\delta\dot{x})^2 + \omega_1^2\omega_2^2(\delta x)^2] - j(\tau)\delta x(\tau)\right\}\right), \qquad (20)$$
$$\delta x(\tau_a) = 0, \quad \delta x(\tau_b) = 0\,,$$

$\delta\dot{x}(\tau_a)=0, \quad \delta\dot{x}(\tau_b)=0,$

where $\delta x(\tau)$ has now a measure like (15) except that δx_{N+1}, δx_0, δv_{N+1}, and δv_1 vanish at the end points, i.e.,

$$\int \mathscr{D}\,\delta x(\tau)=\epsilon\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\frac{dx_n}{\sqrt{2\pi\epsilon}\,\epsilon}\right]\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\times\int_{-\infty}^{\infty}\frac{dp_{N+1}}{2\pi}e^{-i(p_{N+1}\delta x_N-p_1\delta x_1)}. \tag{21}$$

The vanishing of δx_0, δx_{N+1} implies that $\delta x(\tau)$ has only the Fourier components

$$\delta x(\tau)=\sqrt{\frac{2}{\beta}}\sum_{m=1}^{N}\delta x_m\sin\nu_m(\tau-\tau_a), \tag{22}$$

with $\beta\equiv\tau_b-\tau_a$ and frequencies

$$\nu_m=(\pi/\beta)m. \tag{23}$$

In terms of the Fourier components the exponential in the fluctuation factor (20) reads

$$\exp\left\{-\frac{1}{2}\sum_{m=1}^{N}(\Omega_m^2+\omega_1^2)(\Omega_m^2+\omega_2^2)(\delta x_m)^2+\int_{\tau_a}^{\tau_b}d\tau j(\tau)\sqrt{\frac{2}{\beta}}\sum_m\delta x_m\sin\nu_m(\tau-\tau_a)\right\},$$

where

$$\Omega_m^2\equiv(1/\epsilon^2)(2-2\cos\nu_m\epsilon)=(4/\epsilon^2)\sin^2(\nu_m\epsilon/2) \tag{24}$$

are the squared eigenvalues of the differences $[(x_n-x_{n-1})/\epsilon]$ in Fourier space. Since the Fourier series has a unit Jacobian, the measure (21) becomes

$$\int\mathscr{D}\,\delta x(\tau)=\epsilon\prod_{m=1}^{N}\int_{-\infty}^{\infty}\frac{d\delta x_m}{\sqrt{2\pi\epsilon}\,\epsilon}\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\times\int_{-\infty}^{\infty}\frac{dp_{N+1}}{2\pi}\epsilon^{-i(p_{N+1}\delta x_N-p_1\delta x_1)}. \tag{25}$$

Neglecting for a moment the couplings to p_{N+1} and p_1 and to the current $j(\tau)$, the pure δx_m part of the integrals gives the product

$$\epsilon\left[\prod_{m=1}^{N}(\epsilon^2\Omega_m^2+\epsilon^2\omega_1^2)(\epsilon^2\Omega_m^2+\epsilon^2\omega_2^2)\right]^{-1/2}\underset{N\to\infty}{\to}\epsilon\sqrt{\frac{\omega_1\epsilon}{\sinh\omega_1\beta}}\sqrt{\frac{\omega_2\epsilon}{\sinh\omega_2\beta}}. \tag{26}$$

In Fourier space, the couplings to p_{N+1}, p_1, and the current amount to

$$-ip_{N+1}\sqrt{\frac{2}{\beta}}\sum_{m=1}^{N}\delta x_m\sin(\nu_m\epsilon)+ip_1\sqrt{\frac{2}{\beta}}\sum_{m=1}^{N}\delta x_m\sin(\nu_m(\beta-\epsilon))+\int_{\tau_a}^{\tau_b}d\tau j(\tau)\sqrt{\frac{2}{\beta}}\sum_{m=1}^{N}\delta x_n\sin\nu_m(\tau-\tau_a). \tag{27}$$

This can be rewritten as

$$\sqrt{\frac{2}{\beta}}\Bigg[-i(p_{N+1}-p_1)\sum_{m=1,3,5,\ldots}\delta x_m\sin(\nu_m\epsilon)-i(p_{N+1}+p_1)\sum_{m=2,4,6,\ldots}\delta x_m\sin(\nu_m\epsilon)+\int_{\tau_a}^{\tau_b}d\tau j(\tau)\sum_{m=1,2,3,\ldots}\delta x_m\sin(\nu_m(\tau-\tau_a))\Bigg]. \tag{28}$$

In the absence of a current, these terms lead, after the δx_m integrations, to the additional momentum integrals

$$\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\int_{-\infty}^{\infty}\frac{dp_2}{2\pi}\exp\Bigg\{-\frac{\epsilon^2}{2}(p_{N+1}-p_1)^2\times\frac{2}{\beta}\sum_{m=1,3,5,\ldots}\frac{(1/\epsilon^2)\sin^2(\nu_m\epsilon)}{(\Omega_m^2+\omega_1^2)(\Omega_m^2+\omega_2^2)}-\frac{\epsilon^2}{2}(p_{N+1}+p_1)^2\times\frac{2}{\beta}\sum_{m=2,4,6,\ldots}\frac{(1/\epsilon^2)\sin^2(\nu_m\epsilon)}{(\Omega_m^2+\omega_1^2)(\Omega_m^2+\omega_2^2)}\Bigg\}. \tag{29}$$

Due to their fast convergence, the sums can be replaced, for $\epsilon\to0$, by

$$\frac{2}{\beta}\sum_m\frac{\nu_m^2}{(\nu_m^2+\omega_1^2)(\nu_m^2+\omega_2^2)}=\frac{2}{\beta}\frac{1}{\omega_1^2-\omega_2^2}\sum_m\left(\frac{\omega_1^2}{\nu_m^2+\omega_1^2}-\frac{\omega_2^2}{\nu_m^2+\omega_2^2}\right). \tag{30}$$

These sums are equal to

$$D_e=\frac{1}{\beta(\omega_1^2-\omega_2^2)}\left(\frac{\omega_1\beta}{2}\coth\frac{\omega_1\beta}{2}-(12)\right),\quad\text{for even } m,$$
$$D_o\equiv\frac{1}{\beta(\omega_1^2-\omega_2^2)}\left(\frac{\omega_1\beta}{2}\tanh\frac{\omega_1\beta}{2}-(12)\right),\quad\text{for odd } m, \tag{31}$$

such that the integrations over p_{N+1}, p_1 yield the further factor

$$\frac{1}{2\pi}\frac{1}{\epsilon^2}\frac{1}{2}\frac{1}{\sqrt{D_eD_o}}=\frac{1}{2\pi\epsilon^2}\frac{\beta}{2}\frac{|\omega_1^2-\omega_2^2|}{[(\omega_1\beta/2)\coth(\omega_1\beta/2)-(12)]^{1/2}[(\omega_1\beta/2)\tanh(\omega_1\beta/2)-(12)]^{1/2}}$$
$$=\frac{\beta}{2\pi\epsilon^2}\frac{(\omega_1^2-\omega_2^2)\sqrt{\sinh\omega_1\beta\sinh\omega_2\beta}}{\sqrt{(\omega_1^2+\omega_2^2)\sinh\omega_1\beta\sinh\omega_2\beta-2\omega_1\omega_2(\cosh\omega_1\beta\cosh\omega_2\beta-1)}}. \tag{32}$$

If the current is nonzero, the expression (29) is replaced by

$$\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\int_{-\infty}^{\infty}\frac{dp_2}{2\pi}\exp\frac{1}{2}\left\{\frac{2}{\beta}\sum_{m=1,3,5,\ldots}\left(\int_{\tau_a}^{\tau_b}d\tau j(\tau)\sin\nu_m(\tau-\tau_a)-i(p_{N+1}-p_1)\sin\nu_m\epsilon\right)\right.$$

$$\times\left(\int_{\tau_a}^{\tau_b} d\tau' j(\tau')\sin \nu_m(\tau'-\tau_a) - i(p_{N+1}-p_1)\sin \nu_m\epsilon\right)[(\Omega_m^2+\omega_1^2)(\Omega_m^2+\omega_2^2)]^{-1}$$
$$+\frac{2}{\beta}\sum_{m=2,4,5,\ldots}\left(\int_{\tau_a}^{\tau_b} d\tau j(\tau)\sin \nu_m(\tau-\tau_a) - i(p_{N+1}+p_1)\sin \nu_m\epsilon\right)$$
$$\times\left(\int_{\tau_a}^{\tau_b} d\tau' j(\tau')\sin \nu_m(\tau'-\tau_a) - i(p_{N+1}+p_1)\sin \nu_m\epsilon\right)[(\Omega_m^2+\omega_1^2)(\Omega_m^2+\omega_2^2)]^{-1}\Bigg\}.$$

This gives an additional term in (29),

$$\exp\left\{\frac{1}{2}\int_{\tau_a}^{\tau_b} d\tau \int_{\tau_a}^{\tau_b} d\tau' \hat{G}(\tau,\tau') j(\tau) j(\tau')\right\}$$
$$\times\exp\left\{-i\epsilon(p_{N+1}-p_1)\int_{\tau_a}^{\tau_b} d\tau j(\tau) h_o(\tau) - i\epsilon(p_{N+1}+p_1)\int d\tau j(\tau) h_e(\tau)\right\},$$

where

$$\hat{G}(\tau,\tau') = \frac{2}{\beta}\sum_m \frac{\sin \nu_m(\tau-\tau_a)\sin \nu_m(\tau'-\tau_a)}{(\Omega_m^2+\omega_1^2)(\Omega_m^2+\omega_2^2)},$$
$$h_o(\tau) = \frac{2}{\beta}\sum_{m=1,3,5,\ldots}\frac{(1/\epsilon)\sin \nu_m\epsilon \sin \nu_m(\tau-\tau_a)}{(\Omega_m^2+\omega_1^2)(\Omega_m^2-\omega_2^2)},\tag{33}$$
$$h_e(\tau) = \frac{2}{\beta}\sum_{m=2,4,6,\ldots}\frac{(1/\epsilon)\sin \nu_m\epsilon \sin \nu_m(\tau-\tau_a)}{(\Omega_m^2+\omega_1^2)(\Omega_m^2-\omega_2^2)}.$$

If we now integrate out the momenta p_{N+1}, p_1 the external source yields the factor

$$\exp\left\{\frac{1}{2}\int_{\tau_a}^{\tau_b} d\tau \int_{\tau_a}^{\tau_b} d\tau' j(\tau) G(\tau,\tau') j(\tau')\right\},\tag{34}$$

where

$$G(\tau,\tau') = \hat{G}(\tau,\tau') - \frac{h_e(\tau)h_e(\tau')}{D_e} - \frac{h_o(\tau)h_o(\tau')}{D_o}\tag{35}$$

This is the correlation function of the fluctuations $\delta x(\tau)$,

$$\langle \delta x(\tau)\delta x(\tau')\rangle = G(\tau,\tau').\tag{36}$$

Since $\delta x(\tau)$ vanishes at the end points $\tau=\tau_b$, $\tau'=\tau_a$ and has zero velocities, also $G(\tau,\tau')$ must have this property. Indeed, the vanishing of $G(\tau,\tau_a)$ and $G(\tau_b,\tau)$ is trivial to see. The zero velocity at the end points, on the other hand, is a consequence of the two properties [which follows directly from (33) and (31)]:

$$\frac{d}{d\tau'} G(\tau,\tau')|_{\tau'=\tau_a} = h_e(\tau) + h_o(\tau),\tag{37a}$$
$$\frac{d}{d\tau'} h_{\substack{e\\o}}(\tau')|_{\tau'=\tau_a} = D_{\substack{e\\o}}.\tag{37b}$$

Hence

$$\frac{d}{d\tau'} G(\tau,\tau')|_{\tau'=\tau_a} = 0.\tag{38}$$

Collecting all terms, we arrive at the probability distribution

$$(x_b v_b \tau_b | x_a v_a \tau_a) = F(\beta)\exp\left\{-\mathscr{A}_{\rm cl,sf} - \mathscr{A}_{\rm cl,source} + \frac{1}{2}\int_{\tau_a}^{\tau_b} d\tau \int_{\tau_a}^{\tau} d\tau' j(\tau) G(\tau,\tau') j(\tau')\right\},\tag{39}$$

with the fluctuation factor

$$F(\beta) = \frac{1}{2\pi}\sqrt{\omega_1\omega_2}\frac{|\omega_1^2-\omega_2^2|}{\sqrt{(\omega_1^2+\omega_2^2)\sinh \omega_1\beta \sinh \omega_2\beta - 2\omega_1\omega_2(\cosh \omega_1\beta \cosh \omega_2\beta - 1)}}.\tag{40}$$

The terms $\mathscr{A}_{\rm cl,sf}$ and $\mathscr{A}_{\rm cl,source}$ are the only ones that depend on the initial and final variables $x_a v_a$, $x_b v_b$. They will be calculated in the following two sections.

III. THE CLASSICAL ACTION FOR ZERO EXTERNAL SOURCE

The starting point is formula (18). All we have to do is express the quantities $\dot{x}_b, \ddot{x}_b, \dot{x}_a$, and $\ddot{x}_a$ in terms of the initial and final variables $x_b v_b$, $x_b v_b$. For this purpose we invert the matrix relation

$$\begin{pmatrix} x_a \\ x_b \\ x_a \\ x_b \end{pmatrix} = M\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix},\tag{41}$$

with

$$M=\begin{pmatrix} 1 & 0 & 1 & 0 \\ c_1 & s_1 & c_2 & s_2 \\ 0 & \omega_1 & 0 & \omega_2 \\ \omega_1 s_1 & \omega_1 c_1 & \omega_2 s_2 & \omega_2 c_2 \end{pmatrix}, \tag{42}$$

where $c_{\substack{1\\2}} \equiv \cosh \omega_{\substack{1\\2}}\beta$, $s_{\substack{1\\2}} \equiv \sinh \omega_{\substack{1\\2}}\beta$, and find

$$M^{-1} = (1/|M|)R\,, \tag{43}$$

where $|M|$ is the determinant

$$|M| = (\omega_1^2+\omega_2^2)s_1 s_2 - 2\omega_1\omega_2(c_1c_2-1)\,, \tag{44}$$

and thus precisely equal to the expression under the last square root in the fluctuation factor (40). The matrix R is equal to

$$R=\begin{pmatrix} -\omega_1\omega_2(c_1c_2-1)+\omega_2^2 s_1 s_2 & \omega_1\omega_2(c_1-c_2) & -\omega_1 c_1 s_2+\omega_2 s_1 c_2 & \omega_1 s_2-\omega_2 s_1 \\ \omega_1\omega_2 s_1 c_2-\omega_2^2 c_1 s_2 & -\omega_1\omega_2 s_1+\omega_2^2 s_2 & \omega_1 s_1 s_2-\omega_2(c_1c_2-1) & \omega_2(c_1-c_2) \\ \omega_1^2 s_1 s_2-\omega_1\omega_2(c_1c_2-1) & -\omega_1\omega_2(c_1-c_2) & \omega_1 c_1 s_2-\omega_2 s_1 c_2 & -\omega_1 s_2+\omega_2 s_1 \\ -\omega_1^2 s_1 c_2+\omega_1\omega_2 c_1 s_2 & \omega_1^2 s_1-\omega_1\omega_2 s_2 & \omega_1(c_1c_2-1)+\omega_2 s_1 s_2 & -\omega_1(c_1-c_2) \end{pmatrix}. \tag{45}$$

This gives

$$\dot{x}_b = (\omega_1^2 c_1,\omega_1^2 s_1,\omega_2^2 c_2,\omega_2^2 s_2)M^{-1}\begin{pmatrix} x_a \\ x_b \\ v_a \\ v_b \end{pmatrix} = \frac{1}{|M|}\begin{pmatrix} \omega_1^3\omega_2(c_1-c_2)+(12) \\ -\omega_1^3\omega_2(c_1c_2-1)+(12)+2\omega_1^2\omega_2^2 s_1 s_2 \\ -\omega_1^3 s_2+\omega_1^2\omega_2 s_1+(12) \\ \omega_1^3 c_1 s_2-\omega_1^2\omega_2 s_1 c_2+(12) \end{pmatrix}^T \begin{pmatrix} x_a \\ x_b \\ v_a \\ v_b \end{pmatrix}, \tag{46}$$

$$\ddot{x}_a = (\omega_1^2,0,\omega_2^2,0)M^{-1}\begin{pmatrix} x_a \\ x_b \\ v_a \\ v_b \end{pmatrix} = \frac{1}{|M|}\begin{pmatrix} -\omega_1^3\omega_2(c_1c_2-1)+(12)+2\omega_1^2\omega_2^2 s_1 s_2 \\ \omega_1^3\omega_2(c_1-c_2)+(12) \\ -\omega_1^3 c_1 s_2+\omega_1^2\omega_2 s_1 c_2+(12) \\ \omega_1^3 s_2-\omega_1^2\omega_2 s_1+(12) \end{pmatrix}^T \begin{pmatrix} x_a \\ x_b \\ v_a \\ v_b \end{pmatrix}, \tag{47}$$

$$\dddot{x}_b = (\omega_1^3 s_1,\omega_1^3 c_1,\omega_2^3 s_2,\omega_2^3 c_2)M^{-1}\begin{pmatrix} x_a \\ x_b \\ v_a \\ v_b \end{pmatrix} = \frac{1}{|M|}\begin{pmatrix} \omega_1^4\omega_2 s_1+(12) \\ -\omega_1^4\omega_2 s_1 c_2+\omega_1^3\omega_2^2 c_1 s_2+(12) \\ \omega_1^3\omega_2(c_1-c_2)+(12) \\ \omega_1^4 s_1 s_2-\omega_1^3\omega_2(c_1c_2-1)+(12) \end{pmatrix}^T \begin{pmatrix} x_a \\ x_b \\ v_a \\ v_b \end{pmatrix}, \tag{48}$$

and, upon inserting this into (18), the classical action

$$\begin{aligned}\mathscr{A}_{\mathrm{cl,sf}} = (1/2|M|)\{&(\omega_1^2-\omega_2^2)[(\omega_1 c_1 s_2-\omega_2 s_1 c_2)(v_b^2+v_a^2)-2(\omega_1 s_2-\omega_2 s_1)v_b v_a] \\ &-2\omega_1\omega_2[(\omega_1^2+\omega_2^2)(c_1c_2-1)-2\omega_1\omega_2 s_1 s_2](v_b x_b-v_a x_a)+2\omega_1\omega_2(\omega_1^2-\omega_2^2)(c_1-c_2)(v_b x_a-v_a x_b) \\ &+\omega_1\omega_2(\omega_1^2-\omega_2^2)(\omega_1 s_1 c_2-\omega_2 c_1 s_2)(x_b^2+x_a^2)-2\omega_1\omega_2(\omega_1^2-\omega_2^2)(\omega_1 s_1-\omega_2 s_2)x_b x_a\}\,. \end{aligned} \tag{49}$$

In the absence of external currents, this can be inserted into Eq. (39) giving the desired probability distribution. Before we go on to calculating the full $j\neq 0$ contributions, it is useful to first study a few properties of the $j=0$ result.

IV. THE PARTITION FUNCTION AT $j=0$

The quantum statistical partition function of the $j=0$ system is obtained by setting $x_b = x_a \equiv x$, $x_b = v_a \equiv v$, in which case the classical action becomes

$$\mathscr{A}_{\mathrm{cl,sf}} = ax^2+bv^2\,, \tag{50}$$

with

$$\begin{aligned} a &= (1/|M|)(\omega_1^2-\omega_2^2)(\omega_1(c_1-1)s_2-\omega_2(c_2-1)s_1)\,, \\ b &= (1/|M|)(\omega_1^2-\omega_2^2)\omega_1\omega_2 \\ &\quad\times(\omega_1(c_2-1)s_1-\omega_2(c_1-1)s_2)\,, \end{aligned} \tag{51}$$

and forming the trace

$$Z=\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv (xv\tau_b|xv\tau_a)\,. \tag{52}$$

This yields

$$\begin{aligned} Z &= F(\beta)\frac{\pi}{\sqrt{ab}} \\ &= F(\beta)\frac{\pi\sqrt{|M|}}{\sqrt{(\omega_1^2-\omega_2^2)^2\omega_1\omega_2(c_1-1)(c_2-1)}} \end{aligned}$$

$$= \frac{1}{2} \frac{1}{\sqrt{(c_1-1)(c_2-1)}}$$

$$= \frac{1}{2\sinh(\omega_1\beta/2)} \frac{1}{2\sinh(\omega_2\beta/2)}. \quad (53)$$

The result factorizes into the partition functions of the two harmonic oscillators contained in the system. This could also have been obtained directly from (4) by summing over all *periodic* paths, which would have given

$$Z = \prod_{m=0,\pm 2,\pm 4} \frac{1}{[(\epsilon^2\Omega_m^2 + \epsilon^2\omega_1^2)(\epsilon^2\Omega_m^2 + \epsilon^2\omega_2^2)]^{1/2}}$$

$$= \frac{1}{2\sinh(\omega_1\beta/2)} \frac{1}{2\sinh(\omega_2\beta/2)}. \quad (54)$$

In this product, the integer m runs through all even numbers, positive as well as negative, since periodic paths have the Fourier expansion

$$x(\tau) = \frac{1}{\sqrt{\beta}} \sum_{m=0,\pm 2,\pm 4} (e^{i\nu_m\tau} x_m + \text{c.c.}), \quad (55)$$

with $x_m = x^*_{-m}$.

V. LIMITING CASES

Let us check our result (39) at $j=0$ by looking at a couple of limiting cases that have been solved before. Taking $\omega_2 = 0$, $\omega_1 = \omega$, the Hamiltonian (9) reduces to that of a harmonic oscillator in the variable v with an external linear potential ipv. The integral over $\mathscr{D}x$ in (11) forces $p(\tau)$ to be a constant (via the canonical term $\exp \int_{\tau_a}^{\tau_b} d\tau\, ipx$ in the integrand) and the path integral (11) can be written as the Fourier transform

$$(x_b v_b \tau_b | x_a v_a \tau_a) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-ip(x_b - x_a)} (v_b \tau_b | v_a \tau_a)_p \quad (56)$$

of the following probability distribution:

$$(v_b \tau_b | v_a \tau_a)_p \equiv \int \mathscr{D}v(\tau) \times \exp\left[-\int_{\tau_a}^{\tau_b} d\tau \left(\frac{\dot v^2}{2} + \frac{\omega^2}{2} v^2 + ipv\right)\right]. \quad (57)$$

This path integral is well known. It is obtained by a simple shift of the standard oscillator expression[10]

$$(v_b \tau_b | v_a \tau_a) = \int \mathscr{D}v(\tau) \exp\left[-\int_{\tau_a}^{\tau_b} d\tau \left(\frac{\dot v^2}{2} + \frac{\omega^2}{2} v^2\right)\right]$$

$$= \sqrt{\frac{\omega}{2\pi \sinh \omega\beta}} \exp\left\{-\frac{\omega}{2\sinh\omega\beta} \times [\cosh \omega\beta (v_b^2 + v_a^2) - 2v_b v_a]\right\}, \quad (58)$$

with $s \equiv \sinh \omega\beta$, $c \equiv \cosh \omega\beta$, namely,

$$(v_b \tau_b | v_a \tau_a)_p = \sqrt{\frac{\omega}{2\pi s}} \exp\left\{-\frac{\omega}{2s}[c(v_b^2 + v_a^2) - 2v_b v_a] - ip\frac{c-1}{\omega s}(v_b + v_a) - \frac{p^2}{2\omega^2}\left(\beta - 2\frac{c-1}{\omega s}\right)\right\}, \quad (59)$$

whereupon (56) becomes

$$(x_b v_b \tau_b | x_a v_a \tau_a) = \sqrt{\frac{\omega}{2\pi s}} \frac{1}{\sqrt{2\pi\beta}} \frac{\omega}{\sqrt{1-\rho}} \times \exp\left\{-\frac{\omega}{2s}[c(v_b^2 + v_a^2) - 2v_b v_a] - \frac{\omega^2}{2\beta} \frac{1}{1-\rho}\left[x_b - x_a - \frac{\beta}{2}\rho(v_b + v_a)\right]^2\right\}, \quad (60)$$

with

$$\rho \equiv 2\frac{c-1}{\omega\beta s} = \frac{\tanh(\omega\beta/2)}{(\omega\beta/2)}. \quad (61)$$

Taking the trace of (60) with respect to the velocity variable, the distribution acquires the simple form

$$(x_b \tau_b | x_a \tau_a) = \frac{1}{2\sinh(\omega\beta/2)} \times \frac{1}{\sqrt{2\pi\beta/\omega}} e^{-(\omega/2\beta)(x_b - x_a)^2}. \quad (62)$$

The prefactor $1/[2\sinh(\omega\beta/2)]$ accounts for the partition function of the harmonic oscillator associated with the variable v. Apart from that, expression (62) shows the standard mean-square end-to-end distance of a random chain, namely $\langle (x_b - x_a)^2 \rangle = \beta/\omega$. It has the same linear behavior in β as in the absence of the stiffness term $\ddot x^2$.

It is easy to verify that our general expression (39) with (40) and (49) reduces to (60) for $\omega_2 = 0$. Indeed, then

$$|M| \to \omega\omega_2[\omega\beta s - 2(c-1)] = \omega\omega_2\omega\beta s(1-\rho)$$

and

$$\mathscr{A}_{\text{cl,sf}} \to \frac{1}{2\beta s(1-\rho)} \times \{(\omega\beta c - s)(v_b^2 + v_a^2) - 2(\omega\beta - s)v_b v_a - 2\omega(c-1)(v_b x_b - v_a x_a - v_b x_a + v_a x_b) + \omega^2 s(x_b - x_a)^2\}, \quad (63)$$

giving the correct exponent as well as the fluctuation factor in (60).

If we also let $\omega \to 0$, then $1 - \rho \to \frac{1}{12}\omega^2\beta^2$ and the action reduces to the simple expression

$$\mathscr{A}_{\text{cl,sf}} \to (1/2\beta)(v_b - v_a)^2 + (6/\beta^3) \times [x_b - x_a - (\beta/2)(v_b + v_a)]^2, \quad (64)$$

which could have been found directly from the classical orbit

$$x - x_a + v_a\tau + x_2\tau^2 + x_3\tau^3 \quad (65)$$

after adjusting the parameters x_2, x_3 to the initial and final values

$$x_3 = -(2/\beta^3)[x_b - x_a - (\beta/2)(v_b + v_a)],$$

$$x_2 = (3/\beta^2)(x_b - x_a) + (1/2\beta)(v_b - v_a) - (3/2\beta)(v_b + v_a). \quad (66)$$

The transition probability becomes

$$(x_b v_b \tau_b | x_a v_a \tau_a) = (\sqrt{3}/\pi\beta^2) e^{-\mathscr{A}_{\rm cl,sf}}. \quad (67)$$

Another useful limit is that of $\omega_2 \to \omega_1 \equiv \omega$. Setting $\omega_1 = \omega + \epsilon$, $\omega_2 = \omega - \epsilon$, the determinant becomes

$$|M| \to 4\epsilon^2 (s^2 - \omega^2\beta^2). \quad (68)$$

Inserting the limit

$$c_{\substack{1\\2}} = c(1 \pm \epsilon\beta \tanh(\omega\beta)) + (\epsilon^2/2)\beta^2 + \cdots,$$
$$s_{\substack{1\\2}} = s(1 \pm \epsilon\beta \coth(\omega\beta)) + (\epsilon^2/2)\beta^2 + \cdots, \quad (69)$$

and using $\coth(\omega\beta) - \tanh(\omega\beta) = (1/sc)$ we find the classical action

$$\begin{aligned}\mathscr{A}_{\rm cl,1} = \frac{\omega}{s^2 - \omega^2\beta^2} \{ & (sc - \omega\beta)(v_b^2 + v_a^2) \\ & - 2(s - c\omega\beta) v_b v_a \\ & - \omega(s^2 + \omega^2\beta^2)(v_b x_b - v_a x_a) \\ & + 2\omega s (v_b x_a - v_a x_b) + \omega^2 (sc + \omega\beta)(x_b^2 + v_a^2) \\ & - 2\omega^2 (s + c\omega\beta) v_b x_a \}, \end{aligned} \quad (70)$$

and hence

$$(x_b v_b \tau_b | x_a v_a \tau_a) = \frac{1}{\pi} \frac{\omega^2}{\sqrt{s^2 - \omega^2\beta^2}} e^{-\mathscr{A}_{\rm cl,sf}}. \quad (71)$$

In the limit $\omega \to 0$, this reduces again to (67) with (64), as it should.

VI. THE SOURCE TERMS

The source appears in (18) and the last term in (20). First we calculate (18):

$$\mathscr{A}_{\rm cl,source} = -\int_{\tau_a}^{\tau_b} d\tau\, x_{\rm cl}(\tau) j(\tau), \quad (72)$$

where $x_{\rm cl}(\tau)$ is given by (8) with A, B, C, and D expressed in terms of $x_a v_a x_b v_b$ via the matrix M^{-1} of Eq. (42). Hence

$$x_{\rm cl}(\tau) = \frac{1}{|M|} R \begin{pmatrix} \cosh \omega_1(\tau - \tau_a) \\ \sinh \omega_1(\tau - \tau_a) \\ \cosh \omega_2(\tau - \tau_a) \\ \sinh \omega_2(\tau - \tau_a) \end{pmatrix}. \quad (73)$$

In the ordinary harmonic oscillator, the usual way of giving the classical solution is more symmetrical in τ_a and τ_b

$$\begin{aligned} x_{\rm cl} = (1/\sinh \omega\beta)(& x_b \sinh \omega(\tau - \tau_a) \\ & + x_a \sinh \omega(\tau_b - \tau)). \end{aligned} \quad (74)$$

It displays directly the interpolation between x_a and x_b. We can also bring (73) to such a form, which, however, is now much more involved. By expanding $x_{\rm cl}$ into the four solutions

$$\begin{aligned} f_a(\tau) &= \omega_2 \sinh \omega_1(\tau - \tau_a) - \omega_1 \sinh \omega_2(\tau - \tau_a), \\ f_b(\tau) &= \omega_2 \sinh \omega_1(\tau_b - \tau) - \omega_1 \sinh \omega_2(\tau_b - \tau), \\ g_a(\tau) &= \cosh \omega_1(\tau - \tau_a) - \cosh \omega_2(\tau - \tau_a), \\ g_b(\tau) &= \cosh \omega_1(\tau_b - \tau) - \cosh \omega_2(\tau_b - \tau), \end{aligned} \quad (75)$$

which have the boundary properties

$$\begin{aligned} & f_a(\tau_a) = 0, \quad f_a'(\tau_a) = 0, \\ & f_b(\tau_b) = 0, \quad f_b'(\tau_b) = 0, \\ & g_a(\tau_a) = 0, \quad g_b'(\tau_b) = 0, \\ & g_b(\tau_b) = 0, \quad g_b'(\tau_b) = 0, \end{aligned} \quad (76)$$

it is straightforward to form the linear combination, with the correct initial and final values

$$\begin{aligned} x_{\rm cl}(\tau) = & -(1/|M|) \\ & \times \{ [x_b(\omega_1 s_1 - \omega_2 s_2) - v_b(c_1 - c_2)] f_a(\tau) \\ & + [x_a \omega_1 s_1 - \omega_2 s_2) - v_a(c_1 - c_2)] f_b(\tau) \\ & - [x_b \omega_1 \omega_2 (c_1 - c_2) - v_b(\omega_2 s_1 - \omega_1 s_2)] g_a(\tau) \\ & - [x_a \omega_1 \omega_2 (c_1 - c_2) \\ & - v_a(\omega_2 s_1 - \omega_1 s_2)] g_b(\tau) \}. \end{aligned} \quad (77)$$

This may be more useful than (73), for some purposes.

Let us now turn to the fluctuation part of the external source term in (39). Notice that it is sufficient to calculate the odd and even sums

$$\hat{G}_o(\tau,\tau') = \frac{2}{\beta} \sum_{m=1,3,5,\ldots} \frac{\sin \nu_m(\tau - \tau_a) \sin \nu_m(\tau' - \tau_a)}{(\Omega_m^2 + \omega_1^2)(\Omega_m^2 + \omega_2^2)},$$
$$\hat{G}_e(\tau,\tau') = \frac{2}{\beta} \sum_{m=2,4,6,\ldots} \frac{\sin \nu_m(\tau - \tau_a) \sin \nu_m(\tau' - \tau_a)}{(\Omega_m^2 + \omega_1^2)(\Omega_m^2 + \omega_2^2)}. \quad (78)$$

Then

$$\hat{G}(\tau,\tau') = \hat{G}_o(\tau,\tau') + \hat{G}_e(\tau,\tau') \quad (79)$$

and the functions $h_o(\tau)$, $h_e(\tau')$ are simply found from the derivatives [compare (37a)]

$$h_{\substack{o\\e}}(\tau) = \lim_{\tau' \to \tau_a} \frac{\partial}{\partial \tau'} \hat{G}_{\substack{o\\e}}(\tau,\tau'). \quad (80)$$

In the sums (78) we can replace Ω_m^2 by ν_m^2, due to their fast convergence, and write

$$\hat{G}_o(\tau,\tau') = [1/(\omega_2^2 - \omega_1^2)](G_o^{\omega_1}(\tau,\tau') - G_o^{\omega_2}(\tau,\tau')), \quad (81)$$
$$\hat{G}_e(\tau,\tau') = [1/(\omega_2^2 - \omega_1^2)](G_e^{\omega_1}(\tau,\tau') - G_e^{\omega_2}(\tau,\tau')), \quad (82)$$

where

$$\begin{aligned} G_{\substack{o\\e}}^{\omega}(\tau,\tau') &= \frac{2}{\beta} \sum_{\substack{m=1,3,5,\ldots\\2,4,6}} \frac{\sin \nu_m(\tau - \tau_a) \sin \nu_m(\tau' - \tau_a)}{(\nu_m^2 + \omega^2)} \\ &= \pm \frac{2}{\beta} \sum_{\substack{m=1,3,5\\2,4,6}} \frac{\sin \nu_m(\tau_b - \tau) \sin \nu_m(\tau' - \tau_a)}{(\nu_m^2 + \omega^2)} \end{aligned} \quad (83)$$

are the odd and even frequency parts of the correlation function of the ordinary harmonic oscillator. They, in turn, are simply obtained from the standard boson and fermion correlation functions

$$\begin{aligned} G_{\rm B}(\tau) &= \frac{1}{\beta} \sum_{m=0,\pm 2,\pm 4,\ldots} e^{-i\nu_m\tau} \frac{1}{\nu_m^2 + \omega^2} \\ &= \frac{1}{2\omega} \frac{\cosh \omega[\tau - (\beta/2)]}{\sinh(\omega\beta/2)}, \quad \tau \in (0,\beta), \end{aligned} \quad (84)$$

$$G_{\rm F}(\tau) = \frac{1}{\beta} \sum_{m=\pm 1,\pm 3,\pm 4} e^{i\nu_m\tau} \frac{1}{\nu_m^2 + \omega^2}$$

$$= -\frac{1}{2\omega}\frac{\sinh \omega[\tau-(\beta/2)]}{\cosh(\omega\beta/2)}, \quad \tau\in(0,\beta). \tag{85}$$

For $\tau=0$ these coincide with the sums appearing in Eqs. (30) and (31), as they should.

Notice that the right-hand side is valid only for $\tau\in(0,\beta)$. Outside this interval, the functions have to be continued periodically or antiperiodically for G_e or G_o. An explicit representation which shows this property is obtained by rewriting

$$G_B(\tau) = \sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}e^{-i\nu(\tau+l\beta)}\frac{1}{\nu^2+\omega^2}, \tag{86}$$

$$G_F(\tau) = \sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}e^{-i\nu(\tau+l\beta)}e^{\pi i l}\frac{1}{\nu^2+\omega^2}, \tag{87}$$

where the sums over all integer numbers l squeeze the ν integrals into the appropriate sums (84) and (85). Performing the integrals over ν gives

$$G_B(\tau) = \frac{1}{2\omega}\sum_l(\theta(\tau+l\beta)e^{-\omega(\tau+l\beta)} + \theta(\tau-l\beta)e^{\omega(\tau+l\beta)}), \tag{88}$$

$$G_F(\tau) = \frac{1}{2\omega}\sum_l(-)^l(\theta(\tau+l\beta)e^{-\omega(\tau+l\beta)} + \theta(\tau-l\beta)e^{\omega(\tau+l\beta)}). \tag{89}$$

For $\tau\in(0,\beta)$, the sums split into $l=0,1,2,\ldots$ and $l=-1,-2,-3,\ldots$ and can be performed to yield the results (84) and (85). For $\tau\in(\beta,2\beta)$, however, these have to be replaced by

$$G_B(\tau) = \frac{1}{2\omega}\frac{\cosh\omega[\tau-(3\beta/2)]}{\sinh(\omega\beta/2)} \tag{90}$$

$$[\tau\in(\beta,2\beta)].$$

$$G_F(\tau) = \frac{1}{2\omega}\frac{\sinh\omega[\tau-(3\beta/2)]}{\cosh(\omega\beta/2)} \tag{91}$$

When forming the appropriate combinations of these correlation functions in (83) and (84), we have to distinguish the cases $\tau+\tau'<\tau_a+\tau_b$, $\tau+\tau'>\tau_a+\tau_b$. In the first case we find

$$G_e^\omega(\tau,\tau') = -\frac{1}{2\omega\sinh(\omega\beta/2)} \times\sinh\omega\left(\tau-\frac{\tau_b+\tau_a}{2}\right)\sinh\omega(\tau'-\tau_a),$$

$$\text{for } \tau>\tau'\in(\tau_a,\tau_b),\quad \tau+\tau'<\tau_a+\tau_b, \tag{92}$$

$$G_o^\omega(\tau,\tau') = \frac{1}{2\omega\cosh(\omega\beta/2)} \times\cosh\omega\left(\tau-\frac{\tau_b+\tau_a}{2}\right)\sinh\omega(\tau'-\tau_a). \tag{93}$$

In the second case

$$G_e^\omega(\tau,\tau') = \frac{1}{2\omega\sinh(\omega\beta/2)} \times\sinh\omega(\tau_b-\tau)\sinh\omega\left(\tau'-\frac{\tau_a+\tau_b}{2}\right),$$

$$\text{for } \tau>\tau'\in(\tau_a,\tau_b),\quad \tau+\tau'>\tau_a+\tau_b, \tag{94}$$

$$G_o^\omega(\tau,\tau') = -\frac{1}{2\omega\cosh(\omega\beta/2)} \times\sinh\omega(\tau_b-\tau)\cosh\omega\left(\tau'-\frac{\tau_a+\tau_b}{2}\right). \tag{95}$$

As a check we add the even and odd results and find

$$G^\omega(\tau,\tau') = (\omega\sinh\omega\beta)^{-1} \times\sinh\omega(\tau_b-\tau)\sinh\omega(\tau'-\tau_a), \quad \tau>\tau', \tag{96}$$

in either case, which is the correct correlation function

$$G^\omega(\tau,\tau') = \langle\delta x(\tau)\delta x(\tau')\rangle|_{\text{oscill}} = \frac{2}{\beta}\sum_{m=1,2,\ldots}\frac{\sin\nu_m(\tau-\tau_a)\sin\nu_m(\tau'-\tau_a)}{\nu_m^2+\omega^2} \tag{97}$$

appearing in the path integral of the ordinary harmonic oscillator.[11] Inserting (94)–(97) into (82) we find the odd and even parts of the correlation function $\hat{G}(\tau,\tau')$:

$$\hat{G}_e(\tau,\tau') = -\frac{1}{(\omega_2^2-\omega_1^2)}\left(\frac{1}{2\omega_1 s_1}\sinh\omega_1\left(\tau-\frac{\tau_b+\tau_a}{2}\right) \times\sinh\omega_1(\tau'-\tau_a)-(12)\right),$$

$$\text{for } \tau>\tau'\in(\tau_a,\tau_b),\quad \tau+\tau'<\tau_a+\tau_b, \tag{98}$$

$$\hat{G}_o(\tau,\tau') = \frac{1}{(\omega_2^2-\omega_1^2)}\left(\frac{1}{2\omega_1 c_1}\cosh\omega_1\left(\tau-\frac{\tau_b+\tau_a}{2}\right) \times\sinh\omega_1(\tau'-\tau_a)-(12)\right),$$

and

$$\hat{G}_e(\tau,\tau') = \frac{1}{\omega_2^2-\omega_1^2}\left(\frac{1}{2\omega_1 s_1}\sinh\omega_1(\tau_b-\tau) \times\sinh\omega_1\left(\tau'-\frac{\tau_a+\tau_b}{2}\right)-(12)\right),$$

$$\text{for } \tau>\tau'\in(\tau_b,\tau_a),\quad \tau+\tau'>\tau_a+\tau_b, \tag{99}$$

$$\hat{G}_o(\tau,\tau') = -\frac{1}{\omega_2^2-\omega_1^2}\left(\frac{1}{2\omega_1 c_1}\sinh\omega_1(\tau_b-\tau) \times\cosh\omega_1\left(\tau'-\frac{\tau_a+\tau_b}{2}\right)-(12)\right).$$

Adding up the even and odd parts we find, according to formula (79),

$$\hat{G}(\tau,\tau') = -\frac{1}{\omega_1^2-\omega_2^2}\left[\frac{1}{\omega_1 s_1}\sinh\omega_1(\tau_b-\tau) \times\sinh\omega_1(\tau'-\tau_a)-(12)\right] \tag{100}$$

in either case. This is the first part of the correlation function $\langle\delta x(\tau)\delta x(\tau')\rangle$ in Eq. (35).

Since we have treated the even and odd parts separately, it is now easy to find other pieces $h_e(\tau)$, $h_e(\tau')$ from the limits (80)

$$h_e(\tau)=\frac{1}{\omega_1^2-\omega_2^2}\left[\frac{1}{2\sinh(\omega_1\beta/2)}\times\sinh\left(\omega_1\left(\tau-\frac{\tau_a+\tau_b}{2}\right)\right)-(12)\right],$$

$$h_o(\tau)=-\frac{1}{\omega_1^2-\omega_2^2}\left[\frac{1}{2\cosh(\omega_1\beta/2)}\times\cosh\left(\omega_1\left(\tau-\frac{\tau_a+\tau_b}{2}\right)\right)-(12)\right]. \qquad (101)$$

As a cross check, we may form

$$\lim_{\tau\to\tau_a}\frac{\partial}{\partial\tau}h_g(\tau),$$

which gives D_e, D_o, as it should [compare with (37b) and (31)].

Combining (103) and (104) and using D_e, D_o we obtain the complete correlation function of the fluctuations [recall (35)]:

$$\langle\delta x(\tau)\delta x(\tau')\rangle=G(\tau,\tau')=-\frac{1}{\omega_1^2-\omega_2^2}\left\{\left(\frac{1}{\omega_1 s_1}\sinh\omega_1(\tau_b-\tau)\sinh\omega_1(\tau'-\tau_a)-(12)\right)\right.$$
$$+\frac{1}{2}\frac{1}{(\omega_1\coth(\omega_1\beta/2)-(12))}\left(\frac{1}{\sinh(\omega_1\beta/2)}\sinh\omega_1\left(\tau-\frac{\tau_a+\tau_b}{2}\right)-(12)\right)\times\left(\frac{1}{\sinh(\omega_1\beta/2)}\sinh\omega_1\left(\tau'-\frac{\tau_a+\tau_b}{2}\right)-(12)\right)$$
$$\left.+\frac{1}{2}\frac{1}{(\omega_1\tanh(\omega_1\beta/2)-(12))}\left(\frac{1}{\cosh(\omega_1\beta/2)}\cosh\omega_1\left(\tau-\frac{\tau_a+\tau_b}{2}\right)-(12)\right)\times\left(\frac{1}{\cosh(\omega_1\beta/2)}\cosh\omega_1\left(\tau'-\frac{\tau_a+\tau_b}{2}\right)-(12)\right)\right\}. \qquad (102)$$

As a final check we verify once more that this Green's function vanishes at the end points together with its time derivatives. This completes the calculation of the probability distribution $(x_b v_b \tau_b | x_a v_a \tau_a)$. The result is Eq. (39) with the prefactor (40), the classical surface term (49), the classical source term (72) with $x_{\rm cl}(\tau)$ given in (73) or (77), and the fluctuation part of the source term given by the correlation function (102).

VII. LIMITING FORMS OF SOURCE TERMS

For completeness, let us perform the limits $\omega_2\to 0$, $\omega_2\to 0$, $\omega_1\to 0$, and $\omega_1\to\omega_2$ on the source terms. For $\omega_2=0$, $\omega_1=\omega$ the classical solution (77) reduces to

$$x_{\rm cl}=-\frac{1}{\beta\omega(1-\rho)}\times\left\{\left[\left(x_b-\frac{\rho}{2}\beta v_b\right)(\sinh\omega(\tau-\tau_a)-\omega(\tau-\tau_a))+\left(x_a-\frac{\rho}{2}\beta v_a\right)(\sinh\omega(\tau_b-\tau)-\omega(\tau_b-\tau))\right]\right.$$
$$\left.-\left[\left(x_b\frac{\rho}{2}\beta\omega-\frac{v_b}{\omega}\right)\left(1-\frac{\omega\beta}{s}\right)(\cosh\omega(\tau-\tau_a)-1)+\left(x_a\frac{\rho}{2}\beta\omega-\frac{v_a}{\omega}\right)\left(1-\frac{\omega\beta}{s}\right)\times(\cosh\omega(\tau_b-\tau)-1)\right]\right\}. \qquad (103)$$

If also $\omega\to 0$,

$$x_{\rm cl}=-12\left\{\left(x_b-\frac{\beta}{2}v_b\right)\frac{(\tau-\tau_a)^3}{6\beta^3}-(ba)-\left(x_b-\frac{\beta}{6}v_b\right)\frac{(\tau-\tau_a)^2}{4\beta^2}+(ba)\right\}. \qquad (104)$$

In the limit $\omega_1-\omega_2=2\epsilon\to 0$, the functions (75) tend towards

$$f_a(\tau)\to 2\epsilon(\omega(\tau-\tau_a)\cosh\omega(\tau-\tau_a)-\sinh\omega(\tau-\tau_a)),$$

$$g_a(\tau)\to 2\epsilon(\tau-\tau_a)\sinh\omega(\tau-\tau_a),$$

with analogous limits for $f_b(\tau)$, $g_b(\tau)$, and the classical solutions become

$$x_{\rm cl}(\tau)=-\frac{1}{s^2-\omega^2\beta^2}\left\{\left(x_b\left(s+\omega\beta\frac{c}{s}\right)-v_b s\right)(\omega(\tau-\tau_a)\times\cosh(\tau-\tau_a)-\sinh\omega(\tau-\tau_a))-(ab)\right.$$
$$\left.-x_b\left(\omega\beta s+\frac{v_b}{\omega}\left(s-\omega\beta\frac{c}{s}\right)\right)\omega(\tau-\tau_a)\times\sinh\omega(\tau-\tau_a)+(ab)\right\}. \qquad (105)$$

The fluctuation part of the source contribution has the following limits: for $\omega_2\to 0$, $\omega_1=\omega$,

$$\hat{G}(\tau,\tau')=-\frac{1}{\omega^2}\left[\frac{1}{\omega s}\sinh\omega(\tau_b-\tau)\sinh\omega(\tau'-\tau_b)-\frac{(\tau_b-\tau)(\tau'-\tau_a)}{\beta}\right]; \qquad (106)$$

for $\omega_2\to 0$, $\omega_1\to 0$,

$$\hat{G}(\tau,\tau')\to-\frac{1}{6\beta}(\tau_b-\tau)(\tau'-\tau_a)\times[(\tau_b-\tau)^2+(\tau'-\tau_a)^2-\beta^2]; \qquad (107)$$

for $\omega_2\to\omega_1=\omega$,

$$\hat{G}(\tau,\tau')\to\frac{1}{2\omega^3 s}\left[\left(1+\omega\beta\frac{c}{s}\right)\sinh\omega(\tau_b-\tau)\sinh\omega(\tau'-\tau_a)\right.$$
$$-\sinh\omega(\tau_b-\tau)\omega(\tau'-\tau_a)\cosh\omega(\tau'-\tau_a)$$

$$- \omega(\tau_b - \tau)\cosh \omega(\tau_b - \tau)\sinh \omega(\tau' - \tau_a)] ; \quad (108)$$

with the latter reducing properly to (107) in the limit $\omega \to 0$.

The functions $h_e(\tau)$, $h_o(\tau)$ become, for $\omega_2 \to 0$, $\omega_1 = \omega$,

$$h_e(\tau) = \frac{1}{\omega^2}\left(\frac{\sinh \omega(\tau - (\tau_a + \tau_b)/2)}{2\sinh(\omega\beta/2)} - \frac{\tau - (\tau_a + \tau_b)/2}{\omega\beta}\right), \quad (109)$$

$$h_o(\tau) = -\frac{1}{\omega^2}\left(\frac{\cosh \omega(\tau - (\tau_a + \tau_b)/2)}{2\cosh(\omega\beta/2)} - \frac{1}{2}\right); \quad (110)$$

for $\omega_2 \to 0$, $\omega_1 \to 0$,

$$h_e(\tau) = \frac{1}{6\beta}\left(\tau - \frac{\tau_a + \tau_b}{2}\right)\left[\left(\tau - \frac{\tau_a + \tau_b}{2}\right)^2 - \frac{1}{4}\beta^2\right], \quad (111)$$

$$h_o(\tau) = -\frac{1}{4}\left[\left(\tau - \frac{\tau_a + \tau_b}{2}\right)^2 - \frac{1}{4}\beta^2\right]; \quad (112)$$

and for $\omega_2 \to \omega_1 = \omega$,

$$h_e(\tau) = \frac{1}{4\omega^2 \sinh(\omega\beta/2)} \times\left(\omega\left(\tau - \frac{\tau_a + \tau_b}{2}\right)\cosh \omega\left(\tau - \frac{\tau_a + \tau_b}{2}\right) - \frac{\omega\beta}{2}\coth\frac{\omega\beta}{2}\sinh \omega\left(\tau - \frac{\tau_a + \tau_b}{2}\right)\right), \quad (113)$$

$$h_o(\tau) = -\frac{1}{4\omega^2 \cosh(\omega\beta/2)} \times\left(\omega\left(\tau - \frac{\tau_a + \tau_b}{2}\right)\sinh \omega\left(\tau - \frac{\tau_a + \tau_b}{2}\right) - \frac{\omega\beta}{2}\tanh\frac{\omega\beta}{2}\cosh \omega\left(\tau - \frac{\tau_a + \tau_b}{2}\right)\right); \quad (114)$$

and the quantities D_e, D_o, for $\omega_2 \to 0$, $\omega_1 = \omega$,

$$D_e = \frac{1}{\beta\omega^2}\left(\frac{\omega\beta}{2}\coth\frac{\omega\beta}{2} - 1\right), \quad (115)$$

$$D_o = \frac{1}{\beta\omega^2}\frac{\omega\beta}{2}\tanh\frac{\omega\beta}{2}; \quad (116)$$

for $\omega_2 \to 0$, $\omega_1 \to 0$,

$$D_e = \tfrac{1}{12}\beta, \quad (117)$$

$$D_o = \tfrac{1}{4}\beta; \quad (118)$$

and for $\omega_2 \to \omega_1 = \omega$,

$$D_e = \frac{\beta}{8}\left(\frac{\coth(\omega\beta/2)}{\omega\beta/2} - \frac{1}{\sinh^2(\omega\beta/2)}\right), \quad (119)$$

$$D_o = \frac{\beta}{8}\left(\frac{\tanh(\omega\beta/2)}{\omega\beta/2} + \frac{1}{\cosh^2(\omega\beta/2)}\right). \quad (120)$$

Combining these $\hat{G}$, h, and D as required by (35) we obtain the limiting terms of the correlation function $G(\tau,\tau')$.

VIII. SECOND QUANTIZATION

Frequently one is not interested in studying the behavior of a single fluctuating-line-like object but wants to consider grand-canonical ensembles of these. It is then convenient to introduce a single fluctuating field whose Feyman diagrams are capable of representing all the different individual line contributions. For the usual random chain with a Lagrangian $(D/2a)\dot{x}^2$ in D dimensions, it is well known how to achieve this goal. For open chains of a given length L the appropriate field is $\psi(\mathbf{x},\tau)$ and has the action[12]

$$\mathscr{A} = \int_0^D d\tau \int d^D x\{\psi^+(\partial_\tau - \mu)\psi + \psi^+ H(-i\,\partial_x)\psi\}, \quad (121)$$

where $H(\mathbf{p}) = \mathbf{p}^2/(2D/a)$ is the Hamiltonian and μ is the chemical potential of a chain element. For loops of any length L, with a distribution $e^{-m^2 L}$, the fields $\varphi(x)$ depend only on the spatial variable $\mathbf{x}$ and the action is of the Klein–Gordon type[13]

$$\mathscr{A} = \int d^D x\, \varphi(\mathbf{x})(H(-i\partial_x) + m^2)\varphi(\mathbf{x}) = \frac{1}{2}\int d^D x[(\partial\varphi(\mathbf{x}))^2 + m^2\varphi^2(\mathbf{x})]. \quad (122)$$

In the present case where the Lagrangian contains a second time derivative, a second quantization can be achieved by introducing, for open chains of a given length L, a field $\psi(x,v,\tau)$ which depends on position, velocity, and time with an action

$$\mathscr{A} = \int_0^L d\tau \int d^D x \int d^D x \times\{\psi^+(\partial_\tau - \mu)\psi + \psi^+ H(-i\,\partial_x,\mathbf{x},-\mathrm{i}\,\partial_v,\mathbf{v},\tau)\psi\}, \quad (123)$$

where $H(p,x,p_v,v,\tau)$ is a Hamiltonian of the type (9) in D dimensions. For closed chains of any length one has, similarly, a field $\varphi(\mathbf{x},\mathbf{v})$ and an action

$$\mathscr{A} = \frac{1}{2}\int d^D x\, d^D v\, \varphi(\mathbf{x},\mathbf{v})(H(\mathbf{p},\mathbf{x},\mathbf{p}_v,\mathbf{v},\tau) + m^2)\varphi(\mathbf{x},\mathbf{v}).$$

IX. CONCLUSION

We have calculated the exact amplitude for fluctuating orbits $x(\tau)$ governed by the general second-gradient Lagrangian (2). The results is given by Eq. (39) with the fluctuation prefactor (40), the classical action (49), the classical source action (72) and (77), and the fluctuation part of the source given by (102).

ACKNOWLEDGMENT

The author is grateful to Professor B. Zimm for a useful discussion and to Professor N. Kroll and Professor J. Kuti for their hospitality at the University of California, San Diego.

This work was supported in part by the Deutsche Forschungsgemeinschaft under Grant No. Kl 256 and by UCSD/DOE Contract No. DEAT-03-81ER40029.

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