PATH INTEGRAL FOR COULOMB SYSTEM WITH MAGNETIC CHARGES

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We calculate the path integral for the lagrangian $L=\frac{1}{2}m\dot{x}^2+e\dot{x}\cdot A-e\bar{e}/r-v^2/2mr^2$, where $A_i=\bar{g}\epsilon_{ij}x_jx_3/r(x_1^2+x_2^2)$ is the vector potential of a magnetic monopole, $\bar{e}$ its charge, and e the charge of the particle in orbit. In addition, we allow for an arbitrary centrifugal barrier $1/r^2$ potential. After the replacement $e\bar{e}\to e\bar{e}+g\bar{g}$, $e\bar{g}\to e\bar{g}-g\bar{e}$, the results apply to dyonium, the bound state between two electrically and magnetically charged particles.

It is always fun to see old friends in a new dress, especially if this reveals new insights. For this reason, the recalculation of the Green function of the Coulomb problem by different methods [1] has been a popular exercise ever since Schwinger's original solution [2]. Recently, path integration has become a favorite technique of solving once again well-known problems. For the Coulomb case, this was done some years ago [3,4] [#1] and repeated with various modifications [5–7] [#2,3].

In this note we want to generalize our method [3,4] to the case of an electric charge e in orbit around a dyon with charge $\bar{e}$ and magnetic charge $\bar{g}$ [#4]. In order to make the Dirac string invisible, $\bar{g}$ and e fulfil the quantization condition

$$\bar{g}e=q=\text{half-integer}. \tag{1}$$

For the sake of being general, we also add an arbitrary $1/r^2$ potential. We shall calculate the amplitude

$$\langle x_bt_b|x_at_a\rangle=\int\mathscr{D}^3x(t)\,\exp\left[\mathrm{i}\int_{t_a}^{t_b}\mathrm{d}t\left(\tfrac{1}{2}m\dot{x}^2+e\dot{x}\cdot A-\frac{e\bar{e}}{r}-\frac{v^2}{2mr^2}\right)\right], \tag{2}$$

where

$$\int\mathscr{D}x(t)\equiv\lim_{n\to\infty}\frac{1}{\sqrt{2\pi\mathrm{i}\epsilon/m}}\prod_{n=1}^{N}\int_{-\infty}^{\infty}\frac{\mathrm{d}^3x_n}{\sqrt{2\pi\mathrm{i}\epsilon/m}},$$

via path integration. The vector potential A associated with the magnetic charge $\bar{g}$ is given in the abstract.

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[#1] Ref. [4] is the detailed version of ref. [3] including also the two-dimensional case.

[#2] The authors of ref. [5] are more explicit than refs. [3,4] and write down all expressions in the time sliced form but get the correct result only after using a wrong jacobian $\partial(x,s)_j/\partial(u)_j=2^4\bar{r}_j^2$ (see the last line before their eq. (13)). The correct jacobian is $2^4r_j^2$. In fact, construction of the variable F_j^2 must be taken to be equal to r_{j-1}^2 rather than the average between r_i and r_{j-1}.

[#3] In ref. [7] the correct result emerges after using his wrong formula (7), since this author did not consider the fact that $\bar{r}_n^2$ is equal to r_{n-1}^2 by construction. See also ref. [8].

[#4] The problem has a long history, see ref. [9]. For recent discussions including spin, see ref. [10].

Going to the square-root variables

$$x_i = -z^\dagger \sigma_i z, \quad r = z^\dagger z, \tag{3}$$

where σ_i are the Pauli matrices and

$$z_1 = u_1 - \mathrm{i}u_4 = \sqrt{r}\sin\tfrac{1}{2}\theta \exp[-\tfrac{1}{2}\mathrm{i}(\alpha+\varphi)], \qquad z_2 = -u_3 + \mathrm{i}u_2 = -\sqrt{r}\cos\tfrac{1}{2}\theta \exp[-\tfrac{1}{2}\mathrm{i}(\alpha-\varphi)], \tag{4}$$

each point $x_i = (r\cos\theta,\ r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi)$ has as many square roots in the four-dimensional u_μ space as the angle α has values between 0 and 4π. The degeneracy of this mapping is removed by introducing

(1) an auxiliary fourth variable x_4 extending the kinetic term to $\dot{x}_\mu^2 \equiv \dot{x}^2 + \dot{x}_4^2$ and using the path integral in four-dimensional space

$$\int \mathrm{d}(x_4)_\mathrm{b} \int \mathscr{D}x(t) \exp\left(\mathrm{i}\int_{t_\mathrm{a}}^{t_\mathrm{b}} \mathrm{d}t\ \tfrac{1}{2}m\dot{x}^2\right)$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}(x_4)_{N+1}}{\left(\sqrt{2\pi \mathrm{i}\epsilon/m}\right)^4} \prod_{n=1}^{N} \int \frac{\mathrm{d}^4 x_n}{\left(\sqrt{2\pi\mathrm{i}\epsilon/m}\right)^4} \exp\left(\mathrm{i}\epsilon \sum_{n=1}^{N+1} (x_n - x_{n-1})^2/\epsilon^2\right). \tag{5}$$

The integral over the final $(x_4)_{N+1} = (x_4)_\mathrm{b}$ ensures that the extension does not alter the result (see refs. [3,4]).

(2) Mapping intervals $\mathrm{d}x_\mu$ into intervals $\mathrm{d}u_\mu$ by

$$\mathrm{d}x_\mu = 2A_\mu{}^\nu(a)\ \mathrm{d}u_\nu, \tag{6}$$

where

$$A_\mu{}^\nu(u) = \begin{pmatrix} u_3 & u_4 & u_1 & u_2 \\ -u_2 & -u_1 & u_4 & u_3 \\ -u_1 & u_2 & u_3 & -u_4 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}$$

has the inverse $(1/u^2)A^\mathrm{T}$ and the determinant $u^4 = r^2$. Then

$$\dot{x}_\mu^2 = 4u^2\dot{u}^2, \quad \dot{x}_1 x_2 - \dot{x}_2^2 x_1 = 4\left[\left(u_2^2+u_3^2\right)\left(u_4\dot{u}_1 - u_1\dot{u}_4\right) + \left(u_1^2+u_4^2\right)\left(u_3\dot{u}_2 - u_2\dot{u}_3\right)\right],$$

$$x_1^2 + x_2^2 = 4\left(u_2^2+u_3^2\right)\left(u_1^2+u_4^2\right).$$

In the new variables, the lagrangian reads

$$L(u_\mu, \dot{u}_\mu) = \frac{1}{r}\left[\frac{4m}{2}u^4\dot{u}_\mu^2 - q\left(\frac{1}{u_1^2+u_4^2}(u_4\dot{u}_1 - u_1\dot{u}_4) + \frac{1}{u_2^2+u_3^2}(u_3\dot{u}_2 - u_2\dot{u}_3)\right)\right.$$

$$\left. \times\left(u_1^2+u_4^2-u_2^2-u_3^2\right) + e\bar{e} - \frac{v^2}{2mu^2}\right]. \tag{7}$$

In terms of $u = \sqrt{u_\mu^2}$ and the angles θ, φ, α, it looks as follows:

$$L(u,\theta,\varphi,\alpha) = \frac{1}{r}\left\{\frac{4m}{2}u^4\dot{u}^2 + \tfrac{1}{2}mu^6\left[\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\alpha}^2 - 2\left(\dot{\alpha} - \frac{q}{mu^4}\right)\dot{\varphi}\cos\theta\right] + e\bar{e} - \frac{v^2}{2mu^2}\right\}. \tag{8}$$

We now perform the change from t to the path dependent pseudo-time s via the idem factor

$$r_b \int_0^\infty ds \int \frac{dE}{2\pi} \exp[-iE(t_b - t_a)] \exp\left(i\int_0^s ds\, Er(s)\right) = 1, \tag{9}$$

and arrive at the Duru–Kleinert type representation of the Fourier transformed amplitude (2) (see ref. [4] eqs. (100)–(106)):

$$\begin{aligned}\langle x_b | x_a \rangle_E &= \int_0^{4\pi} d\alpha_b \int_0^\infty ds\, e^{-ie\bar{e}s} \langle u_b s | u_a 0 \rangle \\ &= \int_0^{4\pi} d\alpha_b \int_0^\infty ds\, e^{-ie\bar{e}s} \left[\int \mathscr{D}^4 u(s) \exp\left(i\int_0^s ds\, \tfrac{1}{2} M \left\{ u_\mu'^2 + \tfrac{1}{4} u^2 \left[\theta'^2 + \varphi'^2 + \alpha'^2 \right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. - 2(\alpha' - 4q/Mu^2)\varphi' \cos\theta \right] - 4v^2/2Mu^2 + Eu^2 \right\} \right) \right],\end{aligned} \tag{10}$$

where $M = 4m$ and the prime denotes d/ds. We now observe that the lagrangian $L(u, u')$ in this amplitude has a Legendre transform

$$\begin{aligned} H &= p_\mu u_\mu' - L \\ &= (1/2M)\left\{ p_u^2 + (4/u^2)\left[p_\theta^2 + (1/\sin^2\theta)\left(p_\varphi^2 + (p_\alpha + q)^2 + 2(p_\alpha + q)p_\varphi \cos\theta \right) \right] \right\} \\ &\quad + (4/2Mu^2)\left[q^2 - q(p_\alpha + q) + v^2 \right]. \end{aligned} \tag{11}$$

In the canonical version of the path integral (10)

$$\int \mathscr{D}^4 x \int \frac{\mathscr{D}^4 p_u}{2\pi} \exp\left(\tfrac{1}{2} i \int_0^s ds\, (p_u u' + p_\theta \theta' + p_\varphi \varphi' + p_\alpha \alpha' - H) \right),$$

we can therefore easily change the variable of integration p_α into $p_\alpha + q$, thus picking up a phase factor $\exp[-iq(\alpha_f - \alpha_i)]$. Then, since H does not contain α, the α integration can be performed forcing the new p_α to be equal to q. This makes it possible to replace the $1/u^2$ potential in (11) by

$$(4/2Mu^2)(v^2 - q^2).$$

Keeping this in mind, the path integral in the large brackets of eq. (10), may be rewritten

$$\langle \boldsymbol{u}_b s | \boldsymbol{u}_a 0 \rangle = \exp[iq(\alpha_b - \alpha_a)] \int \mathscr{D}^4 u \exp\left(i \int_0^s ds\, \left(\tfrac{1}{2} M u_\mu'^2 - \tfrac{1}{2} M\omega^2 u_\mu^2 - a^2/2Mu^2 \right) \right), \tag{12}$$

where

$$\omega^2 = \sqrt{-2E/M} = \sqrt{-E/2m} \tag{13}$$

and

$$a^2 = 4(v^2 - q^2).$$

This is the amplitude of a four-dimensional harmonic oscillator with an extra $1/u^2$ potential.

For $v = q$ this is immediately solved in closed form [11] with the result

$$\langle \boldsymbol{u}_b s | \boldsymbol{u}_a 0 \rangle = (\sqrt{2\pi i \sin\omega s})^{-4} \exp\left(i \frac{\omega}{2\sin\omega s} \left[(\boldsymbol{u}_b^2 + \boldsymbol{u}_a^2)\cos\omega s - 2\boldsymbol{u}_b \cdot \boldsymbol{u}_a \right] \right). \tag{14}$$

Going from the variable s to $\sigma = e^{-ie\bar{e}s}$, and doing the α_b integration this gives immediately the desired amplitude

$$\langle x_b | x_a \rangle_E = -i \frac{mp_0}{\pi} \int_0^1 d\sigma \frac{\sigma^{-\nu}}{(1-\sigma)^2} I_q\left(2p_0 \frac{\sqrt{2\sigma}}{1-\sigma} \sqrt{x_b \cdot x_a + r_b r_a}\right) \exp\left(-p_0 \frac{1+\sigma}{1-\sigma}(r_b + r_a)\right), \tag{15}$$

where

$$P_0 = \sqrt{-2mE} = \sqrt{-ME/2} = \tfrac{1}{2}M\omega, \quad \nu = ee/2\omega.$$

For $v \neq q$, a little more work is necessary. Here we first have to do the angular integrals. In a D-dimensional generalization of the method of ref. [12], this gives the partial wave expansion

$$\langle u_b s | u_a 0 \rangle = \frac{1}{(u_b u_a)^{D/2}} \sum_{l=0}^{\infty} \langle u_b s | u_a 0 \rangle_l \sum_m Y_{lm}(\hat{u}_b) Y_{lm}(\hat{u}_a), \tag{16}$$

where Y_{lm} are the D-dimensional spherical harmonics (with m denoting all degenerate quantum numbers). They satisfy the completeness relations

$$\sum_l Y_{lm}(\hat{u}_b) Y_{lm}(\hat{u}_a) = \frac{1}{S_D} \frac{2l + D - 2}{D - 2} C_l^{D/2-1}(\hat{u}_b \cdot \hat{u}_a), \tag{17}$$

where $C_l^{(\alpha)}$ are the Gegenbauer polynomials [#5]

$$C_l^{(\alpha)}(\cos \vartheta) = \sum_{r=0}^{n} \frac{\Gamma(\alpha + r)\Gamma(l + \alpha - r)}{n!(n-r)!\Gamma^2(\alpha)} \cos(2r - l)\vartheta \tag{18}$$

and $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the surface of a D sphere. The partial wave amplitudes are given by the radial path integral

$$\langle u_b s | u_a 0 \rangle_l = \int_0^{\infty} \mathscr{D}u \exp\left\{ \frac{i}{h} \int_0^s \left[\left(\tfrac{1}{2}M\dot{u}^2 - \tfrac{1}{2}M\omega^2 u^2\right) - \frac{1}{2Mu^2}\left([l + \tfrac{1}{2}(D-2)]^2 - \tfrac{1}{4} + a^2 \right) \right] \right\}. \tag{19}$$

This can be done [7] giving

$$\langle u_b s | u_a 0 \rangle_l = \frac{M\omega\sqrt{u_b u_a}}{i \sin^2 \omega s} \exp\left[\tfrac{1}{2} i M\omega \operatorname{ctg} \omega s \left(u_b^2 + u_a^2\right)\right] I_{\bar{l}+(D-2)/2}(M u_b u_a / i \sin \omega s), \tag{20}$$

where $\bar{l}$ is chosen such as to make the natural centrifugal barrier associated with this value of angular-momentum include our extra $a^2(2Mu^2)$ potential, i.e.

$$[\bar{l} + (D-2)/2]^2 = [l + (D-2)/2]^2 + 4(v^2 - q^2). \tag{21}$$

We now perform the integral over $\int d\alpha_b \exp[iq(\alpha_b - \alpha_a)]$. Since

$$\hat{u}_b \hat{u}_a = \sqrt{\tfrac{1}{2}(x_b \cdot x_a + r_b r_a)} \cos[(\alpha_b - \alpha_a - \beta)/2],$$

where β is an angle depending only on θ, φ (see footnote on p. 420 of ref. [4]), we can integrate (18) directly

$$\int_0^{4\pi} d\alpha_b \, e^{iq\alpha_b} C_l^{(D/2-1)}(\cos \tfrac{1}{2}\alpha_b) = 4\pi \frac{\Gamma(D/2 - 1 + l + q/2)\Gamma(D/2 - 1 - q/2)}{(l + q/2)!(l - q/2)!\Gamma^2(D/2 - 1)}. \tag{22}$$

[#5] Notice that in four dimensions, $C_l^{(1)}(\cos \vartheta) = \Sigma_{r=0}^{l} \cos(2r - l)\vartheta = \sin[(l+1)\vartheta]/\sin \vartheta$.

Since D is really equal to 4, the right-hand side is equal to 4π for $l = q,\ q+2,\ q+4$. Hence, the integral over α_b in (16) gives

$$\int_0^{4\pi} d\alpha_b \, \langle \boldsymbol{u}_b s \,|\, \boldsymbol{u}_a 0 \rangle = \frac{1}{(u_b u_a)} \frac{M\omega}{\pi i \sin \omega s} \times \exp\{\tfrac{1}{2} i M\omega \operatorname{ctg}[\omega s (u_b^2 + u_a^2)]\} \sum_{l=q,q+2,\dots} 2(l+1) I_{l+1}\left(\frac{M u_b u_a}{i \sin \omega s}\right). \tag{23}$$

As a check, we set $v = q$ such that $l = \bar{l}$ and we can use the identity

$$2(l+1) I_{l+1}(u) = h[I_l(u) - I_{l+2}(u)]$$

to perform the sum

$$\sum_{l=q,q+2,\dots} 2(l+1) I_{l+1}(u) = h I_q(u). \tag{24}$$

This agrees with a direct integration of (14) over α_b.

It is now obvious how the final result (15) changes when allowing for the additional centrifugal barrier $a^2/2Mu^2$: We simply have to to use eq. (24) backwards and replace

$$I_q(h) \to \frac{2}{h} \sum_{l=q,q+2,q+4,\dots} (l+1) I_{l+1}(h), \tag{25}$$

such that

$$\langle x_b | x_a \rangle_E = -i\frac{m}{\pi} \frac{1}{\sqrt{2}\sqrt{\boldsymbol{x}_b \cdot \boldsymbol{x}_a + r_b r_a}} \sum_{l=q,q+2,\dots} (l+1) \times \int_0^1 d\sigma \frac{\sigma^{-\nu-1/2}}{1-\sigma} I_{l+1}\left(2p_0 \frac{\sqrt{2\sigma}}{1-\sigma} \sqrt{\boldsymbol{x}_b \cdot \boldsymbol{x}_a + r_b r_a}\right) \exp\left(-p_0 \frac{1+\sigma}{1-\sigma}(r_b + r_a)\right). \tag{26}$$

The Fourier transform of this is the desired amplitude for the charged particle moving in the field of a dyon.

It goes without saying that the result can trivially be extended to the case that the particle in orbit has itself a magnetic charge $\bar{g}$; in this case the final result merely requires the replacement $e\bar{e} \to e\bar{e} + g\bar{g}$, $e\bar{g} \to e\bar{g} - g\bar{e}$.

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