Derivation of Mass Spectrum and Magnetic Moments from Current Conservation in Relativistic $O(3,2)$ and $O(4,2)$ Theories*

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It is shown that in the framework of the noncompact dynamical group theory, for every choice of the current operator in the representation space, the requirements of current conservation and universality of charge determine the relativistic mass spectrum. The most general currents linear in the algebra and linear in the momenta are investigated, and mass spectra of the form $m=C(j+\frac{1}{2})$ in $O(3,2)$, and of the form $m=Cn$ or $m=Cn(1+a/n^2)^{1/2}$ in $O(4,2)$, are derived with *positive* magnetic moments and with form factors in agreement with experiment. The freedom in the choice of currents, minimal or nonminimal, is discussed and compared with that which exists in the usual Lagrangian field theory.

I. INTRODUCTION

RECENT investigations into the mass spectrum, form factors, and decay properties of hadrons taking into account infinitely many higher spin states provided by a group representation have turned out to have a forerunner in the Majorana equation,[1] put forward at a time when hadrons were hardly known. Gel'fand and Yaglom[2] in 1944 discussed the general mathematical structure of this equation, but it was still not timely to consider physical applications. It is only in recent years that the generalizations of this equation, arrived at from entirely different and independent points of view,[3–5] have been realistic enough to make calculations of hadronic properties and to compare them with experiment.[6,7] At the same time one now has a fairly general and rigorous relativistic framework of a quantum theory of composite particles with new potentialities for further development.

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[1] E. Majorana, Nuovo Cimento **9**, 335 (1932).

[2] I. M. Gel'fand and A. M. Yaglom, Zh. Experim. i Teor. Fiz. **18**, 703 (1948).

[3] The approach closest to Majorana is that of Y. Nambu, Progr. Theoret. Physics Suppl. (Kyoto) **37 & 38**, 368 (1966); Phys. Rev. **160**, 1171 (1967); T. Takabayashi, Progr. Theoret. Phys. Suppl. (Kyoto), 339 (1965).

[4] For the approach starting from the semidirect product of the Poincaré group and internal symmetry group, see P. Budini and C. Fronsdal, Phys. Rev. Letters **14**, 968 (1965); C. Frondsal, in *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna 1965), p. 665; Phys. Rev. **156**, 1653 (1967).

[5] For the approach starting from dynamical groups, see A. O. Barut, in *High Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna 1965), p. 679; A. O. Barut and A. Böhm, Phys. Rev. **139**, B1107 (1965), A. Böhm, *ibid.* **145**, 1212 (1966); N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965); P. Budini, Nuovo Cimento **44A**, 363 (1966); A. O. Barut, Phys. Rev. **156**, 1538 (1967); A. O. Barut and H. Kleinert, *ibid.* **156**, 1541 (1967); in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1967* (W. H. Freeman and Co., San Francisco, 1967), pp. 70–105.

[6] G. Cocho, C. Fronsdal, Harun-Ar-Rashid, and R. White, Phys. Rev. Letters **17**, 275 (1966); H. Leuthwyler, *ibid.* **17**, 156 (1966); W. Rühl, Nuovo Cimento **44**, 572 (1966).

[7] A. O. Barut and H. Kleinert, Phys. Rev. Letters **18**, 754 (1967); H. Kleinert, *ibid.* **18**, 1027 (1967); A. O. Barut and K. C. Tripathy, *ibid.* **19**, 918 (1967); **19**, 1081 (1967).

The main feature of the Majorana equation, as well as of its recent generalizations, is that it contains infinitely many mass and spin states (discrete or continuous) rather than a single one characteristic for the usual quantum field theory. Therefore, quantum systems so described literally possess "structures," whereas single mass equations describe "point particles." The mass spectrum of such a composite system is intimately connected with its external interactions. In previous studies[7] we have mainly considered the problem of transition probabilities and form factors. The main purpose of this paper is to exhibit the relationship between mass spectrum and the charge normalization, on the one hand, and the specification of the currents, that is, external interactions, on the other. In the absence of a model of hadrons in terms of their internal constituents, we describe the composite system group-theoretically by its global quantum numbers and determine the mass spectrum from the consistency with the external interactions. That this idea works was shown recently in the case of the H atom, treated within the framework of $O(4,2)$ group, where the specification of the transformation property of the electromagnetic current determines the mass spectrum via the position of the anomalous threshold.[8]

In Secs. II and III, we briefly recapitulate the general framework of the theory. In Secs. IV and V, we discuss currents and mass spectra in specific $O(3,2)$ and $O(4,2)$ theories. Finally in Sec. VI we discuss the origin of the currents in the Lagrangian form and the problem of minimal coupling for infinite-component wave equations and compare it with the usual Lagrangian theory.

II. GENERAL THEORY

We shall work in an entirely group-theoretical framework. It will become clear that this has its parallels in the approach using infinite-component wave equations.

We start with an irreducible unitary representation D of a group G called "the group of quantum numbers" that labels *all* the states of the system. Let α represent collectively all the quantum numbers and denote the

[8] A. O. Barut and H. Kleinert, Phys. Rev. **160**, 1149 (1967).

states by $|\alpha\rangle$. The representation D is, in general, infinite dimensional. On this Hilbert space $\{|\alpha\rangle\}$ one can introduce, in general, new operators Γ_a which, together with the elements L_A of the Lie algebra of G, generate a larger group $\mathcal{G}$ that has the same representation D of G; i.e., D is also an irreducible representation of $\mathcal{G}$. The group $\mathcal{G}$ contains also the physical transition operators or currents, so that it will be denoted as the "dynamical group."

The angular momentum operators $\mathbf{J}$ are always parts of the algebra of G. We next identify the generators $\mathbf{M}$ of the pure Lorentz transformations, in the simplest case, among the elements of the Lie algebra. The finite pure Lorentz transformations are then represented in our Hilbert space by

$$\exp(i\boldsymbol{\xi}\cdot\mathbf{M}), \quad \tanh\xi=p/E, \tag{2.1}$$

where ξ_i are the relativistic velocities, the three parameters of the pure Lorentz transformations, related to the energy E and momentum p of the particle as shown in Eq. (2.1). The transformations (2.1) allow us to define moving states of the whole system with momentum

$$p^\mu=(m\cosh\xi,\hat{\xi}m\sinh\xi)$$

by

$$|\alpha;p\rangle=\exp(i\boldsymbol{\xi}\cdot\mathbf{M})|\alpha\rangle. \tag{2.2}$$

Note that if M is four-dimensional and α just labels the two spin states, then $|\alpha;p\rangle$ is simply the Dirac spinor $u_\alpha(p)$, so that Eq. (2.2) not only generalizes $u_\alpha(p)$ to infinite-spin components but also to other intrinsic degrees of freedom as well.

Finally, we form with the spinors $|\alpha;p\rangle$ covariant couplings; e.g., a scalar vertex $\langle\alpha';p'|\alpha;p\rangle$, a vector vertex $\langle\alpha';p'|J_\mu|\alpha;p\rangle$, etc., and identify these with the external interactions. More generally, in the case of groups higher than $O(3,1)$, we admit more complicated vertices (i.e., currents) of the form

$$F_\mu=\langle\bar{\alpha}';p'|J_\mu|\bar{\alpha};p\rangle, \tag{2.3}$$

where the new barred states are defined by

$$|\bar{\alpha}\rangle=(1/N_\alpha)\exp(i\theta_\alpha T)|\alpha\rangle, \tag{2.4}$$

and where θ_α are parameters to be determined, N_α is a normalization factor, and T is a combination of rotationally scalar operators in the Lie algebra of $\mathcal{G}$. We refer to Eq. (4) as the "mixing effect" or "tilt," and its physical meaning is that the interactions have a simple form transforming like group generators only in these new states $|\bar{\alpha}\rangle$. Another way of expressing this is that the physical vertex interaction operator (S matrix) is of the form

$$\langle\alpha'|S|\alpha\rangle=(1/N_{\alpha'}N_\alpha)\langle\alpha'|e^{-i\theta_{\alpha'}T}e^{-i\xi'M}J_\mu e^{i\xi M}e^{i\theta_\alpha T}|\alpha\rangle A^\mu.$$

This structure describes correctly the electromagnetic interactions of the H atom, and we study this possibility also for relativistic particle theories. Note that in Eq. (2.3) the tilting operation is done first on the rest states according to (2.4) and *then* the operation of boosting, in that order, so that $\mathfrak{F}_\mu$ is a *bona fide* 4-vector. This is the general framework of the theory. Examples will be given in later sections.

III. CURRENT CONSERVATION AND CHARGE NORMALIZATION

Let J_μ be the current that couples to the electromagnetic field. First the current has to be conserved (i.e., couples to zero-mass photon) for *all* possible transitions, in particular between states with different masses. This requirement gives

$$J_\mu q^\mu=0, \quad q_\mu=p_\mu'-p_\mu. \tag{3.1}$$

It is for the nondiagonal elements (i.e., different masses) that the current conservation will be shown to essentially determine the mass spectrum. The diagonal elements of the zero component of the current for zero momentum transfer give the total charge of the state. Our second requirement is that the charge of all higher states in an irreducible "multiplet" is the same;

$$\langle\bar{\alpha}|J_0|\bar{\alpha}\rangle=q \text{ for all } \alpha. \tag{3.2}$$

We expect similar restrictions on the appropriate universal coupling constants with external scalar and tensor interactions.

We now formulate the requirement (3.1) in a more precise form. Consider the current element between a state $|1\rangle$ of momentum $p_1=(m,0,0,0)$ and a state $|2\rangle$ of momentum $p_2{}^\mu=(m_2\cosh\xi,\hat{\xi}m_2\sinh\xi)$. These states are in general the tilted states introduced in Eq. (2.4). Then according to (2.3),

$$\mathfrak{F}_{\mu_{12}}=\langle\bar{1}|J_\mu\exp(i\boldsymbol{\xi}\cdot\mathbf{M})|\bar{2}\rangle, \tag{3.3}$$

where $|\bar{1}\rangle$ and $|\bar{2}\rangle$ are now states in the representation D of G (i.e., momentum-independent rest-frame states). The requirement of current conservation is simply

$$(p_1-p_2)_\mu\mathfrak{F}^\mu=0 \tag{3.4}$$

or

$$m_1\langle\bar{1}|J_0\exp(i\boldsymbol{\xi}\cdot\mathbf{M})|\bar{2}\rangle - \langle\bar{1}|p_2{}^\mu J_\mu\exp(i\boldsymbol{\xi}\cdot\mathbf{M})|\bar{2}\rangle=0. \tag{3.4$'$}$$

Then, because J_μ is a 4-vector, we have

$$\exp(i\boldsymbol{\xi}\cdot\mathbf{M})m_2J_0\exp(-i\boldsymbol{\xi}\cdot\mathbf{M})=p_2{}^\mu J_\mu. \tag{3.5}$$

Consequently, (3.4′) gives the condition

$$m_1\langle\bar{1}|J_0\exp(i\boldsymbol{\xi}\cdot\mathbf{M})|\bar{2}\rangle=m_2\langle\bar{1}|\exp(i\boldsymbol{\xi}\cdot\mathbf{M})J_0|\bar{2}\rangle. \tag{3.6}$$

In the following sections we shall apply these two fundamental requirements, Eqs. (3.2) and (3.6), to specific $O(3,1)$ and $O(4,1)$ theories and evaluate for specific choices of the currents and normalization of states, the mass spectrum, magnetic moments and form factors.

The tilted states are orthogonal with respect to the metric J_0, for from (3.6) we obtain, in the limit $\xi\to 0$,

$$\langle\bar{1}|J_0|\bar{2}\rangle=\delta_{12}. \tag{3.7}$$

IV. CURRENTS, MASS SPECTRUM, AND MAGNETIC MOMENTS IN $O(3,1)$ THEORY

We consider a model of the hadron resonances in which the group of the quantum numbers, G, is the Lorentz group itself extended by parity: $\{O(3,1),\pi\}$. We assign the hadrons (and their antiparticles, in the case of fermions) of fixed isospin and hypercharge to a single, unitary, irreducible representation of $O(3,1)$, $\{\pi\}$. If we designate by $[j_0j_1]$ the eigenvalues of the two $O(3,1)$ invariant operators $[L^2-M^2=j_0^2+j_1^2-1$ and $\mathbf{L}\cdot\mathbf{M}=-ij_0j_1]$, then the states of spin j, spin component m, and parity η are given by

$$|[j_0j_1]jm\pm\rangle=\tfrac{1}{2}\sqrt{2}[|[j_0j_1]jm\rangle\pm|[-j_0j_1]jm\rangle],$$

with

$$\eta=\pm(-1)^{j-j_0}. \tag{4.1}$$

In the case of (nontrivial) unitary representations, j_0 is an integer or half-integer (the lowest spin) and $j_i=i\nu$ is an arbitrary imaginary number. For the nonunitary Dirac case, however, $[j_0j_1]=[\frac{1}{2},\frac{3}{2}]$.

A. Algebraic Current

It is known that for the unitary representations one can define a *unique* algebraic vector operator Γ_μ for

$$j_0=0,\quad j_1=\tfrac{1}{2},$$
$$j_0=\tfrac{1}{2},\quad j_1=0 \tag{4.2}$$

without doubling the $O(3,1)$ states, and for

$$j_0=\tfrac{1}{2},\quad j_1=i\nu$$

with doubling of the $O(3,1)$ states by parity. On the doubled states $|[\frac{1}{2},i\nu]jm\pm\rangle$, the representation of Γ_μ is explicitly given as follows[9]:

$$\begin{aligned}
&\Gamma^0|jm\pm\rangle\\
&\quad=\pm(j+\tfrac{1}{2})|jm\pm\rangle,\\
&i\Gamma^+|jm\pm\rangle\\
&\quad=\pm[(j-m)(j-m-1)]^{1/2}C_j|j-1,m+1,\pm\rangle\\
&\qquad\mp[(j-m)(j+m+1)]^{1/2}(2j+1)A_j|j,m+1,\mp\rangle\\
&\qquad\mp[(j+m+1)(j+m+2)]^{1/2}C_{j+1}|j+1,m+1,\pm\rangle,\\
&i\Gamma^-|jm\pm\rangle\\
&\quad=\mp[(j+m)(j+m-1)]^{1/2}C_j|j-1,m-1,\pm\rangle\\
&\qquad\mp[(j+m)(j-m+1)]^{1/2}(2j+1)A_j|j,m-1,\mp\rangle\\
&\qquad\pm[(j-m+1)(j-m+2)]^{1/2}C_{j+1}|j+1,m-1,\pm\rangle,\\
&i\Gamma^3|m\pm\rangle\\
&\quad=\pm[j^2-m^2]^{1/2}C_j|j-1,m,\pm\rangle\\
&\qquad\mp m(2j+1)A_j|jm,\mp\rangle\\
&\qquad\pm[(j+1)^2-m^2]^{1/2}C_{j+1}|j+1,m,\pm\rangle,
\end{aligned} \tag{4.3}$$

where

$$C_j=(i/2j)[j^2+\nu^2]^{1/2},\quad A_j=\nu/2j(j+1). \tag{4.4}$$

[9] H. Kleinert, Ph.D. thesis, University of Colorado, 1967 (unpublished).

In the case with no doubling, simply set $\nu=0$ and choose the plus sign in these equations.

We shall require that this algebraic vector operator Γ_μ should always be part of the electromagnetic current of the hadrons, and call it the "algebraic current":

$$J_0{}^\mu=a_0\Gamma^\mu. \tag{4.5}$$

Note also that for $\nu=0$, Γ_μ together with the generators of $O(3,1)$ generate the bigger algebra $\mathcal{G}:O(3,2)$ on the same Hilbert space of D.

B. Nonalgebraic Currents

Besides the algebraic vector operator Γ_μ which exists in the $O(3,1)$ representation alone, we can also construct vector operators out of the Poincaré vectors of the particles in the vertex. Let p and p' be the momenta of the particles, q that of the photon;

$$\begin{aligned}
J_1{}^\mu&=a_1P^\mu=a_1(p'^\mu+p^\mu),\\
J_2{}^\mu&=a_2q^\mu=a_2(p'^\mu-p^\mu),\\
J_3{}^\mu&=a_3L^{\mu\nu}P_\nu,\\
J_4{}^\mu&=a_4L^{\mu\nu}q_\nu.
\end{aligned} \tag{4.6}$$

If we restrict ourselves to current terms that are no more than linear in Γ_μ and $L_{\mu\nu}$, and no more than linear in momenta, the most general current will be a linear combination of the following five terms:

$$G_\mu:(\Gamma_\mu,P_\mu,q_\mu,L_{\mu\nu}P^\nu,L_{\mu\nu}q^\nu). \tag{4.7}$$

The current between states j and j' has the form

$$F_\mu=\langle\bar{j}'|G_\mu|\bar{j}p\rangle. \tag{4.8}$$

Here $|\bar{j}p\rangle$ designates the states boosted to momentum p (and, in general, tilted). In writing the linear combination indicated in (4.7) we meet immediately with a new situation (arising from the fact that we have infinitely many different mass states) which would be trivial in the usual single-mass theories: The coefficient of the first term in (4.7) has a different dimension from those of the others. Therefore, the coefficients of the linear combinations of currents must be functions of the invariants m^2; but because we have many masses they must in fact in general be matrices whose elements are functions of the invariants. Moreover, the condition (3.2) that the total charge of all higher mass states is the same shows also that these coefficients cannot be constant numbers, because the diagonal elements of G_0 for the terms in (4.7) go as

$$(j+\tfrac{1}{2},2m_j,0,0,0).$$

We therefore write the total current in the form

$$G_{j'j}{}^\mu=\sum_{i=0}^{4}e_{j'j}{}^{(i)}J_{(i)}{}^\mu, \tag{4.9}$$

where $e_{j'j}{}^{(i)}$ are the electromagnetic coupling constants for the vertex and may be different for every particle and every current. In the case of the H atom, in fact the coupling "constant" is a matrix and factorizes in the form

$$e_{n'n}=e/N_{n'}N_n,$$

so that the factors can be absorbed into the normalization of states as described in (2.4). We shall assume the same property for our currents which then can be written as

$$G_{j'j}{}^{\mu}=\sum_{i=0}^{4}\frac{e^{(i)}}{N_{j'}{}^{(i)}N_j{}^{(i)}}J_{j'j}{}^{\mu(i)}. \quad (4.10)$$

In the case of $O(3,1)$ there is no tilting [i.e., no scalar operators T defined in (2.4)], so that we are left with a normalization factor $1/N_j$. We therefore have for the most general linear current between the states j and j'

$$G^{\mu}=\frac{a_0}{N_{j'}{}^{\Gamma}N_j{}^{\Gamma}}\langle j'|\Gamma^{\mu}|jp\rangle+\frac{a_1}{N_{j'}{}^{P}N_j{}^{P}}\langle j'|P^{\mu}|jp\rangle$$
$$+\frac{a_2}{N_{j'}{}^{q}N_j{}^{q}}\langle j'|q^{\mu}|jp\rangle+\frac{a_3}{N_{j'}{}^{3}N_j{}^{3}}\langle j'|L^{\mu\nu}P_{\nu}|jp\rangle$$
$$+\frac{a_4}{N_{j'}{}^{4}N_j{}^{4}}\langle j'|L^{\mu\nu}q_{\nu}|jp\rangle.$$

The two fundamental requirements (3.2) and (3.6), i.e.,

$$m_{j'}\langle j'|G^0|j\mathbf{p}\rangle=m_j\langle j',-\mathbf{p}|G^0|j\rangle \quad (3.2')$$

and

$$\langle j|G^0|j\rangle=q, \quad (3.6')$$

give in this case

$$\left\{a_0\frac{m_{j'}(j'+\frac{1}{2})-m_j(j+\frac{1}{2})}{N_{j'}{}^{\Gamma}N_j{}^{\Gamma}}+a_1\frac{m_{j'}{}^2-m_j{}^2}{N_{j'}{}^{P}N_j{}^{P}}+a_2\frac{m_{j'}{}^2-2m_{j'}m_j\cosh\xi+m_j{}^2}{N_{j'}{}^{2}N_j{}^{2}}\right\}\langle j'|jp\rangle$$
$$-a_3\frac{2m_{j'}m_j\sinh\xi}{N_{j'}{}^{3}N_j{}^{3}}\langle j'|L^{03}|jp\rangle=0 \quad (4.11)$$

and

$$a_0\frac{(j+\frac{1}{2})}{N_j{}^{\Gamma 2}}+2a_1\frac{m_j}{N_j{}^{P2}}=q. \quad (4.12)$$

In order that the first requirement holds for arbitrary ξ it is necessary that

$$a_2=a_3=0. \quad (4.13)$$

Thus we have the result that the terms $L^{\mu\nu}P_{\nu}$ and q^{μ} can never be exactly conserved and the term $L^{\mu\nu}q_{\nu}$, which is always conserved, has no effect on the mass spectrum. In analogy with the usual field theory we may call the terms Γ^{μ} and P^{μ} *the minimal currents* and the term $L^{\mu\nu}q_{\nu}$, that comes in with an arbitrary coefficient, as the "anomalous" or the "nonminimal current."

C. Mass Spectrum

Thus our requirements leave us with the following two equations:

$$a_0\frac{m_{j'}(j'+\frac{1}{2})-m_j(j+\frac{1}{2})}{N_{j'}{}^{\Gamma}N_j{}^{\Gamma}}+a_1\frac{m_{j'}{}^2-m_j{}^2}{N_{j'}{}^{P}N_j{}^{P}}=0$$

$$\text{for all } j' \text{ and } j, \quad (4.11')$$

$$a_0\frac{(j+\frac{1}{2})}{N_{j'}{}^{\Gamma 2}}+2a_1\frac{m_j}{N_j{}^{P2}}=q \text{ for all } j. \quad (4.12')$$

In the special case when $a_1=0$, it follows that we can describe only charged particles,

$$N^{\Gamma}=(j+\tfrac{1}{2})^{1/2}, \quad a_0=q \quad (4.14)$$

and the mass spectrum

$$m_j=C(j+\tfrac{1}{2})^{-1}, \quad (4.15)$$

which is the result already obtained by Majorana from a wave equation.[1]

In the general case we can choose from the second condition of (4.12), as one possible simple solution,

$$N_j{}^{\Gamma}=(j+\tfrac{1}{2})^{1/2}, \quad N_j{}^{P}=m_j{}^{1/2}; \quad (4.16)$$

then $a_0+2a_1=q$, and a solution of Eq. (4.11′) is clearly

$$m_j=C(j+\tfrac{1}{2}),$$

therefore,

$$a_0=-a_1;$$

then

$$a_0=-q, \quad a_1=q. \quad (4.17)$$

Again we can only describe charged particles with charge q, but the spectrum is now inverted with respect to the pure Majorana case and is more physical as far as applications to hadrons are concerned. A spectrum of the type (4.17) has recently been written on the basis of a wave equation by Takabayashi.[10] We see that this is also the result under the most general linear current. The solution that we have given of the coupled equations (4.11′) and (4.12′) corresponds to two currents of opposite signs, each one giving a constant charge to all states.

D. Magnetic Moments and Form Factors

The explicit form of the transition form factors in terms of the states (4.1) is given by

$$F^{\mu}=(1/N'N)\langle[j_0j_1]jm\pm|G^{\mu}e^{i\xi M_3}|[j_0j_1]jm\pm\rangle. \quad (4.18)$$

In order to evaluate these matrix elements explicitly it is convenient to use the matrix elements of particular finite $O(3,1)$ transformations. We denote

$$\langle[j_0j_1]j'm|e^{i\xi M_3}|[j_0\nu]jm\rangle=B_m{}^{j'j}(\xi[j_0\nu]). \quad (4.19)$$

[10] T. Takabayashi, Nagoya University Report, 1967 (unpublished).

For $j'\geqslant j=m$, the functions $B_j^{j'j}(\xi[j_0\nu])$ have a particularly simple form[11];

$$B_j^{j'j}(\xi[j_0\nu])=\mathfrak{N}_{(j_0\nu)}{}^{j'j}(\sinh\xi)^{(j'-j)}e^{i\nu\xi}e^{-\xi(j'+j_0+1)}$$
$$\times F(j+1-i\nu,j'+1+j_0;2j'+2;1-e^{-2\xi}),$$

with

$$\mathfrak{N}^{j'j}(j_0\nu)=2^{j'-j}\left[\frac{(j'+j_0)!(j'-j_0)!(j'+j)!}{(j+j_0)!(j-j_0)!(j'-j)!}\frac{(2j+1)!}{(2j'+1)!}\right.$$
$$\left.\times\frac{[(j+1)^2+\nu^2]\cdots[j'^2+\nu^2]}{(2j')!}\right]^{1/2}. \quad (4.20)$$

The matrix elements needed in Eq. (4.18) involve the combinations

$$\langle j'j\pm|e^{i\xi M_3}|jj+\rangle\equiv B_j^{\pm j'j}(\xi[j_0\nu])$$
$$=\tfrac{1}{2}[B_{j(j_0)}{}^{j'j}\pm B_{j(-j_0)}{}^{j'j}]. \quad (4.21)$$

Using these results we find for the minimal currents Γ_μ and P_μ, after some calculation, for the ground state and for our solution (4.16), with

$$a_0=-a_1=-q, \quad C=M=\text{nucleon mass},$$

the following current components:

$$\mathfrak{F}^0=\langle\tfrac{1}{2}\tfrac{1}{2}+|G^0e^{i\xi M_3}|\tfrac{1}{2}\tfrac{1}{2}+\rangle=q\cosh\xi\, B_{\frac12}{}^{+\frac12\frac12}(\xi[\tfrac{1}{2}\nu]),$$
$$\mathfrak{F}^3=a_0\{\langle\tfrac{1}{2}\tfrac{1}{2}+|\Gamma^3|\tfrac{1}{2}\tfrac{1}{2}-\rangle B_{\frac12}{}^{-\frac12\frac12}+\langle\tfrac{1}{2}\tfrac{1}{2}+|\Gamma^3|\tfrac{3}{2}\tfrac{1}{2}+\rangle B_{\frac12}{}^{+\frac32\frac12}\}$$
$$+a_1\langle\tfrac{1}{2}\tfrac{1}{2}+|e^{i\xi M_3}|\tfrac{1}{2}\tfrac{1}{2}+\rangle$$
$$=q\sinh\xi\, B_{\frac12}{}^{+\frac12\frac12}-q(\tfrac{1}{3}\sqrt{2}(9/4+\nu^2)^{1/2}B_{\frac12}{}^{+\frac32\frac12}-\tfrac{2}{3}i\nu B_{\frac12}{}^{-\frac12\frac12}),$$
$$\mathfrak{F}^1=a_0\{\langle\tfrac{1}{2},-\tfrac{1}{2},+|\Gamma^1|\tfrac{1}{2}\tfrac{1}{2}-\rangle B_{\frac12}{}^{-\frac12\frac12}$$
$$+\langle\tfrac{1}{2},-\tfrac{1}{2}+|\Gamma^1|\tfrac{3}{2}\tfrac{1}{2}+\rangle B_{\frac12}{}^{+\frac32\frac12}\}$$
$$=-q(-\tfrac{1}{6}\sqrt{2}(9/4+\nu^2)^{1/2}B_{\frac12}{}^{+\frac32\frac12}-\tfrac{2}{3}i\nu B_{\frac12}{}^{-\frac12\frac12}). \quad (4.22)$$

The relation between these components and the usual nucleon form factors is

$$\mathfrak{F}^0=G_E(t)\cosh\tfrac{1}{2}\xi,$$
$$\mathfrak{F}^3=G_E(t)\sinh\tfrac{1}{2}\xi,$$
$$\mathfrak{F}^1=G_M(t)\sinh\tfrac{1}{2}\xi. \quad (4.23)$$

If we use the expansion of the B functions for small ξ,

$$B_{\frac12}{}^{+\frac12\frac12}(\xi)\approx 1,$$
$$B_{\frac12}{}^{-\frac12\frac12}(\xi)\approx -\tfrac{1}{3}i\nu\xi,$$
$$B_{\frac12}{}^{+\frac32\frac12}(\xi)\approx\tfrac{1}{3}\sqrt{2}(9/4+\nu^2)^{1/2}\xi, \quad (4.24)$$

we obtain

$$G_E(0)=q, \quad (4.25)$$

so that the electric form factor comes out automatically to be correctly normalized to charge q, and

$$\mu=G_M(0)=q(\tfrac{1}{2}+\tfrac{2}{3}\nu^2), \quad (4.26)$$

so that $G_M(0)/q$ is now *positive*. We remark, as has been noted before,[5] that with the algebraic current alone we get

$$\mu=q(-\tfrac{1}{2}-\tfrac{2}{3}\nu^2), \quad (4.27)$$

i.e., a negative magnetic moment; in particular, the Majorana case ($\nu=0$) gives a magnetic moment of $-\frac{1}{2}$.

The value of ν^2 from (4.26) fitted to the proton magnetic moment is 3.43 and fitted to the isoscalar form factor ($q=\frac{1}{2}$) is 0.571. A ν value different from zero was also necessary to fit the decay rates of baryons.[7]

Thus from both the mass spectrum and from the sign of the magnetic moment we show the existence of the second "convective" current P_μ. The form factors as compared to the pure algebraic currents are more slowly decreasing. For example, the electric form factor goes as

$$G_E(\xi)=q\frac{\cosh\xi}{\cosh\frac{1}{2}\xi}B_{\frac12}{}^{+\frac12\frac12}(\xi[\tfrac{1}{2}\nu]),$$
$$t=(m-m')^2-2mm'(\cosh\xi-1), \quad (4.28)$$

which for $\nu=0$ is too slow a decrease compared to the experimental behavior, and for $\nu\neq 0$, it has zeros coming from the B functions. For these reasons we discuss now the more realistic $O(4,1)$ theories. It should be remarked that the neutron form factors as well as a complete fit to the experimental form factors *within* $O(3,1)$ can only be obtained by using the anomalous current term $L_{\mu\nu}q^\nu$. (See also the discussion in Sec. V B).

V. CURRENTS, MASS SPECTRUM, AND MAGNETIC MOMENTS IN $O(4,1)$–$O(4,2)$ THEORIES

Because of the observed spectrum of baryons with fixed internal quantum numbers and because of the t dependence of the form factors, the group $O(3,2)$ is not large enough and therefore the group $O(4,2)$ has been considered for this purpose.[9,12] The group of the quantum numbers G is now $O(4,1)$, again extended by parity. The maximal compact subgroup is $O(4)$, with representations of dimension n^2 ($2n^2$ with parity doubling). The $O(4,1)$ states are labeled by $|njm,\pm\rangle$. A simple fermion representation can be obtained by combining $l=j-\frac{1}{2}$, with $s=\frac{1}{2}$, by

$$|njm\pm\rangle=(-1)^m(2j+1)^{1/2}\begin{pmatrix}\frac{1}{2} & l & j\\ r & m-r & -m\end{pmatrix}$$
$$\times[a_r{}^\dagger+(-1)^l ib_r{}^\dagger]|nl,m-r\rangle, \quad (5.1)$$

where $l=j-\frac{1}{2}$, and

$$|nlm\rangle=(-1)^m(2l+1)^{1/2}$$
$$\times\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & l\\ \frac{1}{2}(n_2-n_1+m) & \frac{1}{2}(n_1-n_2-m) & -m\end{pmatrix}$$
$$\times|n_1n_2m\rangle \quad (5.2)$$

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[11] S. Ström, Arkiv Fysik **29**, 467 (1965).

[12] A. O. Barut and H. Kleinert, Phys. Rev. **161**, 1464 (1967).

and

$$|n_1n_2m\rangle=[n_1!(n_2+|m|)!n_2!(n_1+|m|)!]^{1/2}\times\begin{cases}a_1^{\dagger n_2+m}a_2^{\dagger n_1}b_1^{\dagger n_1+m}b_2^{\dagger n_2}|0\rangle, & m>0\\ a_1^{\dagger n_2}a_2^{\dagger n_1-m}b_1^{\dagger n_1}b_2^{\dagger n_2-m}|0\rangle, & m<0.\end{cases} \tag{5.3}$$

On this Hilbert space one can define new operators L_{i6} ($i=1,2,3$), L_{46}, and L_{56}, which together with the elements of the Lie algebra of $O(4,1)$ generate the Lie algebra of $O(4,2)$. In other words, the representation of the $O(4,1)$ group that we are considering is also an irreducible unitary representation of $O(4,2)$, the conformal group [isomorphic to $SU(2,2)$].

We now choose L_{ij} and L_{i4} as the angular momentum and the analog of Lenz vector operators. Without loss of generality we can also choose the boosters to be

$$M_i=L_{i5}. \tag{5.4}$$

Among the remaining operators we choose a vector operator Γ_μ and the scalar tilting operator T;

$$\Gamma_\mu=(L_{56},L_{i6}),\quad T=L_{45}. \tag{5.5}$$

The generator L_{56} has the eigenvalue n is therefore diagonal. The theory now contains the tilting angles θ_n as parameters to be determined from current conservation as well.

A. Mass Spectrum

We are now in a position to write the fundamental equations (3.2) and (3.6) for the most general linear current of the type (4.7)–(4.10), now between the tilted states

$$\mathfrak{F}^\mu=\sum_{i=0}^{4}a_i\frac{1}{N_{j'}{}^iN_j{}^i}\langle\bar n'j'm'|G_i{}^\mu e^{i\xi M_3}|\bar njm\rangle. \tag{5.6}$$

Again only the two minimal currents Γ_μ and P_μ contribute, and we obtain from the constancy of charge the equation

$$a_0\frac{n\cosh\theta_n}{(N_n{}^\Gamma)^2}+a_1\frac{2m_n}{(N_n{}^P)^2}=q, \tag{5.7}$$

and from the current conservation

$$m_{n'}\langle\bar n'|\Gamma^0e^{i\xi M_3}|\bar n\rangle=m_n\langle\bar n'|e^{i\xi M_3}\Gamma^0|\bar n\rangle \tag{5.8}$$

the equation

$$a_0(1/N_{n'}{}^\Gamma N_n{}^\Gamma)(m_{n'}n'\cosh\theta_{n'}-m_nn\cosh\theta_n+\Delta)=(a_1/N_{n'}{}^PN_n{}^P)(m_n{}^2-m_{n'}{}^2), \tag{5.9}$$

where

$$\Delta=\frac{m_{n'}\sinh\theta_{n'}\langle n'|L_{46}T_{n'n}|n\rangle-m_n\sinh\theta_n\langle n'|T_{n'n}L_{46}|n\rangle}{\langle n'|T_{n'n}|n\rangle},$$

with

$$T_{n'n}=e^{-i\theta_{n'}L_{45}}e^{i\xi L_{35}}e^{i\theta_nL_{45}}.$$

First of all, if no tilting is present, i.e., $\cosh\theta_n=1$, $\sinh\theta_n=0$, we can solve the equations (5.7) and (5.9), with

$$N_n{}^\Gamma=n^{1/2},\quad N_n{}^P=m_n{}^{1/2},\quad a_0+2a_1=q$$

and obtain the mass spectrum

$$m_n=Cn \tag{5.10}$$

for $a_0=-a_1$. Again the spectrum is physical and is inverted as compared to the one obtained from an algebraic current alone, namely,

$$m_n=Cn^{-1}. \tag{5.11}$$

To see the effect of the tilting we now solve the general equation (5.9). From $\operatorname{Im}\Delta=0$ we obtain, first the masses in terms of the tilting angles

$$m_n=\lambda\cosh\theta_n/\sinh\theta_n, \tag{5.12}$$

then

$$\Delta=-\lambda(n'\sinh\theta_{n'}-n\sinh\theta_n). \tag{5.13}$$

Hence we must satisfy the simpler equation

$$(a_0/N_{n'}{}^\Gamma N_n{}^\Gamma)\{m_{n'}n'\cosh\theta_{n'}-m_nn\cosh\theta_n-\lambda(n'\sinh\theta_{n'}-n\sinh\theta_n)\}=(a_1/N_{n'}{}^PN_n{}^P)(m_n{}^2-m_{n'}{}^2). \tag{5.14}$$

From the first condition (5.7) we choose

$$N_n{}^\Gamma=(n\cosh\theta_n)^{1/2},\quad N_n{}^P=(m_n)^{1/2};$$

hence

$$a_0+2a_1=q. \tag{5.15}$$

Then in order to bring the second condition (5.12) into a recursion type we set

$$c^{1/2}N_n{}^\Gamma=N_n{}^P\Rightarrow m_n=Cn\cosh\theta_n. \tag{5.16}$$

Hence

$$a_0(Cn^2\cosh^2\theta_n-\lambda n\sinh\theta_n)+a_1Cn^2\cosh^2\theta_n=\text{const}\equiv K,$$

which gives with $a_0=-a_1=-q$ the tilting angles

$$\sinh\theta_n=(K/q\lambda)n^{-1}, \tag{5.17}$$

that are no longer free parameters. From (5.12), (5.16), and (5.17) there is a relation between the constants q, C, K, and λ, namely,

$$C=q\lambda^2/K, \tag{5.18}$$

so that the final mass formula can be written as

$$m_n=Cn[1+(K/qc)n^{-2}]^{1/2}. \tag{5.19}$$

For completeness we also discuss the effect of tilting on the algebraic current alone. The relevant equations are now

$$m_nn\cosh\theta_n-\lambda n\sinh\theta_n=K \tag{5.20}$$

and

$$m_n=\lambda\cosh\theta_n/\sinh\theta_n, \tag{5.12'}$$

from which one obtains the mass spectrum

$$m_n=K[\lambda^2/K^2+1/n^2]^{1/2}. \tag{5.21}$$

B. Magnetic Moments and Form Factors

The magnetic moments and the form factors are calculated from Eq. (5.6) as in the case of $O(3,1)$. We obtain for the ground state

$$\mathfrak{F}^0=\left[\frac{a_0}{N_\Gamma{}^2}\left(\tfrac{3}{2}\cosh\theta-2\sinh\theta\tanh\theta\tanh^2(\tfrac{1}{2}\beta)\right)+\frac{a_1}{N_P{}^2}2m\cosh^2(\tfrac{1}{2}\xi)\right]\frac{\cosh(\frac{1}{2}\xi)}{\cosh^4(\frac{1}{2}\beta)},$$

$$\mathfrak{F}^3=\frac{\sinh(\frac{1}{2}\xi)}{\cosh(\frac{1}{2}\xi)}\mathfrak{F}^0,$$

$$\mathfrak{F}^1=-\tfrac{1}{2}\frac{a_0}{N_\Gamma{}^2}\cosh\theta\frac{\sinh(\frac{1}{2}\xi)}{\cosh^4(\frac{1}{2}\beta)},\tag{5.22}$$

with

$$\cosh^2\tfrac{1}{2}\beta=(1-\cosh^2\theta t/4m^2),$$

which, when compared with (4.24), give the following expressions for the form factors:

$$G_E(t)=q(2\cosh^2\tfrac{1}{2}\xi-1+\tfrac{4}{3}\tanh^2\theta\tanh^2\tfrac{1}{2}\beta)\times\cosh^{-4}\tfrac{1}{2}\beta,$$
$$G_M(t)=q\tfrac{1}{3}\cosh^{-4}\tfrac{1}{2}\beta,\tag{5.23}$$

so that at zero momentum transfer we have

$$G_E(0)=q,\tag{5.24}$$

and the magnetic moment is given by

$$\mu=\tfrac{1}{3}\text{ in units of }q/2mc.\tag{5.25}$$

Thus the magnetic moment is now positive, in contrast to the pure algebraic current which gives always a negative magnetic moment. More detailed comparison with the magnetic moments and form factors of proton, neutron, and higher baryon states have been given elsewhere.[13]

C. Galilean-Invariant $O(0,4)$: Theory

It is instructive to treat the case of the H atom described by a Galilean-invariant $O(4,2)$ theory[8,9] with the same methods used for relativistic theories. Thus we want to derive the mass spectrum of the H-like quantum systems and the form factors from our fundamental requirements of the universality of charge for all states and the current conservation.

In this theory the current is given by

$$\Gamma^\mu=(L_{56}-L_{46},L_{i6}),\tag{5.26}$$

while the generators of the pure Galilean transformations (Galilean boosters) are

$$M_i=L_{i5}-L_{i4}.\tag{5.27}$$

[13] A. O. Barut, D. Corrigan and H. Kleinert, Phys. Rev. Letters **20**, 167 (1968).

The components of M_i commute, as they should. The new tilted states defined by Eq. (2.4) are given by

$$|\tilde{n}lm\rangle=n^{-1}e^{-i\theta_nL_{45}}|nlm\rangle,\tag{5.28}$$

with the parameters θ_n to be determined.

The condition of the universality of charge for all states is expressed by

$$\langle n'|e^{-i\theta_{n'}L_{45}}(L_{56}-L_{46})e^{i\theta_nL_{45}}|n\rangle=\delta_{n'n}q,\tag{5.29}$$

which also shows the orthogonality of the original states $|n\rangle$ and $|n'\rangle$ with respect to the "metric" $e^{-i\theta_{n'}L_{45}}\Gamma_0e^{i\theta_nL_{45}}$. This equation holds if and only if

$$\theta_n=\ln n.\tag{5.30}$$

The equation for the current conservation can be written in the form

$$\langle n'|e^{i\theta_nL_{45}}[(m_{n'}-p^0)\Gamma^0-p^3\Gamma^3]e^{i\xi M_3}|n\rangle=0,\tag{5.31}$$

and gives

$$\frac{\mathfrak{F}^3}{\mathfrak{F}^0}=\frac{1}{q}\left(-\frac{1}{2n^2}+\frac{1}{2n'^2}+\frac{q^2}{2m_e}\right)=\frac{1}{q}\left(E_n-E_{n'}+\frac{q^2}{2m_e}\right),\tag{5.32}$$

which is just the type of equation we expect from a Galilean-invariant theory, where the momentum transfer is to the electron. The electric form factors have been given elsewhere.[8]

VI. RELATION TO LAGRANGIANS AND INFINITE-COMPONENT WAVE EQUATIONS

For completeness we shall give in this section the relation of the previous calculations to the infinite-dimensional relativistically invariant wave equations. The wave function $\Psi_\alpha(x)$, where α takes an infinite number of values, corresponds to the states of the $O(3,1)$ or the $O(4,1)$ representation that we have considered.

A. $O(3,1)$: Theory

Our restriction in Sec. IV B to current terms linear in the Lie algebra and linear in the momenta implies that the most general Lagrangian we can write is

$$\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}_1+\mathfrak{L}_2,\tag{6.1}$$

with

$$\mathfrak{L}_0=\Psi^\dagger\Gamma^\mu i\partial_\mu\Psi-\gamma\Psi^\dagger\Psi,$$
$$\mathfrak{L}_1=\alpha(\partial_\mu\Psi^\dagger)(\partial^\mu\Psi),$$
$$\mathfrak{L}_2=i\beta(\partial_\mu\Psi^\dagger)L^{\mu\nu}(\partial_\nu\Psi).\tag{6.2}$$

The wave equation corresponding to $\mathfrak{L}_0$ is clearly

$$(\Gamma^\mu P_\mu-\gamma)\Psi=0,\tag{6.3}$$

and the corresponding conserved current is that which we have called the algebraic current, i.e.,

$$J_0{}^\mu = \Psi^\dagger \Gamma^\mu \Psi \,. \tag{6.4}$$

The mass spectrum obtained from (6.3) is given by (4.15) and this corresponds to the Majorana theory if the $O(3,1)$ representation has $\nu=0$.

The Lagrangian $\mathcal{L}_0+\mathcal{L}_1$ has a wave equation

$$(\Gamma^\mu P_\mu - \alpha P_\mu P^\mu - \gamma)\Psi = 0\,, \tag{6.5}$$

with the current

$$J^\mu = \Psi^\dagger \Gamma^\mu \Psi - \alpha \Psi^\dagger P^\mu \Psi\,, \tag{6.6}$$

and gives the inverted mass spectrum (4.17); the fields Ψ here must correspond to the normalized states as discussed in Eq. (4.10) and following.

The addition of the term $\mathcal{L}_2$ to the Lagrangian does not change the wave equation. It does change, however, the current by the amount

$$i\beta \Psi^\dagger L^{\mu\nu} q_\nu \Psi \tag{6.7}$$

and adds β to the magnetic moment of the ground state. The situation here is quite analogous to that encountered in ordinary Lagrangian theory, where two Lagrangians differing by a divergence term lead to the same field equations as well as same integrated quantities, i.e., total charge, energy, total linear and angular momentum, but to quite different densities. This analogy was the motivation of our naming the current $\Gamma^\mu - \alpha P^\mu$ "minimal," and the current $L^{\mu\nu} q_\nu$ "nonmimimal" in Sec. IV.

B. $O(4,2)$: Theory

The important feature of this theory as distinct from the $O(3,1)$ theory is the introduction of new fields defined by

$$\bar{\Psi}_n(x) = N^{-1} e^{i\theta_n L_{45}} \Psi_n(x)\,, \tag{6.8}$$

where θ_n are parameters to be determined and L_{45} is a scalar generator of $O(4,2)$.

Some simple choices of Lagrangians in terms of the new states $\bar{\Psi}_n(x)$ contain an $\mathcal{L}_0$ term;

$$\mathcal{L}_0 = \bar{\Psi}^\dagger (\Gamma^\mu i\partial_\mu + \beta S - \gamma) \bar{\Psi}\,, \tag{6.9}$$

where Γ^μ is a vector in the $O(4,2)$ Lie algebra, β and γ are constants, and S is a scalar $O(4,2)$ generator whose presence in the Lagrangian is necessary in order to diagonalize the corresponding wave equation in the rest frame, as will be shown below. A simple term that one might add to the Lagrangian $\mathcal{L}_0$ is

$$\mathcal{L}_1 = \alpha (\partial_\mu \bar{\Psi}^\dagger)(\partial^\mu \bar{\Psi})\,, \tag{6.10}$$

and gives rise to the current $\Gamma^\mu - \alpha P^\mu$ discussed in Sec. V. One might also add terms like

$$(\partial_\mu \bar{\Psi}^\dagger) S (\partial^\mu \Psi)\,,\quad (\partial_\mu \bar{\Psi}^\dagger) L^{\mu\nu} (\partial_\nu \bar{\Psi}), \cdots. \tag{6.11}$$

The Lagrangian $\mathcal{L}_0$ alone leads to the wave equation

$$(\Gamma^\mu P_\mu + \beta S - \gamma)\bar{\Psi} = 0\,. \tag{6.12}$$

This equation is diagonalized in the rest frame by choosing $\tanh\theta_n = \beta/m_n$ and leads exactly to Eq. (5.20) and with the $\tanh\theta = \beta/m$ to the mass spectrum (5.21).

The addition of $\mathcal{L}_1$ term gives

$$[(1/N_\Gamma{}^2)(\Gamma^\mu P_\mu + \beta S) - (\alpha/N_p{}^2) P_\mu P^\mu - \gamma]\bar{\Psi} = 0\,. \tag{6.13}$$

Again, the requirement that (6.13) be diagonalized for $p^\mu = (m,\mathbf{0})$ requires, with the choice (5.16) for the normalization constants N_Γ and N_p, Eq. (5.12) and the mass spectrum (5.19). Note, however, that if one starts directly from the wave equation, it may have other solutions than those physical ones considered in the previous sections, for example, solutions with spacelike momenta.