The Two Gauge Fields of Elasticity and Plasticity[a]

H. KLEINERT

Free University of Berlin
Institute for Elementary Particle Theory
Arnimallee 14
1000 Berlin 33, Federal Republic of Germany

1. INTRODUCTION

There are a variety of physical systems that are characterized by:

1. The existence of long-range modes which dominate the thermodynamics at low temperatures.
2. The existence of short-range linelike disturbances of order, called vortex or defect lines, which are activated at higher temperatures and lead, ultimately, to the destruction of the ordered state in one or more phase transitions.

In recent years, there has been a great deal of progress in developing a statistical mechanics of such systems using only these two modes as elementary excitations. In this development, two gauge structures have played an important role.

The first gauge structure emerges when formulating the simplest "harmonic model" of vortices or defects with long-range interactions. The long-range modes are described by some smooth displacement field $u(\mathbf{x})$. Vortices or defect lines arise if $u(\mathbf{x})$ is a multivalued field and can have jumps across surfaces. The boundaries of these surfaces are the defect lines. The precise position of the surfaces is irrelevant. This irrelevance manifests itself in the existence of a gauge invariance. The gauge field involved will be called "plastic gauge field" or "defect gauge field."

The second gauge structure is found when constructing a disorder field theory of vortex or defect lines. The long-range modes of the system cause long-range interactions between such lines. In order to include them into the disorder field theory, a local coupling is needed. Such a local coupling emerges after rewriting the long-range modes in terms of a gauge field, to be called "elastic gauge field" or "stress gauge field." It permits bringing the disorder field theory to the same form as the Ginzburg-Landau theory of superconductivity, albeit with a different meaning of the field quantities. Instead of order, the field describes the *disorder* of defect lines; and instead of magnetism, the gauge potential accounts for the long-range elastic forces.

Historically, the need for developing a statistical mechanics of defects was first recognized 30 years ago. In 1952, Shockley found that it would be useful to study the phase transition of melting as a proliferation of closed dislocation lines.[1] Three years later, Feynman made the same suggestion with respect to the superfluid transition and

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vortex lines.[2] As we understand it now, Feyman was right, Shockley's proposal needs some refinement.

For superfluid transitions the vortex mechanism was clarified first in two dimensions by Berenzinskii, Kosterlitz, and Thouless.[3] They showed that the breakdown of quasi-long-range order was really due to the unbinding of vortex antivortex pairs. In three dimensions, Feynman's program was carried through by the author, who developed a complex disorder field theory for the grand-canonical ensemble of vortex lines.[4] The long-range forces between the lines were correctly included by means of a local gauge field, the stress gauge field.

The plastic gauge fields are probably most useful as a starting point when setting up a theory of defects with elastic forces. The field quantities appearing in it have the most transparent physical interpretation. Functional integral techniques can be used to bring the partition function to other equivalent forms. In particular, we can arrive at the above-mentioned field theory of defect coupled to an elastic gauge field.

It is the purpose of this paper to exhibit the two-fold gauge theory of elasticity and plasticity, following this approach, for two typical systems: superfluid ^{4}He and the crystalline solid. In doing so we shall adopt a uniform language which helps in stressing the structural parallels between the two different physical systems. Giving preference to a "crystalline language," we shall call the hydrodynamic energy of superflow "elastic" or "stress" energy, and the vortex lines "defect lines."

2. DEFECT GAUGE FIELDS IN SUPERFLUIDS

The superfluid is described by an order parameter, a phase exp $(iu(\mathbf{x}))$ with an angle $u(\mathbf{x})$ called *displacement*. The gradient $\partial_i u(\mathbf{x})$ is the superfluid velocity to be called *distortion*. At zero temperature, the hydrodynamic energy of the superfluid or *elastic* energy is given by

$$E = \frac{1}{2} \int d^3x \, (\partial_i u(\mathbf{x}))^2 \tag{1}$$

If the system is heated, vortex lines appear. A vortex line can be described by introducing a *Volterra cutting surface S* over which the phase jumps by 2π, i.e.,

$$(\partial_i u^{\mathrm{P}}(\mathbf{x})) = 2\pi \, \delta_i(S) \tag{2}$$

This singular distortion is called *plastic distortion*. We have put the derivative in parentheses since $(\partial_i u^{\mathrm{P}}(\mathbf{x}))$ *cannot* be written as the derivative of a single-valued scalar field. It has to be treated like a three-component vector field. Only a multivalued $u(\mathbf{x})$ can have such a derivative. The multivaluedness[5] shows up when going around the boundary of the surface S and forming the Burgers circuit integral

$$\int_B dx_i (\partial_i u^{\mathrm{P}}(\mathbf{x})) = \int_B du^{\mathrm{P}}(\mathbf{x}) = 2\pi \tag{3}$$

We see that the vortex line is characterized by a *vortex density* which is singular along the boundary line L of the surface S.

$$\alpha_i(\mathbf{x}) = \epsilon_{ij\epsilon} \, \partial_i (\partial_k u^{\mathrm{P}}(\mathbf{x}) = 2\pi \, \epsilon_{ij\epsilon} \, \partial_i \, \delta_k(S) = 2\pi \, \delta_i(L) \tag{4}$$

Notice that only the line L is a physical observable. The position of the surface has no relevance. It can be changed arbitrarily in space as long as the boundary L is kept fixed. A shift from S to S' produces a change in $\alpha_i(\mathbf{x})$ as follows

$$\begin{aligned}(\partial_i u^P(\mathbf{x})) = 2\pi\,\delta_i(S) \longrightarrow (\partial_i u^P(\mathbf{x}))' &= 2\pi\,\delta_i(S') \\ &= 2\pi\,\delta_i(S) + 2\pi(\delta_i(S') - \delta_i(S)) \\ &= (\partial_i u^P(\mathbf{x})) + \partial_i N(\mathbf{x}) \end{aligned} \tag{5}$$

where

$$N(\mathbf{x}) = -\,2\pi\,\delta(V) \tag{6}$$

and V is the closed volume element over which S has swept. Thus, changes in S amount to local gauge transformations of $(\partial_i u^P(\mathbf{x}))$. Under them, the curl $\alpha_i(\mathbf{x})$ of $(\partial_i u^P(\mathbf{x}))$ remains invariant. For this reason, the plastic disortion field will be called a *defect gauge field* and the transformation (**5**) a *defect gauge transformation.*

The elastic energy of an ensemble of lines is given by the deviation of the total distortion from the plastic one as

$$E_\varphi = \int d^3x\,\frac{1}{2}\,(\partial_i u(\mathbf{x}) - (\partial_i u^P(\mathbf{x})))^2 \tag{7}$$

Under a defect gauge transformation, the total displacement field $u(\mathbf{x})$ changes by

$$u(\mathbf{x}) \longrightarrow u(\mathbf{x}) + N(\mathbf{x}) \tag{8}$$

Such a change is of no physical relevance since N is a multiple of 2π and only $e^{iu(\mathbf{x})}$ is observable.

In the presence of vortex lines, the superflow velocity is given by

$$v_i(\mathbf{x}) = \partial_i u(\mathbf{x}) - (\partial_i u^P(\mathbf{x})) \tag{9}$$

This quantity is a *defect gauge invariant.* It is single valued and observable.

The grand-canonical ensemble of fluctuating vortex lines including their proper long-range forces is described by the partition function

$$Z = \int Du(\mathbf{x}) \int D(\partial_i u^P(\mathbf{x}))\,\Phi[(\partial_i u^P)]\,e^{-(\beta/2)\int d^3x(\partial_i u - (\partial_i u^P(\mathbf{x})))^2} \tag{10}$$

where the symbol $D(\partial_i u^P(\mathbf{x}))$ implies a summation over all Volterra surfaces S. The functional $\Phi[(\partial_i u^P)]$ has to be introduced in order to avoid an infinite overall factor, due to the gauge degeneracy of the integrand. This amounts to fixing a *particular way* of constructing the Volterra surface S for every L. A particularly simple choice *would be the transverse defect gauge*

$$\Phi[(\partial_i u^P)] = \prod_{\mathbf{x}} \delta\,(\partial_i(\partial_i u^P(\mathbf{x}))) \equiv \delta[\partial_i(\partial_i u^P)] \tag{11}$$

With it, $\partial_i u$ and $(\partial_i u^P)$ decouple and $u(\mathbf{x})$ can directly be integrated to give

$$Z = \prod_x \sqrt{\frac{2\pi}{\beta}}\,\det\,(-\partial^2) \int D(\partial_i u^P)\,\delta[\partial_i(\partial_i u^P)]\,e^{-(\beta/2)\int d^3x(\partial_i u^P)^2} \tag{12}$$

The factor $\sqrt{2\pi/\beta}$ contributes ½ to the specific heat, in accordance with Dulong-Petit's law.

In the fixed transverse gauge, the integral over $(\partial_i u^p)$ can easily be replaced by an integral over the manifestly defect gauge invariant vortex density $\alpha_i(\mathbf{x})$. In the transverse gauge, it is related to $(\partial_i u^p)^2$ by

$$\alpha_i(\mathbf{x})^2 = \partial_k(\partial_i u^p)\,\partial_k(\partial_i u^p) - (\partial_i(\partial_i u^p))^2$$
$$= \partial_k(\partial_i u^p)\,\partial_k(\partial_i u^p)$$

such that Z becomes

$$Z = \prod_{\mathbf{x}} \sqrt{\frac{2\pi}{\beta}} \det(-\partial^2) \int D\alpha_i(\mathbf{x})\, \delta\,[\partial_i\alpha_i]\, e^{-(\beta/2)\int d^3x\alpha_i(\mathbf{x})1/(-\partial^2)\alpha_i(\mathbf{x})} \quad (13)$$

where $1/(-\partial^2)\ \alpha_i(\mathbf{x}) \equiv \int d^3x'\ U(\mathbf{x} - \mathbf{x}')\alpha(\mathbf{x}')$ and $U(\mathbf{x} - \mathbf{x}') = 1/4\pi R$ is the Coulomb-Green's function ($R = (\mathbf{x} - \mathbf{x}')$).

There is one immediate objection to this derivation: the transverse gauge (**11**) cannot be realized for any shape of the Volterra surfaces. Only a gauge of the axial type can, say,

$$(\partial_3 u^p) = 0$$

Fortunately, however, the mistake has no consequences in the derivation of (**13**) since the difference between the two gauges becomes relevant only in the presence of external volume forces.[5]

3. STRESS GAUGE FIELDS IN SUPERFLUIDS

Besides the defect gauge invariance, the superfluid harbors another gauge structure associated with the supercurrent.[6] It is revealed by rewriting the path integral (**10**) in the canonical form

$$Z = \int \frac{Db_i(\mathbf{x})}{\sqrt{2\pi\beta}} \int Du(\mathbf{x}) \int D(\partial_i u^p(\mathbf{x}))\Phi[(\partial_i u^p)]\, e^{-(1/2\beta)\int d^3x b_i^2(\mathbf{x}) + i\int d^3x b_i(\partial_i u - (\partial_i u^p))} \quad (14)$$

which is obviously the same as (**10**) by a quadratic completion. Integrating out the $u(\mathbf{x})$ field produces a δ-functional $\delta[\partial_i b_i]$ such that $b_i(\mathbf{x})$ forms closed field lines, just like a magnetic field. Taking advantage of this analogy we can introduce a vector potential $a_i(\mathbf{x})$ via

$$\mathbf{b}(x) = (\partial \times \mathbf{a})(\mathbf{x}) \quad (15)$$

This relation displays invariance under *stress gauge transformations*

$$a_i(\mathbf{x}) \rightarrow a_i(\mathbf{x}) + \partial_i\Lambda(\mathbf{x}) \quad (16)$$

In terms of $\mathbf{a}$ the path integral (**14**) becomes

$$Z = \int \frac{Da_i}{\sqrt{2\pi\beta}} \delta[\partial_i a_i]\, Z_d[a]$$
$$= \int \frac{Da_i}{\sqrt{2\pi\beta}} \delta[\partial_i a_i] \int D(\partial_i u^p)\Phi[(\partial_i u^p)]\, e^{-i\int d^3x(\partial\times\mathbf{a})\cdot(\partial u^p)} \quad (17)$$

A partial integration brings the defect factor to the form

$$Z_d[a] \equiv \int D\alpha_i(\mathbf{x})\delta[\partial_i\alpha_i]\, e^{-i\int d^3x \mathbf{a}\cdot\boldsymbol{\alpha}(\mathbf{x})} \tag{18}$$

If we insert this into (**17**) and integrate out the a field we recover once more (**13**). Inserting into (**18**) the explicit decomposition (**4**) of α_i according to lines we can write

$$Z_d[a] = \sum_{\{L\}} e^{-i\int_L dx_i a_i(x)} \tag{19}$$

where $\Sigma_{\{L\}}$ denotes the sum over all closed line configurations.

4. DISORDER FIELD THEORY OF SUPERFLUIDS

The properties of such a partition function can best be studied by transforming it to a fluctuating complex disorder field theory

$$Z_d[a] = \int D\varphi D\varphi^+ e^{-\int d^3\{1/2[(\partial - _ia)\varphi|^2 + (m^2/2)|\varphi|^2 + (g/4)|\varphi|^4 + \ldots\}} \tag{20}$$

That this is possible follows from the usual duality between fluctuating particle orbits and fields.[5] The orbits are now the vortex lines L.

Inserted into (**17**), the exponent becomes a field energy of the Ginzburg-Landau type

$$\frac{1}{T}E = \frac{1}{2\beta}\int d^3x\, (\partial \times \boldsymbol{a})^2$$
$$+ \int d^3x \left\{\frac{1}{2}|(\partial - i\mathbf{a})\varphi|^2 + \frac{m^2}{2}|\varphi|^2 + \frac{g}{4}|\varphi|^4 + \ldots\right\} \tag{21}$$

The coupling g parameterizes the steric repulsion between vortex lines.

The mass term $m^2 \propto (\epsilon/T - S)$ becomes negative *above* a certain critical temperature $T_c = \epsilon/S$ where the entropy S per line element exceeds the energy ϵ per temperature. There, (**21**) has a second-order phase transition and the field φ takes a nonzero expectation $|\varphi| = \sqrt{-m^2/g}$ as a signal for the proliferation of vortex lines. This explains the name *disorder field.* As a consequence, the vector potential acquires a mass term $\frac{1}{2}|\varphi|^2\mathbf{a}^2$. When added to $\frac{1}{2}\beta\, (\partial \times \mathbf{a})^2$, we see that in the normal state the vector potential has a finite penetration depth $\xi = 1/\sqrt{\beta|\phi|^2}$. This is the *disorder version of the Meissner effect* in superconductivity.

It has been claimed that the second-order transition in a pure $|\varphi|^4$ theory becomes first order when coupled to a gauge field.[7] Recently, the author showed that this is true only below a certain tricritical ratio $K_t = \sqrt{g/\beta} \approx 0.7/\sqrt{2}$. For the vortices in ^{4}He, this amounts to a very small steric repulsion between the lines.

Let us now follow the same line of approach and study the situation in a crystal.

5. DEFECT GAUGE FIELDS IN CRYSTALS

In crystals, the elastic energy can be expanded into gradients of the displacement field $u_i(\mathbf{x})$ as follows

$$E_{el} = \int d^3x \, [\mu u_{ij}^2 + \frac{\lambda}{2} u_{ii}^2 + 2\mu l^2 (\partial_j \omega_i)^2] \tag{22}$$

where

$$u_{ij} = (\partial_i u_j + \partial_j u_i)(\mathbf{x}) \tag{23}$$

$$\omega_i = \frac{1}{2} \epsilon_{ijk} \partial_j u_k(\mathbf{x}) \tag{24}$$

are the strain and rotation fields, respectively. We have omitted all higher gradients of $u_i(\mathbf{x})$ which are inessential for our purpose, but included the lowest gradient of ω_i which is essential for the discussion.

In the presence of defects, the energy is given by the differences between u_{ij} and the plastic strain $u_{ij}^P = 1/2((\partial_i u_j^P) + (\partial_i u_j^P))$ and between $\partial_i \omega_j$ and the rotational part of the plastic distortion $\partial_i 1/2 \epsilon_{jkl} (\partial_k u_l^P)$ plus an additional *independent*[9] plastic contortion $\phi_{ij}^P \equiv (\partial_i \omega_j^P)$:

$$E_{el} = \int d^3x \left\{ \mu (u_{ij} - u_{ij}^P)^2 + \frac{\lambda}{2} (u_{ii} - u_{ii}^P)^2 \right.$$
$$+ 2\mu l^2 \left[\left(\partial_i \omega_j - \partial_i \frac{1}{2} \epsilon_{jkl} (\partial_k u_l^P) - (\partial_i \omega_j^P) \right)^2 \right.$$
$$\left.\left. + \epsilon \left(\partial_i \omega_j - \partial_i \frac{1}{2} \epsilon_{jkl} (\partial_k u_l^P) - (\partial_i \omega_j^P) \right) \left(\partial_j \omega_i - \partial_j \frac{1}{2} \epsilon_{ikl} (\partial_k u_l^P) - (\partial_j \omega_i^P) \right) \right] \right\} \tag{25}$$

We have added another term $\epsilon(\partial_i \omega_j - \partial_i \, ½ \, \epsilon_{jkl} \, (\partial_k u_l^P) - (\partial_i \omega_j^P)) \, (\partial_j \omega_i - \partial_j \, ½ \, \epsilon_{ikl} \, (\partial_k u_l^P) - (\partial_j \omega_i^P)$, which would be zero in the absence of defects but which can now be present, since for multivalued fields, $\partial_i \omega_i^P = ½ \, \epsilon_{ijk} \partial_i (\partial_j u_k^P) \neq 0$. A dislocation line is constructed by taking a Volterra cutting surface S and letting the plastic distortion have a jump across it, i.e.,

$$(\partial_i u_j^P) = b_i \delta_i(S) \tag{26}$$

where b_i is the Burgers vector. The energy is invariant under the *dislocation gauge transformations*[5]

$$\begin{aligned} (\partial_i u_j^P) \longrightarrow (\partial_i u_j^P)' &= \delta_i(S') b_j \\ &= \delta_i(S) b_i + (\delta_i(S') - \delta_i(S)) b_i \\ &= (\partial_i u_j^P) + \partial_i N_j(\mathbf{x}) \end{aligned} \tag{27}$$

with

$$N_i(\mathbf{x}) = -\,\delta(V) b_i$$

when changing simultaneously

$$u_i(\mathbf{x}) \longrightarrow u_i(\mathbf{x}) + N_i(\mathbf{x}) \tag{28}$$

These latter transformations reflect the multivaluedness of $u_i(\mathbf{x})$: the rest position of an atom is undefined up to a multiple of a lattice vector b_i.

A disclination line is given by a jump in the rotation field

$$(\partial_i \omega_j^p) = \delta_i(S)\Omega_j \tag{29}$$

The defect lines are the boundaries of the surfaces S and show up when forming the curls

$$\alpha_{ij}(\mathbf{x}) = \epsilon_{ikl}\partial_k(\partial_l u_j^p) = \delta_i(L) b_j \tag{30}$$

$$\Theta_{ij}(\mathbf{x}) = \epsilon_{ikl}\partial_k(\partial_l \omega_j^p) = \delta_i(L)\Omega_j \tag{31}$$

They are called *dislocation* and *disclination densities.* It is useful to define a combination of these, the total defect density

$$\eta_{ij}(\mathbf{x}) = \epsilon_{ikl}\epsilon_{jmn}\partial_k\partial_m u_{ln}^p \tag{32}$$

Writing[9]

$$\begin{aligned}\alpha_{in}(\mathbf{x}) &= \epsilon_{ikl}\partial_k(u_{ln} + \epsilon_{ln}\omega_m) \\ &= \epsilon_{ikl}\partial_k(\partial_l u_n^p) + \delta_{in}(\partial_k \omega_k^p) - (\partial_n \omega_i^p)\end{aligned} \tag{33}$$

and applying $\epsilon_{jmn}\partial_m$ we find the relation

$$\eta_{ij}(\mathbf{x}) = \Theta_{ij} + \tfrac{1}{2}\,(\epsilon_{imn}\partial_m \alpha_{jn} + (ij)) + \tfrac{1}{2}\,\epsilon_{ijn}\partial_m \alpha_{mn} \tag{34}$$

It is usually postulated that there are no jumps in any higher derivatives. From this it follows that

$$\partial_i \Theta_{ij} = 0 \tag{35}$$

which states that disclination lines are closed. Applying to (27) we find that

$$\partial_i \alpha_{in} = -\epsilon_{npq}\Theta_{pq} \tag{36}$$

This implies that whenever dislocation lines contain sources, these are due to an antisymmetric part of the disclination density.

The energy (25) is furthermore invariant under *disclination gauge transformations*[5]

$$\begin{aligned}(\partial_i \omega_j^p) &\longrightarrow (\partial_i \omega_j^p) + (\delta_i(S') - \delta_i(S))\,\Omega_j = (\partial_i \omega_j^p) + \partial_i M_j \\ (\partial_i u_j^p) &\longrightarrow (\partial_i u_j^p) - \epsilon_{ijk}\, M_k\end{aligned} \tag{37}$$

with

$$M_k = -\,\delta(V)\Omega_k$$

The partition function of dislocations and disclinations under stress reads

$$Z = \int Du_i(\mathbf{x}) \int D(\partial_i u_l^p(\mathbf{x}))\, \Phi\,[\partial_i u_l^p] \int D\omega_i(\mathbf{x}) \int D(\partial_j \omega_i^p(\mathbf{x}))\, \Phi\,[\partial_j \omega_i^p] \times \exp\{-\beta E_{el}\} \tag{38}$$

We can now choose a gauge in which $(\partial_i u_j^p) = (\partial_j u_i^p) \equiv (\partial_i \overset{s}{u}{}_j^p)$ (via M_k) and in which $\partial_i u_{ij}^p = 0$ (via $N_i(\mathbf{x})$). Then the Boltzmann factor is

$$\exp - \beta \int d^3\mathbf{x}\{\mu(u_{ij}^2 + (\partial_i \mathring{u}_j^p)^2) + \frac{\lambda}{2}(u_{ii}^2 + 2u_i\partial_i(\partial_l u_l^p) + (\partial_l u_l^p)^2)$$
$$+ 2\mu l^2[(\partial_i\omega_j)^2 - 2\omega_j\partial_i(\partial_i\omega_j^p) + (\partial_i\omega_j^p)^2$$
$$+ \epsilon(-2\omega_j\partial_i(\partial_j\omega_i^p) + (\partial_i\omega_j^p)(\partial_j\omega_i^p))]\} \tag{39}$$

Forgetting for a moment the l^2 terms, we can perform the u_i integrals and find[b]

$$Z = \prod_{\mathbf{x}} \sqrt{\frac{2\pi}{\beta}}^3 \det(-(\lambda + 2\mu)\partial^2)^{-1/2}\det(-\mu\partial^2)^{-1}$$
$$\cdot \int D\partial_j u_i^p \Phi[\partial_j u_i^p] \int D(\partial_j\omega_i^p)\Phi[\partial_j\omega_i^p]$$
$$\cdot \exp -\beta \int d^3x d^3x' \{\mu(\partial_i u_i^p)^2\delta^{(3)}(\mathbf{x} - \mathbf{x}')$$
$$- \frac{\lambda^2}{2} G_{ii'}(\mathbf{x} - \mathbf{x}')\partial_i(\partial_l u_l^p(\mathbf{x})\,\partial_{i'}(\partial_l u_l^p(\mathbf{x}'))\} \tag{40}$$

where $G_{ij}(q) = (1/\mu q^2)\,[(\delta_{ij} - q_iq_j/q^2) + \mu/(\lambda + 2\mu)q_iq_j/q^2]$ is the elastic Green's function. In the transverse gauge we can calculate

$$\eta_{ij} = \partial^2\delta_{ij}\, u_{ll}^p - \partial^2 u_{ij}^p - \partial_i\partial_j u_{ll}^p$$
$$u_{ij}^p = -\frac{1}{\partial^2}(\eta_{ij} - (\delta_{ij} - \partial_i\partial_j/\partial^2)\eta_{ll}) \tag{41}$$

such that the exponential becomes

$$\exp - \beta\mu \int d^3x\,(\eta_{ij}(\mathbf{x})\frac{1}{\partial^4}\eta_{ij}(\mathbf{x}) + \frac{\nu}{1-\nu}\eta_{ll}(\mathbf{x})\frac{1}{\partial^4}\eta_{ll}(\mathbf{x})) \tag{42}$$

where $\nu = \lambda/(2\lambda + 2\mu)$.

Inserting only the α_{li} part of (**34**) leads to the well-known Blin $1/R$ law for the energy between dislocation lines

$$\exp - \beta\frac{\mu}{2}\int d^3x\alpha_{lj}(\mathbf{x})\left(\frac{1}{-\partial^2}\left(\delta_{lj'}\delta_{jl'} - \frac{1}{2}\delta_{lj}\delta_{l'j'}\right) + \frac{1}{1-\nu}\epsilon_{ljk}\epsilon_{l'j'k'}\partial_k\partial_{k'}/\partial^4\right)\alpha_{l'j'}(\mathbf{x}) \tag{43}$$

In addition, (**42**) specifies also the long-range forces of the form $-R/8\pi$ between disclination lines [via (**34**), (**31**)], as well as the mixed interactions of the form log R between dislocations and disclinations.

[b]For unisotropic materials the exponential would read

$$\exp - \frac{\beta}{2}\sum_{\mathbf{q}}(c_{klk'l'} - q_j\mathbf{q}_{j'}c_{ijkl}c_{i'j'k'l'}G_{ii'}(\mathbf{q}))\left[\frac{1}{\mathbf{q}^2}(\eta_{kl} - \delta_{kl}\eta_{rr}) + \frac{\mathbf{q}_k\mathbf{q}_l}{\mathbf{q}^4}\eta_{rr}\right]$$
$$\times\left[\frac{1}{\mathbf{q}^2}(\eta_{k'l'} - \delta_{k'l'}\eta_{r'r'}) + \frac{\mathbf{q}_{k'}\mathbf{q}_{l'}}{\mathbf{q}^4}\eta_{r'r'}\right]$$

where $G_{ii'}(\mathbf{q})$ is the inverse of $c_{iji'j'}q_jq_{j'}$.

Including the l^2 term without ϵ modifies the exponent in (**42**) to

$$-\beta \int d^3x d^3x' \{(\mu(\partial_j u_i^p)^2 + 2\mu l^2(\partial_j\omega_i^p)^2)\delta^{(3)}(\mathbf{x}-\mathbf{x}')$$
$$- \tfrac{1}{2} G_{ii'}(\mathbf{x}-\mathbf{x})(\lambda\partial_i\partial_l u_l^p + 2\mu l^2 \epsilon_{ikn}\partial_n\partial_j(\partial_j\omega_k^p))$$
$$\times (\lambda\partial_{i'}\partial_l u_l^p + 2\mu l^2 \epsilon_{i'kn}\partial_n\partial_j(\partial_j\omega_k^p))\} \quad (\mathbf{44})$$

where $G_{ij'}(q) = 1/(\mu q^2(1+l^2q^2))(\delta_{ij} - q_iq_j'/q^2) + 1/(2\mu+\lambda)q^2\, q_iq_{i'}/q^2$. In momentum space this becomes

$$-\beta\sum_{\mathbf{q}}\left\{\frac{\mu}{\mathbf{q}^4}\left(|\eta_{ij}|^2 + \frac{\nu}{1-\nu}|\eta_{ll}|^2\right)\right.$$
$$+ 2\mu l^2\left[|(\partial_j\omega_i^p)|^2 - \frac{1}{\mathbf{q}^2}|\partial_j(\partial_j\omega_i^p)|^2 + \frac{1}{\mathbf{q}^4}|\partial_j\partial_i(\partial_j\omega_i^p)|^2\right]$$
$$\left. + \frac{2\mu l^2}{1+l^2\mathbf{q}^2}\left[\frac{1}{\mathbf{q}^2}|\partial_i(\partial_j\omega_i^p)|^2 - \frac{1}{\mathbf{q}^4}|\partial_j\partial_i(\partial_j\omega_i^p)|^2\right]\right\} \quad (\mathbf{45})$$

We now use

$$\sum_{\mathbf{q}}|\Theta_{\mathbf{q}}|^2 = \sum_{\mathbf{q}}[\mathbf{q}^2|(\partial_j\omega_i^p)|^2 - |\partial_j(\partial_j\omega_i^p)|^2]$$
$$(\delta_{ij}\partial^2 - \partial_i\partial_j)\alpha_{ij} + \epsilon_{kil}\partial_k\Theta_{il} = 2\,\partial_i\partial_j(\partial_i\omega_j^p) \quad (\mathbf{46})$$

to write the second term as

$$-\beta\sum_{\mathbf{q}} 2\mu l^2\left(\frac{1}{\mathbf{q}^2}|\Theta_{ij}|^2 + \frac{1}{4}\frac{1}{\mathbf{q}^4}|(\delta_{ij}\partial^2 - \partial_i\partial_j)\alpha_{ij} + \epsilon_{kil}\partial\Theta_{il}|^2\right) \quad (\mathbf{47})$$

and the third term as

$$-\beta\sum_{\mathbf{q}} 2\mu l^2 \frac{1}{1+l^2\mathbf{q}^2}\frac{1}{\mathbf{q}^6}|(\delta_{ij}\partial^2 - \partial_i\partial_j)\partial_k\alpha_{ik}|^2 \quad (\mathbf{48})$$

Including finally the ϵ terms there is one further contribution

$$-\beta\sum_{\mathbf{q}} 2\mu l^2\left[\frac{\epsilon}{\mathbf{q}^2}\left(|\Theta_{ij}|^2 - \frac{1}{\mathbf{q}^2}|\partial_j\Theta_{ij}|^2 - |\Theta_{ll}|^2\right)\right.$$
$$- \frac{\epsilon^2 l^2}{1+l^2\mathbf{q}^2}\frac{1}{\mathbf{q}^2}|\partial_j\Theta_{ij}|^2 + \frac{1}{4\mathbf{q}^2}|\partial^2\alpha_{ll} - \partial_i\partial_j\alpha_{ij} + \epsilon_{ilk}\partial_i\,\theta_{lk}|$$
$$+ \frac{\epsilon}{4\mathbf{q}^2}|(\partial^2\delta_{ij} - \partial_i\partial_j)\alpha_{ij} + \epsilon_{kil}\Theta_{il}|^2$$
$$\left. + \frac{\epsilon}{1+l^2\mathbf{q}^2}\frac{1}{\mathbf{q}^2}((\partial_{ij}\partial^2 - \partial_i\partial_j)\partial_k\alpha_{jk}^*\,\epsilon_{ipq}\Theta_{pq} + \text{c.c.})\right] \quad (\mathbf{49})$$

This Boltzmann factor gives the elastic energy for *all* defects, dislocations, and disclinations due to second gradient elasticity.[c]

[c]For more details, see Reference 23.

The path integral over the defect fields $(\partial_i u_j^p)$, $(\partial_i \omega_j^p)$ may now be replaced by integrals over the defect densities α_{ij}, Θ_{ij}.

$$\int D\alpha_{ij}(\mathbf{x})\,\delta[\partial_i\alpha_{ij} + \epsilon_{ikl}\Theta_{kl}] \int D\Theta_{ij}(\mathbf{x})\,\delta[\partial_i\Theta_{ij}] \tag{50}$$

Notice also that classical linear elasticity is degenerate in that it cannot distinguish the different characters of the defects. It only depends on the combination $\eta_{ij}(\mathbf{x})$ of α_{ij} and Θ_{ij} which picks only three independent combinations of the 12 independent components of α_{ij}, Θ_{ij}. The higher gradient terms of elasticity lift this degeneracy. In writing down (**47**), (**49**) we have chosen to rewrite the longitudinal parts $\partial_i\alpha_{ij}$ of α_{ij} as disclinations $-\epsilon_{jkl}\Theta_{kl}$. The remaining energy depends only on the divergenceless part of α_{ij}, α_{ij}^T which satisfies $\partial_i\alpha_{ij}^T = o$. Hence we can write the $\epsilon = o$ part of Z as

$$Z = \sqrt{\frac{2\pi}{\beta}}^{3} \det\left(-(\lambda + 2\mu)\partial^2\right)^{-1/2} \det\left(-\mu\partial^2(1 - l^2\partial^2)\right)^{-1}$$

$$\int D\alpha_{ij}^T\delta(\partial_i\alpha_{ij}^T) \int D\Theta_{ij}\delta(\partial_i\Theta_{ij})$$

$$\exp\left\{-\beta\mu\sum_{\mathbf{q}}\frac{1}{\mathbf{q}^4}\left(|\eta_{ij}|^2 + \frac{\nu}{1-\nu}|\eta_{ll}|^2\right)\right.$$

$$-2\beta\mu l^2\sum_{\mathbf{q}}\left[\frac{1}{\mathbf{q}^2}|\Theta_{ij}|^2 + \frac{1}{2}\frac{1}{\mathbf{q}^4}|\delta_{ij}\partial^2\alpha_{ij}^T + \epsilon_{kil}\partial_k\Theta_{il}|^2\right]$$

$$\left.-2\beta\mu l^2\sum_{\mathbf{q}}\frac{1}{1 + l^2\mathbf{q}^2}\frac{1}{\mathbf{q}^2}|\partial_k\alpha_{ki}^T|^2\right\} \tag{51}$$

with a similar replacement of α_{ij} by α_{ij}^T in the ϵ term (**49**).

6. STRESS GAUGE FIELDS IN CRYSTALS

As in superfluid, there exists a second gauge structure associated with the stress energy. In order to see this we bring the partition function (**38**) to the canonical form

$$Z = \prod_{i,j}\int\frac{D\sigma_{ij}^s}{\sqrt{2\pi\beta\mu}}\prod_{i,j}\int D\sigma_{ij}^a\prod_{i,j}\int\frac{D\tau_{ij}}{\sqrt{8\pi\beta\mu l^2}}\int Du_i(\mathbf{x})\int D\omega_i(\mathbf{x})$$

$$\int D(\partial_j u_i^p)\Phi[(\partial_j u_i^p)] \int D(\partial_i\omega_j^p)\Phi[(\partial_i\omega_j^p)]$$

$$\exp\left\{-\frac{1}{\beta}\int d^3x\left[\frac{1}{4\mu}\left(\sigma_{ij}^{s2} - \frac{\nu}{1+\nu}\sigma_{ll}^{s2}\right) + \frac{1}{8\mu l^2}(\delta_1\tau_{ij}^2 + \delta_2\tau_{ij}\tau_{ji})\right]\right.$$

$$+ i\int d^3x\sigma_{ij}(\partial_i u_j - \epsilon_{ijk}\omega_k - (\partial_i u_j^p))$$

$$\left.+ i\int d^3x\tau_{ij}(\partial_i\omega_j - (\partial_i\omega_j^p))\right\} \tag{52}$$

where $\delta_1 \equiv 1/(1-\epsilon)$, $\delta_2 = -\epsilon\delta_1$. We have found it convenient to treat ω_i as an independent integration variable whose connection with $\frac{1}{2}\,\epsilon_{ijk}\,\partial_j u_k$ is enforced by the integration over the antisymmetric part σ^a_{ij} of σ_{ij}. The energy depends only on the symmetric part σ^s_{ij}.

Integrating out u_i and ω_i gives the conservation laws

$$\partial_i \tau_{ij}$$

$$\partial_i \tau_{ij} = -\epsilon_{jkl}\,\sigma_{kl} \tag{53}$$

which are the stress analogues of the defect conservation laws (35) and (36). We can now introduce stress gauge fields A_{il} and h_{il} which guarantee (53), by writing

$$\sigma_{ij} = \epsilon_{ikl}\partial_k A_{lj}$$

$$\tau_{ij} = \epsilon_{ikl}\partial_k h_{lj} + \delta_{ij}A_{ll} - A_{ji} \tag{54}$$

These decompositions are invariant under the stress gauge transformations[d]

$$A_{lj} \rightarrow A_{lj} + \partial_l \Lambda_j$$

$$h_{lj} \rightarrow h_{lj} - \epsilon_{ljk}\Lambda_k \tag{55}$$

and

$$h_{lj} \rightarrow h_{lj} + \partial_l \xi_j \tag{56}$$

which are the analogues of (27), (37). In terms of A_{lj} and h_{lj}, the coupling to the defects in (51) reads

$$\exp\left\{-i\sum_{\mathbf{x}} A_{lj}[\epsilon_{lik}\partial_k(\partial_i u^p_j) + \delta_{lj}(\partial_n\omega^p_n) - (\partial_j\omega^p_l)] - i\sum_{\mathbf{x}} h_{lj}[\epsilon_{lki}\partial_k(\partial_i\omega^p_j)]\right\} \tag{57}$$

This coupling is invariant under the defect gauge transformations (27), (37). In particular, (37) can be used to make $(\partial_i u^p_j)$ symmetric. Remembering (33) and (31), we recognize the sources as α_{li} and Θ_{li}, and the coupling to the ensemble of defect lines is given by

$$\exp\left\{-i\int d^3x\,(A_{lj}\alpha_{lj} + h_{lj}\Theta_{lj})\right\} = \exp\left\{-ib_j\int dx_l\,A_{lj} - i\Omega_j\int dx_l\,h_{lj}\right\} \tag{58}$$

Integrating out the A_{lj}, h_{lj} fields in (52) leads again to a Boltzmann factor (45), (49) with a sum over defect densities (50).

7. DISORDER FIELD THEORY OF CRYSTALS

The analogy with (18), (19) allows us to introduce disorder fields, one associated with every fundamental Burgers' vector b_i and one with every Frank vector Ω_i. If we

[d]Notice that with ω_i being treated as an independent variable, the energy in (52) is invariant under the defect gauge transformations

$$(\partial_i u^p_j) \rightarrow (\partial_i u^p_j) + \partial_i N_j, \qquad u_i \rightarrow u_i + N_i$$

the last replacement following from the δ-functional produced by $\int d\sigma^a_{ij}$.

neglect the coupling between dislocation and disclination lines in (50), we can write directly

$$Z = \int DA_{lj} \int Dh_{lj}\Phi[A]\Phi[h_{lj}] \int D\varphi_b D\varphi_b^+ \, D\varphi_\Omega D\varphi_\Omega^+$$

$$\exp\left\{ -\sum_{\mathbf{x},\mathbf{b}} \frac{1}{2} |(\partial_l - ib_j A_{lj})\varphi_b|^2 + \frac{m_b^2}{2}|\varphi_b|^2 + \frac{g_{bb'}}{4}|\varphi_b|^2|\varphi_{b'}|^2 + \dots \right.$$

$$\left. - \sum_{\mathbf{x},\Omega} \frac{1}{2} |(\partial_l - i\Omega_j h_{lj})\varphi_\Omega|^2 + \frac{m_\Omega^2}{2}|\varphi_\Omega|^2 + \frac{g_{\Omega\Omega'}}{4}|\varphi_{\Omega'}|^2|\varphi_{\Omega'}|^2 + \dots \right\} \quad (59)$$

Together with the elastic part of the partition function, this would represent the disorder field theory of defect lines in a crystal under stress. The coupling (36) makes things somewhat more complicated and the reader is referred to Reference 5 for a detailed discussion (see also Reference 11).

8. THE MELTING TRANSITION

As an application of the disorder field theory (59) involving the elastic gauge fields, consider the melting transition. Above a certain temperature, the dislocation lines proliferate, $|\varphi_b|$ becomes nonzero and screens the stress fields in a disorder version of the Meissner effect.[12] Stress can no longer penetrate into the disordered state. This weakens the forces between disclination lines from $-R$ to $1/R$. If the core energy of the disclination lines comes mostly from the elastic stress field around the line, an increase in $|\varphi_b|$ is sufficient to weaken the energy such as to also make the disclination lines proliferate. If we minimize the disclination energy, plot the minimum as a function of $|\varphi_b|$, and add this function to the dislocation potential, the result is a curve with a protrusion towards the $|\varphi_b|$ axis. This changes the second-order transition of the pure φ_b theory into a first-order transition.[11]

This backfeeding mechanism is the novel feature when going from the vortex-induced phase transition to the defect-induced one. The superfluid has only one type of defect line which can only proliferate in a second-order transition. The crystal, on the other hand, has two types of lines with different long-range interactions, one with $1/R$ and the other with R. The proliferation of one type screens the forces of the other and triggers also their proliferation.

For Monte Carlo studies of the system the original defect gauge theory (25) is more suitable. It can easily be formulated on a lattice by changing integrals to sums and letting the gauge fields $(\partial_j u_i^p)$ be multiples of the lattice spacing a.

Taking $l = \infty$, for simplicity, one arrives at the periodic Gaussian model[10,14]

$$Z = \prod_{\mathbf{x},i} \int_{-\infty}^{\infty} du_i(\mathbf{x}) \sum_{\{n_{ij}(\mathbf{x})\}} \Phi\,[n_{ij}]$$

$$\exp -\frac{\beta}{4}\left[\sum_{\mathbf{x},ij} \mu(\nabla_i u_j + \nabla_j u_i - 4\pi n_{ij})^2 + 2\lambda \sum_{\mathbf{x},i} (\nabla_i u_i - 2\pi n_{ii})^2\right]$$

By a Villain approximation, this can be substituted by another simpler model of the

XY type, in which the periodic Gaussian is replaced by an exponential of a cosine.[13] This is simpler to simulate. At $l = 0$, the core energies of the disclination lines come purely from linear elasticity and the transition is indeed of first order, in three[15] as well as in two dimensions.[16] The latter result is in contradiction with theoretical considerations by Halperin and Nelson.[17] The flaw in their argument lies in an unphysical choice of the core energies.[18]

CONCLUSION

The two gauge structures seem to be a universal feature of all systems with long-range modes and linelike defects. The methods presented here can be applied to a variety of physical systems such as liquid crystals,[19] magnetic superconductors,[20] pion condensates,[21] etc.

It is worth pointing out that, by introducing random quenched disorder into these theories, it is possible to gain access to glassy systems.[22]

SUMMARY

We have stressed the relevance of two kinds of gauge fields in many body systems. One is associated with long-range elastic forces, the other with vortex or defect lines caused by plastic deformations. The elastic gauge field is indispensible for constructing a disorder field theory of defects under stress (which is of the Ginzburg-Landau type). The plastic gauge field is useful for finding an order field theory of stresses disturbed by defects (which is of the XY model type).

As a particular technical result we use the plastic gauge invariance to calculate the forces between crystalline defect lines in the presence of higher gradients in the elastic energy.

REFERENCES AND NOTES

1. SHOCKLEY, W. 1952. *In* L'Etat Solid, Proc. Neuvieme-lousail de Physique, Brussels. R. Stoops, Ed. Institute International de Physique. Solvay, Brussels.
2. FEYNMAN, R. P. 1955. *In* Progress in Low Temperature Physics, C. J. Gortes, Ed. **1**. North-Holland. Amsterdam, the Netherlands; POPOV, V. N. 1973 Sov. Phys. J. Exp. Theor. Phys. **37**: 341.
3. BEREZINSKII, V. L. 1971. Sov. Phys. J. Exp. Theor. Phys. **32**: 493; KOSTERLITZ, J. M. & D. J. THOULESS. J. Phys. **C6**: 1181; 1978. Prog. Low Temp. Phys. **7B**: 371; KOSTERLITZ, J. M. 1974. J. Phys. **C7**: 1046.
4. KLEINERT, H. 1982. Phys. Lett. **93A**: 82.
5. For more details see KLEINERT, H. 1986. Gauge Theory of Stresses and Defects. Gordon and Breach. New York, N.Y.
6. The two gauge field structures were first pointed out in KLEINERT, H. 1983. Phys. Lett. **A97**: 51. See also KLEINERT, H. 1982. Lett. Nuovo Cimento **35**: 41.
7. HALPERIN, B. I., T. C. LUBENSKY. S. K. MA. 1974. Phys. Rev. Lett. **32**: 292; HALPERIN, B. I. & T. C. LUBENSKY. 1974. Solid State Commun. **14**: 997.
8. KLEINERT, H. 1982. Lett. Nuovo Cimento **35**: 405.
9. KRÖNER, E. 1981. Les Houches Lectures 1980. *In* The Physics of Defects. R. Balian *et al.*, Eds. North-Holland. Amsterdam, the Netherlands. See also MURA, T. 1972. Arch. Mech. **24**: 499.

10. KLEINERT, H. 1983. Phys. Lett. **96A:** 302.
11. KLEINERT, H. 1983. Phys. Lett. **95A:** 493.
12. KLEINERT, H. 1983. Lett. Nuovo Cimento **34:** 471.
13. KLEINERT, H. 1983. Lett. Nuovo Cimento **37:** 425.
14. KLEINERT, H. 1982. Phys. Lett. **91A:** 295.
15. JACOBS, L. & H. KLEINERT. 1984. J. Phys. **A17:** 361.
16. JANKE, W. & H. KLEINERT. 1984. Phys. Lett. **105A:** 134.
17. NELSON, D. R. & B. I. HALPERIN. 1979. Phys. Rev. **B19:** 2457; YOUNG, A. P. 1979. Phys. Rev. **B19:** 1855; NELSON, D. R. 1982. Phys. Rev. **B26:** 269.
18. KLEINERT, H. 1983. Phys. Lett. **95A:** 381.
19. KLEINERT, H. 1983. J. Phys. Paris **44:** 353.
20. KLEINERT, H. 1982. Phys. Lett. **90A:** 259.
21. KLEINERT, H. 1982. Lett. Nuovo Cimento **34:** 103, and lectures presented at the 1983 Conference on High Energy Nuclear Physics at Lake Balaton, published by Budapest University Press, J. Erö, Ed.
22. KLEINERT, H. 1984. Phys. Lett. **A101:** 224.
23. KLEINERT, H. 1985. Lett. Nuovo Cimento **43:** 261.