RENORMALIZATION OF CURVATURE ELASTIC CONSTANTS FOR ELASTIC AND FLUID MEMBRANES

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Received 9 September 1986; revised manuscript received 28 November 1986; accepted for publication 8 December 1986

We study the fluctuations of membranes with area and curvature elasticity and calculate the renormalization of the curvature elastic constants due to thermal fluctuations. For the mean curvature elastic constant the result is the same as obtained previously for "ideal membranes" which resist only to curvature deformations. The renormalization of the gaussian curvature, on the other hand, depends on the elastic constants. In an incompressible membrane, it is five times weaker than in an ideal membrane.

The behavior of a wide variety of physical systems is governed by fluctuations of tensionless two-dimensional interfaces, called membranes [1–10]. Examples are red blood cells, bilipid vesicles, and soap layers separating oil and water regions in microemulsions. It is therefore important to study such fluctuations. Up to now, investigations have been confined to what may be called "ideal" or "mathematical membranes". These are infinitely thin surfaces resisting *only* to curvature deformations, with an energy [1,2]

$$E_{\text{curv}}=\tfrac{1}{2}\kappa_0\int d^2\xi\,\sqrt{g}\,(c_1+c_2-c_0)^2+\tfrac{1}{2}\bar{\kappa}_0\int d^2\xi\,\sqrt{g}\,c_1c_2\,, \tag{1}$$

where c_1, c_2 are the two principal curvatures, c_0 is the spontaneous curvature, and $g_{ij}(\xi)=\partial_iX^a(\xi)\partial_jX^a(\xi)$ is the metric of the surface in the parametrization $X^a(\xi^i)$ ($a=1, 2, 3$; $i=1, 2$). Since the argument will not depend on c_0, we shall set it equal to zero.

The energy (1) is reparametrization invariant. As a consequence, the path integral which governs the fluctuations

$$Z=\prod_{\xi}\int dX^a\,g^{3/4}(\xi)\,\exp(-T^{-1}E_{\text{curv}}) \tag{2}$$

requires two gauge fixing δ-functionals. In ref. [9] we have investigated the softening of the curvature elastic constants in two gauges, the *Gauss map* and the *normal gauge*. In both gauges we found that κ, $\bar{\kappa}$ change with temperature as

$$\kappa=\kappa_0-T\times\tfrac{1}{2}\alpha L\,,\quad \bar{\kappa}=\bar{\kappa}_0-T\times\tfrac{1}{2}\bar{\alpha}L\,, \tag{3}$$

with $\alpha=3$, $\bar{\alpha}=-\frac{10}{3}$, $L=(1/2\pi)\ln(k_{\max}/k_{\min})$ (the first value agreeing with the original calculation in ref. [4]).

The question arises as to whether the result (3) depends on the idealization of the membrane to be an object controlled only by (1). Physical membranes are made of elastic material or they behave like an almost incompressible fluid. Thus, in equilibrium, they carry two more material parameters, the moduli of compression and shear, K and μ, respectively. In terms of these, the elastic energy may be parametrized as follows [13],

$$E_{\text{elastic}}=\int d^2\xi\,\sqrt{g_0}\,(\tfrac{1}{2}K\alpha_v^2+\mu\alpha_s^2)\,, \tag{4}$$

where α_v, α_s are the volume and shear distortions, respectively. If λ_1, λ_2 are the principal extension ratios, then

$$\alpha_v=\lambda_1\lambda_2-1\,,\quad \alpha_s=\frac{\lambda_1-\lambda_2}{\sqrt{2\lambda_1\lambda_2}}\,.$$

As far as fluctuations are concerned, the following parametrization of the energies (1) and (3) is most convenient. Let $X_0^a(\xi)$ be the equilibrium configuration and $X^a(\xi)=X_0^a(\xi)+\delta X^a(\xi)$ an arbitrary neigh-

[1] Supported in part by Deutsche Forschungsgemeinschaft under grant no. Kl 256.

boring configuration. We can then use the metric $g_{ij}(\xi)$ to define the connection $\Gamma_{ijk}=\frac{1}{2}(\partial_i g_{jk}+\partial_j g_{ik}-\partial_k g_{ij})$ and covariant derivatives (for a vector v_j it is $D_i v_j=\partial_i v_j-\Gamma_{ijk}v^k$). Decomposing $\delta X^a(\xi)$ into the tangential and normal components with respect to the equilibrium configuration

$$\delta X^a(\xi)=\tau^i\partial_i X_0^a(\xi)+\nu N_0^a(\xi)$$
$$(\boldsymbol{N}_0=(1/\sqrt{g_0})\partial_1\boldsymbol{X}_0\times\partial_2\boldsymbol{X}_0)\ ,$$

we can expand (indices are raised and lowered with g_0^{ij}, g_{0ij})

$$g_{ij}=g_{0ij}+\Delta g_{ij}=g_{0ij}+\tau_{ij}+\tau_{ji}+\nu_i\nu_j+\tau_i{}^l\tau_{jl}\ , \tag{5}$$

where

$$\nu_i\equiv D_i^0\nu+C_0{}^i{}_j\tau^j\ ,$$

$$\tau_i{}^j\equiv D_i^0\tau^j-C_{0i}{}^j\nu\ , \tag{6}$$

with $C_{0i}{}^j$ being the extrinsic curvature matrix

$$C_{0i}{}^j=N_0^a D_i^0 D_0^j X_0^a\ ,$$

whose trace $C_0\equiv C_{0i}{}^j$ and determinant

$$K_0\equiv\tfrac{1}{2}[(C_{0i}{}^i)^2-C_{0i}{}^jC_{0j}{}^i]$$

are equal to c_1+c_2 and c_1c_2, respectively.

Hence, with (5), we calculate

$$\alpha_v=-1+\sqrt{g/g_0}$$
$$=\tfrac{1}{2}\Delta g_i{}^i-\tfrac{1}{4}\Delta g_i{}^j\Delta g_j{}^i+\tfrac{1}{8}(\Delta g_i{}^i)^2+\ldots$$
$$=\tau_i{}^i+\tfrac{1}{2}\nu_i\nu^i+\tfrac{1}{2}[(\tau_i{}^i)^2-\tau_i{}^j\tau_j{}^i]+\ldots,$$

$$\alpha_s^2=\tfrac{1}{4}(\Delta g_i{}^j-\tfrac{1}{2}\delta_i{}^j\Delta g_k{}^k)^2+\ldots$$
$$=\tfrac{1}{4}(\tau_i{}^j+\tau^j{}_i)^2-\tfrac{1}{2}\alpha_v^2+\ldots\ . \tag{7}$$

The elastic energy becomes, therefore, to lowest order

$$E_{\rm el}=\int d^2\xi\,\sqrt{g_0}\,[\tfrac{1}{2}(K-\mu)(\tau_i{}^i)^2+\tfrac{1}{4}\mu(\tau_i{}^j+\tau^j{}_i)^2]\ . \tag{8}$$

For the mean curvature energy

$$\tfrac{1}{2}\kappa_0\int d^2\xi\,\sqrt{g}\,C^2=\tfrac{1}{2}\kappa_0\int d^2\xi\,\sqrt{g}\,(N^aD^2X^a)^2 \tag{9}$$

we find the expansion

$$\tfrac{1}{2}\kappa_0\int d^2\xi\,\sqrt{g_0}\,[C_0^2+2C_0(D_0^2+\tfrac{1}{2}C_0^2-2K_0)\times(\nu-\tau^iD_i^0\nu-\tfrac{1}{2}C_{0ij}\tau^i\tau^j)+\nu(D_0^2)^2\nu$$
$$+2C_0C_0^{ij}(D_i^0\nu D_j^0\nu+2\nu D_i^0D_j^0\nu)-\tfrac{1}{2}C_0^2(D_0\nu)^2$$
$$-4K_0\nu D_0^2\nu+(C_0^2-K_0)(C_0^2-4K_0)\nu^2$$
$$+2C_0(D_0^kC_0)\nu D_k^0\nu+\ldots]\ . \tag{10}$$

This expression is obtained by expanding eq. (9) first in powers of ν only (at $\tau^i=0$) as done in ref. [7], then observing that the invariance of free energy under reparametrizations $\delta X^a(\xi)=-\lambda^i(\xi)\partial_iX^a(\xi)$ implies invariance under changing ν and τ^i by $\delta\nu=-\lambda^i\nu_i$, $\delta\tau^i=-\lambda^i-\lambda^j\tau_j{}^i$. As a consequence, ν, τ^i can be replaced by $\tilde\nu=\nu-\tau^iD_i^0\nu-\tfrac{1}{2}C_{0ij}\tau^i\tau^j+\ldots$, $\tilde\tau^i=0$ and this is what leads to (10).

We are now ready to calculate the fluctuation effects. The measure of integration was shown in ref. [9] to be

$$\int\mathscr{D}^3X\,g^{3/4}(\xi)=\prod_\xi\int d\nu(\xi)\,g^{-1/4}(\xi)g_0^{+1/2}(\xi)\times\prod_\xi\int d^2\tau^i\,g(\xi)\ .$$

In the elastic energy, the integration over τ^i gives, after some algebra, a fluctuation energy

$$E_{\rm el}^{\rm fl}=\tfrac{1}{2}T\,{\rm Tr}\ln[-(K+\mu)(D_0^2)\ -\mu R_0]+\tfrac{1}{2}T\,{\rm Tr}\ln[-\mu(D_0^2)\ -\mu R_0]\ , \tag{11}$$

where R_0 is the gaussian curvature of the background configuration. After subtracting the energy of a flat reference membrane of the same area, we obtain the logarithmically divergent contribution

$$\Delta E_{\rm el}^{\rm fl}=\tfrac{1}{2}TL\int d^2\xi\,\sqrt{g_0}\times\left[-2\frac{R_0}{6}-\left(\frac{\mu}{K+\mu}+1\right)R_0\right]. \tag{12}$$

Apart from that, there remains a ν dependent energy from the elastic energy, which is reached at some τ^i of the order of $D^{-1}C\nu$, and is of the order $C^2\nu^2$.

Consider now the fluctuation of the curvature energy (10). As far as the quadratic terms in ν are concerned, we extract the propagator $\langle\nu\nu\rangle=(T/\kappa)(D_0^2)^{-2}$ and obtain the same renormalization as for the ideal membrane (see ref. [7]). Since we are considering only smooth curvatures ($DC\approx 0$), the D_0^2 piece of the term in (10) is a pure boundary term, just as in the ideal case.

The only new terms involving $\tau\nu$ and $\tau\tau$ are accompanied by at least three powers of C_0. After expressing them in terms of ν at the minimum of the elastic energy, they are of the order $C_0^4\nu^2$.

Thus the renormalization of κ is the same as given for the mathematical membrane in refs. [7] and [9]. In contrast, due to (12), gaussian curvature renormalizes differently depending on the elastic constants, namely with

$$\bar{\alpha}=-4+\frac{4}{6}+\left[\frac{4}{6}+2\left(\frac{\mu}{K+\mu}+1\right)\right]$$
$$=-\frac{2}{3}+2\frac{\mu}{K+\mu}\approx-\frac{2}{3},$$

rather than $-4+\frac{4}{6}=-\frac{10}{3}$ found previously for the mathematical membrane [7,9]. As a consequence, spherical incompressible bilayer vesicles have the distribution [7] $N^{a}\mathrm{e}^{-\mathrm{const}\times N}$ with $a=7\rho^2-6\rho+\frac{8}{3}$ where ρ is the ratio of liquid crystal bending to elastic bending energy, (the first being caused by the rod-like shape of the molecules, the second by the difference in the extensions of the two monolayers). Experimentally, a lies half-way between 1 and 2 such that ρ appears to be around $\frac{1}{2}$.

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