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Spontaneous Generation of String Tension and Quark Potential

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The quark potential of a string with extrinsic curvature is calculated in the limit $d \to \infty$, via a saddle-point approximation. The saddle point has an anisotropic gap matrix λ^{ij} ($\neq \lambda g^{ij}$ in contrast to an unjustified assumption in all previous discussions). The anisotropy enters the nonleading $-(d-2)\pi c/24R$ part of the potential. c is calculated, the entire potential is plotted, and simple analytic expressions are found which approximate all quantities involved very closely.

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Recently, Polyakov and the author[1] have independently pointed out that various aspects of string fluctuations call for an additional term in the action. It is proportional to the square of the extrinsic curvature

$$A_K = \frac{1}{2\alpha} \int d^2\xi \sqrt{g} K^2, \tag{1}$$

where g_{ij} is the intrinsic metric, $\partial_i x^\mu \partial_j x^\mu$, associated with the surface $x^\mu(\xi^i)$ $(i=0,1)$, and $K^2 \equiv (D^2 x)^2$ (D_i=covariant derivative). Even though the ensuing higher-derivative theory seems, at first sight, to be beset by even more ghosts than the original Nambu-Goto string, the Euclidean formulation has a very attractive feature: It shares with quantum chromodynamics not only the infrared confinement but also the ultraviolet freedom. The theory has therefore become a new focus of interest.[2]

A particularly important quantity of any string is the static quark potential $v(R)$ since this can eventually be compared with experimental meson spectra.[3] First attempts to calculate it were undertaken by various authors,[4,5] who used the techniques by which Alvarez[6] found correctly the potential of the ordinary Nambu-Goto string. The calculation involves a gap matrix in which the energy is to be extremized. In Alvarez's case it was found to be proportional to g^{ij}. We shall call this situation *isotropic*. The isotropy of Alvarez's λ^{ij} led all previous authors[4,5] to assume an isotropic λ^{ij} also for a string with curvature. This assumption, however, is not justified. We shall see that the anisotropy is quite large and influences the potential drastically. In particular, the nonleading $1/R$ part of the potential can change sign, depending on the amount of extrinsic curvature in the total action.

Consider first a surface which has no Nambu-Goto tension at all and is governed completely by our action (1). Since K^2 involves four derivatives of $x^\mu(\xi)$, its fluctuations are so violent that the theory generates itself a mass spontaneously, to be called $\bar{\lambda}^{1/2}$. As a function of an arbitrarily chosen mass scale μ this has, at the one-loop level, the renormalization-group-invariant form (d=dimensionality of space)[1] $\bar{\lambda} \propto \mu^2 \exp(-\text{const}/\alpha)$. It plays the role of the dimensionally transmuted coupling constant associated with the dimensionless couplings α. The mass $\bar{\lambda}^{1/2}$ gives rise to a spontaneous string tension and causes permanent quark confinement.

It has often been stressed that the spontaneous generation of a length scale in four-dimensional non-Abelian gauge theories can be viewed as an analog of a similar process in the two-dimensional nonlinear σ model.[7] Also there, the theory starts out massless. Its action is [$n^\mu = (n^1, \ldots, n^N)$]

$$A = \frac{1}{2g} \int d^2\xi [(\partial_i n)^2 + \lambda(n^2 - 1)]. \tag{2}$$

For $N \geq 3$, violent fluctuations prevent the massless modes from surviving. They acquire spontaneously a mass which reads, in the renormalization-group-invariant form (at the one-loop level),

$$\bar{\lambda}=\mu^2 e^{-4\pi/g(\mu^2)}. \qquad (3)$$

It now appears that the analogy is much closer than initially thought. Indeed, let us write the action (1) in the Gauss gap where $x^\mu(\xi)=(\xi^0,\xi^1,x^2,x^3,\ldots,x^{d-1})$,

$$A_K=\frac{1}{2\alpha}\int d^2\xi\sqrt{g}\,[(D_i t_i)^2+\lambda^{ij}(t_i t_j-g_{ij}+\delta_{ij})], \qquad (4)$$

where $t_i^a=\partial_i x^a$ $(a=2,\ldots,d-1)$ are components of the tangent vectors of the surface. Apart from the presence of a fluctuating g_{ij} and the i indices of t_i, this action is practically the same as (3). It is therefore not surprising to find the same type of spontaneous mass generation. As in the σ model, the quantity λ^{ij} acquires a renormalization-group-invariant nonzero value

$$\lambda^{ij}=\bar{\lambda}\delta^{ij}=\mu^2\exp\{-[2/(d-2)][4\pi/\alpha(\mu^2)]+1\}. \qquad (5)$$

This is the signal for the spontaneous generation of a string tension.

Consider the action (4) and integrate out the x^a fluctuations. Assuming a flat intrinsic geometry we make the Ansatz $g_{ij}=\rho_i\delta_{ij}$ and find

$$A_{\rm fl}=\frac{d-2}{2}\int d^2\xi(\rho_0\rho_1)^{1/2}\left[\int\frac{d^2q}{(2\pi)^2}\ln(q^4+\rho^i\lambda^{ij}q_iq_j)-\frac{1}{2\tilde{\alpha}}\lambda^{ij}(g_{ij}-\delta_{ij})\right] \qquad (6)$$

[with $\tilde{\alpha}\equiv 2\alpha(d-2)$] to be extremized in $\lambda^{ij},\rho_0,\rho_1$, where we have set $q_0\equiv k_0/\sqrt{\rho_0}$, $q_1\equiv k_1/\sqrt{\rho_1}$. These are intrinsic momenta. For a finite extrinsic (=physical) distance $R_{\rm ext}$ between the quarks, q_1 takes the values $n\pi/R$ $(n=1,\ldots,\infty)$ where $R=\sqrt{\rho_1}R_{\rm ext}$ is the intrinsic distance between the ends. In the following we shall assume λ^{ij} to have the general form $\lambda^{ij}=\lambda^i g^{ij}=\lambda^i/\rho_i\delta^{ij}$. The infinite system is isotropic, $\lambda^0=\lambda^1=\tilde{\lambda}$ for $R\to\infty$, and the quantity in brackets in (6) becomes

$$\int\frac{d^2q}{(2\pi)^2}\ln(q^4+\tilde{\lambda}q^2)-\frac{\tilde{\lambda}}{2\tilde{\alpha}}+\frac{1}{2\tilde{\alpha}}\left[\frac{\lambda_0}{\rho_0}+\frac{\lambda_1}{\rho_1}\right].$$

The first two terms can be regularized by working in $2+\epsilon$ intrinsic dimensions. Absorbing the $1/\epsilon$ infinity as usual in $1/\tilde{\alpha}$, we can introduce the dimensionally transmuted coupling constant $\bar{\lambda}$ and write the bracket as

$$f^{R=\infty}=f_0(\tilde{\lambda})-\tilde{\lambda}/4\pi+(1/2\tilde{\alpha})(\lambda_0/\rho_0+\lambda_1/\rho_1), \qquad (7)$$

where $f_0(\tilde{\lambda})\equiv-(\tilde{\lambda}/4\pi)[\ln(\tilde{\lambda}/\bar{\lambda})-1]$. The multiplicative renormalization of $\tilde{\alpha}$ is compensated by a corresponding one of ρ_0,ρ_1 and λ_0,λ_1 so that all expressions are finite. Extremization of the action

$$A_{\rm fl}=\tfrac{1}{2}(d-2)R_{\rm ext}\beta_{\rm ext}(\rho_0\rho_1)^{1/2}f^{R=\infty}$$

in $\rho_0,\rho_1,\tilde{\lambda}$ gives $f^{R=\infty}=f_0$ and $\tilde{\lambda}=\bar{\lambda}$. The $R=\infty$ values of $\rho_0=\rho_1\equiv\bar{\rho}$ satisfy $1/\tilde{\alpha}\bar{\rho}=1/4\pi$. A further finite renormalization of $\tilde{\lambda},\bar{\lambda}$ by $\bar{\rho}$ makes $\rho_0\to\rho_1\to 1$ for $R\to\infty$. With $\tilde{\lambda}=\bar{\lambda}\neq 0$, the surface has acquired spontaneously a string tension

$$M_{\rm sp}^2=\tfrac{1}{2}(d-2)\bar{\lambda}/4\pi.$$

An extra Nambu-Goto action $M_{\rm NG}^2\int d^2\xi\sqrt{g}$ adds to $f^{R=\infty}$ a constant $\tilde{M}_{\rm NG}^2=[2/(d-2)]M_{\rm NG}^2$. Then the extremization of $A_{\rm fl}$ in $\rho_0,\rho_1,\tilde{\lambda}$ gives $f^{R=\infty}=2\tilde{M}_{\rm NG}^2+f_0$ with a gap equation

$$\tilde{M}_{\rm NG}^2/\tilde{\lambda}-\ln(\tilde{\lambda}/\bar{\lambda})/4\pi=0$$

which is solved by $\tilde{\lambda}=\bar{\lambda}_v\equiv\bar{\lambda}e^v$. Here we have introduced a parameter v, to be called "normality," so that $\tilde{M}_{\rm NG}^2\equiv(\bar{\lambda}/4\pi)e^v v$. It measures the relative amount of $\tilde{M}_{\rm NG}^2$ with respect to the spontaneous $\tilde{M}_{\rm sp}^2=\bar{\lambda}/4\pi$. The total string tension is

$$M_{\rm tot}^2=\frac{d-2}{2}f^{R=\infty}\bigg|_{\lambda=\bar{\lambda}_v}=\frac{d-2}{2}\frac{\bar{\lambda}_v}{4\pi}(1+v).$$

Thus physics requires $v>-1$.

In a finite system, we set $\tilde{\lambda}\equiv(\lambda_0+\lambda_1)/2$, $\delta\equiv(\lambda_1-\lambda_0)/2\tilde{\lambda}$ and find that $f^{R=\infty}$ is to be supplemented by

$$\Delta f^R=\frac{\tilde{\lambda}}{4\pi}\left\{\ln\left[\frac{\tilde{\lambda}R^2}{\bar{\lambda}\bar{R}^2}\right]-\frac{2}{3\tilde{\lambda}_R}+\frac{4}{\tilde{\lambda}_R}\sum_{n=1}^{\infty}\left[(n^2+\tilde{\lambda}_R)^{1/2}-n-\frac{\tilde{\lambda}_R}{2n}\right]\right\}$$
$$=\frac{\tilde{\lambda}}{4\pi}\left\{-\frac{1}{3\tilde{\lambda}_R}-\frac{2}{(\tilde{\lambda}_R)^{1/2}}-\frac{4}{\pi}\sum_{n=1}^{\infty}K_1(2\pi\tilde{\lambda}_R{}^{1/2}\tilde{n})/\tilde{n}\right\} \qquad (8)$$

[with $\bar{R}\equiv 2\pi e^{-\gamma}/\bar{\lambda}^{1/2}$, $\gamma\approx 0.577215$ = Euler's number, $\tilde{\lambda}_R\equiv\tilde{\lambda}R^2/\pi^2$, $K_1(z)\equiv$ modified Bessel function], due to the finite size of the system, plus a term

$$\Delta f^{\rm an}=\frac{\tilde{\lambda}}{4\pi}\frac{4}{\tilde{\lambda}_R}\sum_{n=1}^{\infty}[A_n^+ +A_n^- -(n^2+\tilde{\lambda}_R)^{1/2}-n]+\frac{\Delta\lambda}{8\pi}, \qquad (9)$$

with

$$A_n^{\pm}=\{n^2+\tfrac{1}{2}\tilde{\lambda}_R(1-\delta)[1\pm(1-4\delta n^2/\tilde{\lambda}_R(1-\delta)^2)^{1/2}]\}^{1/2}$$

due to the anisotropy of the gap. The total action is $[(d-2)/2]R_{\rm ext}\beta_{\rm ext}$ times

$$\tilde{\alpha}\equiv(\rho_0\rho_1)^{1/2}[f^{R=\infty}+\Delta f^R+\Delta f^{\rm an}+(1/2\tilde{\alpha})(\lambda_0/\rho_0-\lambda_1/\rho_1)]. \tag{10}$$

Extremization in $\rho_0,\rho_1,\tilde{\lambda},\delta$ gives the gap equations

$$\left(\frac{\bar{\lambda}_v}{\tilde{\lambda}}-1\right)v-\ln\left(\frac{\tilde{\lambda}}{\bar{\lambda}_v}\right)-\frac{1}{\tilde{\lambda}_R^{1/2}}+4\sum_{\tilde{n}=1}^{\infty}K_0(2\pi\tilde{\lambda}_R^{1/2}\tilde{n})+\left.\frac{\partial\Delta f^{\rm an}}{\partial\tilde{\lambda}}\right|_{R,\delta}=0, \tag{11}$$

$$-\frac{1}{4\pi}\frac{1}{1-\delta^2}\left\{\delta\left(1+v\frac{\bar{\lambda}_v}{\tilde{\lambda}}\right)+\left[\frac{4\pi}{\tilde{\lambda}}f^R_{\rm tot}-v\frac{\bar{\lambda}_v}{\tilde{\lambda}}-1\right]\right\}+\left.\frac{\partial\Delta f^{\rm an}}{\partial\delta}\right|_{R,\tilde{\lambda}}=0, \tag{12}$$

and the equations for ρ_0,ρ_1

$$\rho_0^{-1}=[(1+v)(1+\delta)]^{-1}(4\pi/\tilde{\lambda})f^R_{\rm tot},\quad \rho_1^{-1}=[(1+v)(1+\delta)]^{-1}[2+2v\bar{\lambda}_v/\tilde{\lambda}-(4\pi/\tilde{\lambda})f^R_{\rm tot}], \tag{13}$$

where

$$f^R_{\rm tot}\equiv[2\tilde{M}^2_{\rm NG}+f_0(\tilde{\lambda})+\Delta f^R+\Delta f^{\rm an}]_{\rm extremum}=\frac{\tilde{\lambda}}{4\pi}\left[v\frac{\bar{\lambda}_v}{\tilde{\lambda}}+1-\frac{1}{\tilde{\lambda}_R^{1/2}}-\frac{1}{3\tilde{\lambda}_R}-\frac{4}{\pi}\sum_{\tilde{n}=1}^{\infty}\frac{K_1(2\pi\tilde{\lambda}_R^{1/2}\tilde{n})}{\tilde{n}}\right], \tag{14}$$

so that $\tilde{\alpha}=(\rho_0\rho_1)^{1/2}f^R_{\rm tot}$. When performing $\partial/\partial\rho_1$ we have to remember that ρ_1 occurs in $\tilde{\lambda}_R=\tilde{\lambda}R^2/\pi^2=\tilde{\lambda}R^2_{\rm ext}\rho_1/\pi^2$.

The solution of these equations is quite simple. We take $\delta=0$ and find for various $\tilde{\lambda}_R$ the corresponding value for $\tilde{\lambda}$ from Eq. (11). For this we calculate a first $\delta\neq0$ from (12). This is reinserted into (11) to obtain an improved value of $\tilde{\lambda}$. After a few iterations, the procedure converges. The quark potential is $v=[(d-2)/2]R_{\rm ext}\tilde{\alpha}$ with $R_{\rm ext}=R\rho_1^{-1/2}=\pi(\tilde{\lambda}/\tilde{\lambda}_R\rho_1)^{-1/2}$. In Fig. 1 we have plotted the reduced potential

$$\tilde{v}=(R_{\rm ext}/\bar{R}_{{\rm ext},v})(\rho_0\rho_1)^{1/2}f^R_{\rm tot}4\pi/\bar{\lambda}_v(1+v)$$

as a function of $R_{\rm ext}/\bar{R}_{{\rm ext},v}$ where

$$\bar{R}_{{\rm ext},v}\equiv2\pi e^{-\gamma}[\bar{\lambda}_v(1+v)]^{-1/2}$$
$$=e^{-\gamma}[(d-2)\pi/2M^2_{\rm tot}]^{1/2}$$

is the natural length scale of the system, for $v=0$ (corresponding to a purely spontaneous string) ($\tilde{v}$ is normalized to behave asymptotically as $R_{\rm ext}/\bar{R}_{{\rm ext},v}$+const). We have also displayed the gaps $\tilde{\lambda},\lambda_0,\lambda_1$ as functions of $\bar{R}/R$. For comparison we show as dashed lines an approximate solution of Eqs. (11)-(14) which arises when we drop the Bessel functions (they are exponentially small for large R), and when we approximate $\Delta f^{\rm an}$ by the leading terms $(\tilde{\lambda}/4\pi)(\delta/\tilde{\lambda}_R^{1/2}-\delta^2/4)$. Then the equations can easily be solved analytically. The approximation is seen to be very good. The dash-dotted lines are the corresponding curves which would result under the unjustified assumption $\delta=0$. Actually, in this case, the approximation of dropping the Bessel functions is so good that the curves cannot be distinguished on the plot. They can easily be calculated in closed form. For various $\tilde{\lambda}/\bar{\lambda}$ we have

$$\tilde{\lambda}_R=[(\bar{\lambda}_v/\tilde{\lambda}-1)v-\ln(\tilde{\lambda}/\bar{\lambda}_v)]^{-2}$$

(where $\bar{\lambda}_v/\bar{\lambda}=e^v$), and

$$f^{R,\rm app}_{\rm tot}=(\tilde{\lambda}/4\pi)[v\bar{\lambda}_v/\tilde{\lambda}+1-\tilde{\lambda}_R^{-1/2}-1/3\tilde{\lambda}_R],$$

$$\rho_{0,1}^{-1}=(1+v)^{-1}[v\bar{\lambda}_v/\tilde{\lambda}+1\mp\tilde{\lambda}_R^{-1/2}\mp1/3\tilde{\lambda}_R].$$

The quark potential is given by the simple expression

$$\tilde{v}^{\rm app}=(R_{\rm ext}/\bar{R}_{{\rm ext},v})(\tilde{\lambda}/\bar{\lambda}_v)[v\bar{\lambda}_v/\tilde{\lambda}+1-\tilde{\lambda}_R^{-1/2}-1/3\tilde{\lambda}_R]^{1/2}[v\bar{\lambda}_v/\tilde{\lambda}+1+\tilde{\lambda}_R^{-1/2}+1/3\tilde{\lambda}_R]^{-1/2}, \tag{15}$$

where

$$R_{\rm ext}/\bar{R}_{{\rm ext},v}=(R/\bar{R})[e^v(1+v)/\rho_1]^{1/2}$$

$$=[\tilde{\lambda}_R\bar{\lambda}_v(1+v)/\tilde{\lambda}]^{1/2}e^{\gamma}/2.$$

For $v\to\infty$, this expression goes smoothly over into the Alvarez potential[6] of the Nambu-Goto string (shown in Fig. 1).

It is easy to calculate the nonleading asymptotic term

$$\Delta v=-[(d-2)/2](\pi/12R_{\rm ext})c, \tag{16}$$

$$c=1-2v/3(1+v)^2-12/(3+2v). \tag{17}$$

Thus, while the pure Nambu-Goto string ($v=\infty$) has $c=1$, this parameter decreases for less and less normality and changes sign at $v\approx5.1031$. At this place it is equal[8] to the c value of the so-called Ramond string.

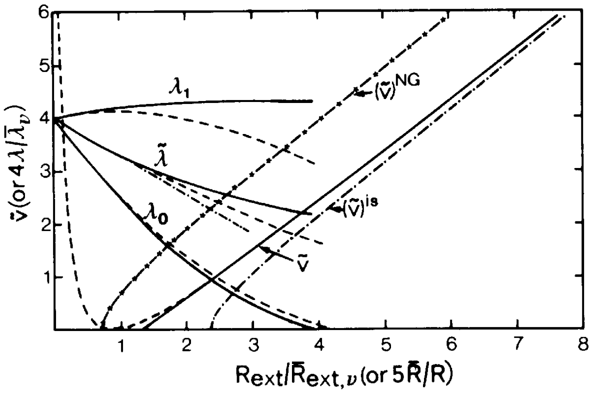

FIG. 1. The reduced quark string potential $\tilde{v}$ for a purely spontaneous string ($\nu=0$), as a function of $R_{ext}/\bar{R}_{ext,\nu}$ with the scale explained in the text. The dashed curve is a simple but quite good approximation. The curves starting from the top left are the gaps $\tilde{\lambda},\lambda_0,\lambda_1$, as functions of $5\bar{R}/R$. The dash-dotted curves are the corresponding solutions assuming isotropy in which case we have found a closed-form approximation which lies right on top of the exact one, on the scale of the figure. For $\nu \to \infty$, the potential approaches that of the pure Nambu-Goto string which we have also plotted (dash-starred line).

For a purely spontaneous string, $\nu=0$, c is equal to -3.

Let us conclude by mentioning that anisotropic gaps have been known to be of practical importance in superfluids with a tensorial order parameter.[9]

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