SPONTANEOUS QUANTUM GRAVITY: A SOLUBLE MODEL

H. KLEINERT [1]

Institut für Theorie der Elementarteilchen, Freie Universität Berlin, Arnimallee 14, 1000 Berlin 33, Germany

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We present a model of gravity in which the Planck mass is generated spontaneously by a similar mechanism that is responsible for the tension in a recent membrane model of strings. The action describes very floppy fluctuations of the physical space in some large flat embedding space. The coupling constant is dimensionless but the fluctuations are so violent that they produce spontaneously a mass which plays the role of the dimensionally transmuted coupling constant and can be identified with the Planck mass.

In 1967, Sakharov suggested that Einstein gravity should eventually be understood as a "metric elasticity of space" caused by the fluctuations of all elementary particle fields [1] [#1]. With the recent idea that these are excitations of a fundamental string [3], Sakharov's suggestion had led to the derivation of gravity from string fluctuations. In this approach, the Planck mass M_P appears as an input parameter which characterizes the mass splitting between the elementary particles in the string model.

Even though this approach appears to be mathematically consistent, it is frustrating from the point of view that the fundamental mass M_P should depend on an infinity of (probably forever) unobservable excited states of elementary particles (in this respect it is not much better than a brute-force cutoff beyond which a theory can never be tested at all). In addition, there is something unsatisfactory in the string model of elementary particles itself. Strings are supposed to be just another way of describing the forces between some unknown subelementary constituents which, in turn, should be due to some non-abelian gauge fields. These, however, obey a dimensionless action and the string tension arises from violent gauge field fluctuations as the dimensionally transmuted coupling constant of the system. From this point of view, it would be more attractive to find another formulation of gravity in which the Planck mass arises spontaneously. Indeed, this idea has been pursued in a number of works [4].

It is the purpose of this note to propose a new simple model for such a mechanism. In contrast to earlier attempts, the fluctuations are not those of the elementary particles but of *curved space itself*, assumed to take place in some unknown space of larger dimension d with $d \gg 4$. The model is based on a generalization of an action which has recently been used to generate the string tension spontaneously [5,6],

$$\mathcal{A}=\frac{1}{2\alpha}\int \mathrm{d}^2\xi\,\sqrt{g}\,\mathrm{D}^2x\,\mathrm{D}^2x\,, \tag{1}$$

where $x^a(\xi^i)$ ($a=1, ..., d$; $i=1, 2$) describes the position of the string and $g_{ij}=\partial_i x \partial_j x$ the metric. This action has the attractive feature that α is dimensionless and asymptotically free. This is why there is a spontaneous generation of string tension as the dimensionally transmuted coupling constant of the system. Thus it has a great structural similarity with non-abelian gauge theories underlying the string. Such an action controls the physics of membranes [7] and plays an important role for cell walls, lipid vesicles, and microemulsions. Its possible relevance to string physics was recognized only last year [5] and is beginning to attract an increasing amount of research.

Since in physics the same type of mechanism

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[#1] For a general discussion see ref. [2].

appears often at different places and since the string action, with tension M, may be viewed as the two-dimensional analogue of gravity, with Planck mass M_P, it is suggestive to assume a similar mechanism to generate the Planck mass in gravity. Our starting point is the simplest four-dimensional action which describes extrinsic fluctuations of space and has a dimensionless coupling constant [#2]

$$\mathscr{A}=\frac{1}{2\alpha}\int d^4\xi\,\sqrt{g}\,D^\mu D^2 x D_\mu D^2 x\,. \qquad (2)$$

Here $x^a(\xi^\mu)$ $(a=1, ..., d;\ \mu=1, ..., 4)$ gives the position of the physical space in some unknown embedding space with metric $g_{\mu,\nu}=\partial_\mu x\partial_\nu x$, and D_μ is the covariant derivative. This type of action has previously been studied in the context of "floppy membranes" [8] by which we define surfaces with neither tension nor curvature stiffness) and it was noted that α is asymptotically free in the ultraviolet. Thus we expect the fluctuations to generate also here a mass spontaneously. In order to show this we use the same technique as for the non-linear $O(N)$ model [9]. We rewrite (1) as an action of *independent* fields x and g, and enforce the relation $g_{\mu\nu}=\partial_\mu x\partial_\mu x$ by a lagrangian multiplier $\lambda^{\mu\nu}$:

$$\mathscr{A}=\frac{1}{2\alpha}\int d^2\xi\,\sqrt{g}[D^\mu D^2 x D_\mu D^2 x$$
$$-\lambda^{\mu\nu}(\partial_\mu x\partial_\nu x-g_{\mu\nu})]\,. \qquad (3)$$

In the sequel we shall assume d to be very large. But just as in the σ-model, where the large-N spontaneous mass appears as early as $N=3$, our result is hoped to be valid for any $d>4$. Anyhow, nothing is known about d and it could well be infinite. Integrating out the x fluctuations we arrive at the purely intrinsic action

$$\mathscr{A}=\frac{d}{2}\left(\mathrm{tr}\ln(D^6-D_\mu\lambda^{\mu\nu}D_\nu)-\int d^4\xi\,\sqrt{g}\,\frac{\lambda^{\mu\nu}g_{\mu\nu}}{\tilde\alpha}\right), \qquad (4)$$

where $\tilde\alpha\equiv d\alpha/4$. For $d\to\infty$, this action has to be maximized in $\lambda^{\mu\nu}$. The result will be some functional

$$\lambda^{\mu\nu}=\lambda^{\mu\nu}[g_{\lambda\kappa}]\,. \qquad (5)$$

The final action involves only the metric and is supposed to describe gravity.

When calculating the trace log of D^2 there is a quartic divergence of the type $\propto\int d^4\xi\,\sqrt{g}$ (cosmological term), a quadratic one $\propto\int d^4\xi\,\sqrt{g}R$ (Einstein term), and a logarithmic one $\propto\int d^4\xi\sqrt{g}R^2$ (Cartan term). These have to be removed by corresponding counterterms. For a space of constant curvature R, we can assume the maximum to have the form $\lambda^{\mu\nu}=\lambda g^{\mu\nu}$ with a constant λ and find the gap equation

$$\frac{1}{\tilde\alpha}=\frac{1}{D^4+\lambda}(0,0)=\frac{1}{-2i\sqrt{\lambda}}(G_{i\sqrt{\lambda}}-G_{-i\sqrt{\lambda}})\,, \qquad (6)$$

where G_{m^2} is the Green's function $(-D^2+m^2)^{-1}$ $(x=0,\ y=0)$ which is known [10] to be equal to

$$G_{m^2}=\frac{m^2-2K}{8\pi^2(d-4)}-\frac{3K}{16\pi^2}$$
$$+\frac{m^2-2K}{16\pi^2}\left(\ln\frac{K}{4\pi}+\gamma-1+\psi(\tfrac{3}{2}+i\rho)+\psi(\tfrac{3}{2}-i\rho)\right) \qquad (7)$$

(with $K\equiv R/12$ the inverse square radius of the space and $\rho^2\equiv m^2/K-\frac{9}{4}$). For small K, the bracket has the expansion $\ln(m^2/4\pi)+\gamma-1-4K/3m^2$ and the gap equation starts out as

$$\frac{1}{\tilde\alpha}=-\frac{\mu^{(d-4)}}{8\pi^2(d-4)}-\frac{1}{32\pi^2}\left(\ln\frac{\lambda}{4\pi\mu^4}+\frac{\gamma-1}{2}\right)$$
$$+\frac{K}{16\pi\sqrt{\lambda}}+...\,, \qquad (8)$$

where μ is an arbitrary mass scale.

This is the place where we can introduce the dimensionally transmuted coupling constant

$$\bar\lambda\equiv 4\pi\mu^4\exp\{-32\pi^2[\tilde\alpha^{-1}+4\mu^{d-4}/(d-4)]$$
$$-(\gamma-1)/2\}\,,$$

so that eq. (8) reads

$$0=-\frac{1}{32\pi^2}\left(\log\frac{\lambda}{\bar\lambda}-\frac{8\pi}{9}\frac{1}{\sqrt{\lambda_K}}\right)+...\,, \qquad (9)$$

[#2] Since we only want to demonstrate the basic mechanism we shall ignore all other invariants of the same dimension. We also do not write down the intrinsic counterterms.

where λ_K is the ratio $\lambda_K \equiv 16\lambda/81K^2$. The total renormalized action is $\mathscr{A}=\frac{1}{2}d\int d^2\xi\,\sqrt{g}f$ with a "free energy density"

$$f=\frac{\lambda}{32\pi^2}\left(1-\log\frac{\lambda}{\bar\lambda}+\frac{16\pi}{9\sqrt{\lambda_K}}\right)+\dots . \tag{10}$$

An additional finite cosmological counterterm is necessary to make $f=0$ at $K=0$ so that the total f becomes

$$f_{\mathrm{tot}}=\frac{\lambda}{32\pi^2}\left(1-\frac{\bar\lambda}{\lambda}-\log\frac{\lambda}{\bar\lambda}+\frac{16\pi}{9\sqrt{\lambda_K}}\right)+\dots . \tag{11}$$

Inserting the gap equation this has the small-K behavior

$$f_{\mathrm{tot}}=\frac{\bar\lambda}{32\pi^2}\frac{16\pi}{9\sqrt{\lambda_K}}+\dots=\frac{\sqrt{\bar\lambda}}{16\pi^2}\frac{\pi}{6}R+\dots . \tag{12}$$

Thus, in this limit we have obtained a spontaneous Einstein action. Writing $\mathscr{A}$ as $(1/16\pi^2)(c^2M_{\mathrm{P}}^2/\hbar^2)\times\int d^4\xi\,\sqrt{g}\,R$ with the Planck mass

$$M_{\mathrm{P}}=\sqrt{c\hbar/G}\approx 2.177\times10^{-5}\,\mathrm{g}$$
$$\approx 1.221\times10^{19}\,\mathrm{GeV}/c^2\,,$$

we identify

$$M_{\mathrm{P}}^2=\tfrac{1}{2}d\sqrt{\bar\lambda}\,\pi/6\,. \tag{13}$$

For larger K, the theory predicts the appearance of higher powers of the curvature with well-defined coefficients. Let us study the behavior for very large curvature. Then

$$\tfrac{3}{2}+i\rho\approx 3-m^2/3K,\quad \tfrac{3}{2}-i\rho\approx m^2/3K\,,$$

and

$$\psi(\tfrac{3}{2}+i\rho)+\psi(\tfrac{3}{2}-i\rho)$$
$$\approx -2\gamma+\tfrac{11}{6}-3K/m^2+\tfrac{49}{108}m^2/K+\dots ,$$

so that

$$G_m^2\xrightarrow[K\to\infty]{}\frac{m^2-2K}{8\pi^2(d-4)}-\frac{3K}{16\pi^2}$$
$$+\frac{m^2-2K}{16\pi^2}\left(\ln\frac{K}{4\pi}+\gamma-1-2\gamma\right.$$
$$\left.+\frac{11}{6}-\frac{3K}{m^2}+\frac{49}{108}\frac{m^2}{K}+\dots\right). \tag{14}$$

Hence the gap equation becomes

$$0=\frac{1}{32\pi^2}\left(-\ln\frac{\lambda}{\bar\lambda}-\ln\frac{K^2}{\lambda}+4\gamma-\frac{50}{27}+\frac{12K^2}{\lambda}\right)+\dots . \tag{15}$$

Thus, for $K\to\infty$, λ grows like

$$\lambda\approx 12K^2/[\log(K^2/\lambda)-4\gamma+\tfrac{50}{27}]\,.$$

The free energy density has the large-K limit

$$f_{\mathrm{tot}}\to\frac{12}{32\pi^2}K^2\ln K^2\,. \tag{16}$$

This growth of f_{tot} at large K will strongly dampen any singular solution of the field equations. In particular, the physics at the center of black holes will be quite different from that following Einstein's equations. For completeness, we write down the full gap equation

$$0=\frac{1}{32\pi^2}\left[-\ln\frac{\lambda}{\bar\lambda}-\ln\frac{K^2}{\lambda}+2\ln 16+4\gamma\right.$$
$$-\sum_{n=1}^{\infty}\frac{1}{n}\left(4+\frac{1}{2}(1-n^2)\frac{16}{(n^2-9)^2+81\lambda_K}\right.$$
$$\left.\left.-n^2(1-4n^2)\frac{16}{(4n^2-9)^2+81\lambda_K}\right)\right], \tag{17}$$

and the full free energy density

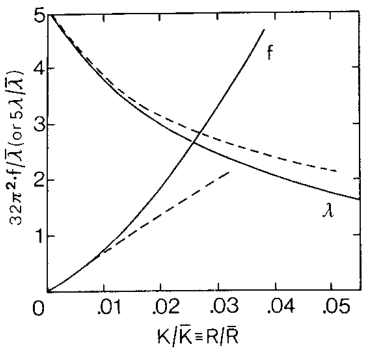

Fig. 1. The free energy density $32\pi^2 f_{\mathrm{tot}}/\bar\lambda$ as a function of $K/\bar K=R/\bar R$ where $\bar K\equiv\bar R/12\equiv 16\exp(2\gamma)\sqrt{\bar\lambda}$ and $\sqrt{\bar\lambda}$ is related to the Planck mass via $M_{\mathrm{P}}^2=\frac{1}{2}d\sqrt{\bar\lambda}\pi/6$. The upper curve shows the gap function $\lambda=\lambda(K)$. The dashed curves are the approximations (9), (12).

$$
\begin{aligned}
f_{\mathrm{tot}} = \frac{\lambda}{32\pi^2}\Bigg[& \frac{\bar{\lambda}}{\lambda} - \ln\frac{\lambda}{\bar{\lambda}} - \ln\frac{K^2}{\lambda} + 2\ln 16 + 4\gamma \\
& + 2\frac{K^2}{\lambda}\Bigg(-\frac{9}{2} + \sum_{R=1}^{\infty}\frac{1}{n}\big(-\tfrac{9}{4}(1+2n^2) + 2\cdot\tfrac{81}{16}(1-\lambda_K) \\
& - \tfrac{1}{4}n^2(1-n^2)\ln\{[(n^2-9)^2 + 81\lambda_K]/n^4\} \\
& + \tfrac{1}{2}n^2(1-4n^2)\ln\{[(4n^2-9)^2 \\
& + 81\lambda_K]/16n^4\}\big)\Bigg)\Bigg] + \text{const.}\,,
\end{aligned} \tag{18}
$$

where the constant is such as to achieve $f_{\mathrm{tot}}=0$ at $K=0$. Fig. 1 shows a plot of the gap λ and the free energy density as a function of $K/\bar{K}=R/\bar{R}$ where $\bar{K}\equiv\bar{R}/12\equiv\sqrt{\bar{\bar{\lambda}}}16\exp(2\gamma)\approx 50.755\sqrt{\bar{\lambda}}=193.870$ M_{P}^2/d.

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