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Gravity as a Theory of Defects in a Crystal with Only Second Gradient Elasticity

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Abstract. We show that a crystal with defects in which the lowest order elastic constants vanish, behaves like an Einstein universe. The role of the Einstein curvature tensor is played by the conserved defect tensor.

Gravitation als Theorie von Defekten im Kristall mit Elastizität rein zweiter Ordnung

Inhaltsübersicht. Es wird gezeigt, daß ein Kristall mit Defekten, in welchem die niedrigste Ordnung der elastischen Konstanten verschwindet, sich wie ein Einstein-Universum verhält. Die Rolle des Einsteinschen Krümmungstensors wird dabei von der tensoriellen Erhaltungsgröße des Defektes übernommen.

It is well-known that crystal defects can be described by geometry [1, 2]. If $\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}^P(\boldsymbol{x})$ is the nonholonomic mapping, corresponding to the plastic deformation of an ideal crystal into a crystal with defects, the connection is at the linearized level

$$\Gamma_{ijk} = \partial_i \partial_i u_k^P, \tag{1}$$

such that the torsion tensor reads

$$S_{ijk} = \frac{1}{2} (\partial_i \partial_j - \partial_j \partial_i) u_k^P, \tag{2}$$

and the curvature tensor becomes

$$R_{ijkl} = (\partial_i \partial_j - \partial_j \partial_i) \partial_k u_l{}^P. \tag{3}$$

A crystal with defects has an elastic partition function [1])

$$Z = \int \mathscr{D}\boldsymbol{u}(\boldsymbol{x}) e^{-\frac{\mu}{4} \int dx (\partial_i u_j + \partial_j u_i - u_{ij}{}^P)^2}, \tag{4}$$

where $u_i(\boldsymbol{x})$ is the total displacement field and $u_{ij}{}^P = (\partial_i u_j{}^P + \partial_j u_i{}^P)/2$ the plastic strain tensor. The energy density is invariant under "defect gauge transformations" [3, 4]

$$u_{ij}{}^P \to u_{ij}{}^P + \partial_i \xi_j + \partial_j \xi_i. \tag{5}$$

Physically, they express the fact that defects cannot be modified by elastic distortions of the crystal (only by plastic ones).

The defect density is therefore a gauge invariant object, namely [1]

$$\mu_{ij} = \varepsilon_{ikl} \varepsilon_{jmn} \partial_k \partial_m u_{ln}{}^P. \tag{6}$$

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1) We neglect the second elastic constant, for simplicity of the argument.

If distances in a crystal are measured by counting the number of steps along atoms, the plastic strain tensor $u_{ln}{}^P$ takes the role of the metric tensor g_{ln} and μ_{il} is recognized as the three dimensional version of the Einstein tensor

$$\mu_{ij} \equiv G_{ij} = \frac{1}{4} \varepsilon_{ikl}\varepsilon_{imn} R_{klmn}. \tag{7}$$

If we go to the transverse gauge

$$\partial_i u_{ij}{}^P = 0, \tag{8}$$

we can integrate out the elastic fluctuations in (4) and remain with a defect system with elastic interactions

$$Z \propto e^{-\frac{\mu}{4}\int dx (u_{ij}{}^P)^2}. \tag{9}$$

Rewriting (6) in the form

$$\begin{aligned}\mu_{ij} = \partial^2(\delta_{ij}u_{ll}{}^P - u_{ij}{}^P) - \partial_i\partial_j u_{ll}{}^P \\ + (\partial_i\partial_k u_{kj}{}^P + (ij)) - \delta_{ij}\partial_l\partial_n u_{ln}{}^P,\end{aligned} \tag{10}$$

and using the gauge (8) we identify $\mu_{ij} = -\partial^2 u_{ij}{}^P$ and see that (9) yields the well-known elastic energy

$$Z \propto e^{-\frac{\mu}{4}\int dx \mu_{ij} \frac{1}{(-\partial^2)^2} \mu_{ij}} \tag{11}$$

between defects (Blin's law). The interactions have a long range and are of continuous nature (they grow like R). This has the consequence that disclinations and antidisclinations are permanently bound together. The lowest bound state is a dislocation line. The binding is relaxed only during the melting process [5] when dislocations proliferate, cause a Meissner screening of the elastic forces [6], and soften the interaction energy to

$$\frac{1}{-\partial^2} \approx (4\pi R)^{-1}.$$

Consider now the Einstein universe and suppose that it is a "world crystal" in which distances are measured by counting atoms along crystalline directions. Then the metric g_{ij} is given by the plastic strain tensor of this crystal and the energy momentum tensor is equal to the Einstein curvature tensor which, in turn, is equal to the defect tensor μ_{ij}. The elastic interaction of defects must coincide with Newton's law such that Z must be

$$Z \propto e^{-\frac{\mu}{4}\int dx \mu_{ij} \frac{1}{-\partial^2} \mu_{ij}}. \tag{12}$$

Working our way backwards we see that such an interaction would arise from a plastically deformed world crystal with an elastic energy

$$Z = \int \mathcal{D}\boldsymbol{u}(\boldsymbol{x})\, e^{-\frac{\mu}{4}\int dx (\partial_k(\partial_i u_j + \partial_j u_i - u_{ij}{}^P))^2} \tag{13}$$

An elastic interaction which behaves like $(\partial^2 u_i)^2$ is present in any Cosserat continuum [1, 7].

Hence, we arrive at the conclusion that it is possible to view the Einstein universe as a Cosserat continuum with defects in which the normal elastic constants are accidentally zero.

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