Gravity as a Theory of Defects in a Crystal with Only Second Gradient Elasticity

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Abstract. We show that a crystal with defects in which the lowest order elastic constants vanish, behaves like an Einstein universe. The role of the Einstein curvature tensor is played by the conserved defect tensor.

Gravitation als Theorie von Defekten im Kristall mit Elastizität rein zweiter Ordnung


It is well-known that crystal defects can be described by geometry [1, 2]. If \( x \rightarrow x' = x + u^P(x) \) is the nonholonomic mapping, corresponding to the plastic deformation of an ideal crystal into a crystal with defects, the connection is at the linearized level

\[
\Gamma_{ijk} = \dot{\epsilon}_i \dot{\epsilon}_j u_k^P, \tag{1}
\]

such that the torsion tensor reads

\[
S_{ijk} = \frac{1}{2} (\dot{\epsilon}_i \dot{\epsilon}_j - \dot{\epsilon}_j \dot{\epsilon}_i) u_k^P, \tag{2}
\]

and the curvature tensor becomes

\[
R_{ijkl} = (\dot{\epsilon}_i \dot{\epsilon}_j - \dot{\epsilon}_j \dot{\epsilon}_i) \dot{\epsilon}_k u_l^P. \tag{3}
\]

A crystal with defects has an elastic partition function \(^1\)

\[
Z = \int \mathcal{D}u(x) e^{-\frac{1}{2} \int dx (\varepsilon_{ij} u^P_{ij} + \dot{\epsilon}_i \dot{\epsilon}_j - u^P_{ij})^2}, \tag{4}
\]

where \( u_k(x) \) is the total displacement field and \( u_{ij}^P = (\dot{\epsilon}_i u_j^P + \dot{\epsilon}_j u_i^P)/2 \) the plastic strain tensor. The energy density is invariant under "defect gauge transformations" [3, 4]

\[
u_{ij}^P \rightarrow u_{ij}^P + \dot{\epsilon}_i \xi_j + \dot{\epsilon}_j \xi_i. \tag{5}\]

Physically, they express the fact that defects cannot be modified by elastic distortions of the crystal (only by plastic ones).

The defect density is therefore a gauge invariant object, namely [1]

\[
\mu_{ij} = \varepsilon_{ikl} \varepsilon_{mn} \dot{\epsilon}_k \dot{\epsilon}_m u^P_{ln}. \tag{6}
\]

\(^1\) We neglect the second elastic constant, for simplicity of the argument.
If distances in a crystal are measured by counting the number of steps along atoms, the plastic strain tensor \( u_{in} \) takes the role of the metric tensor \( g_{in} \) and \( \mu_{ij} \) is recognized as the three dimensional version of the Einstein tensor

\[
\mu_{ij} = G_{ij} = \frac{1}{4} \varepsilon_{ikl} \varepsilon_{imn} R_{klnn}.
\] (7)

If we go to the transverse gauge

\[
\varepsilon_{k} u_{ij} = 0,
\] (8)

we can integrate out the elastic fluctuations in (4) and remain with a defect system with elastic interactions

\[
Z \propto e^{-\frac{\mu}{4} \int dx (\varepsilon_{ij} u_{ij})^2}.
\] (9)

Rewriting (6) in the form

\[
\mu_{ij} = \varepsilon^2 (\delta_{ij} u_{ij} - u_{ij}) - \varepsilon_{k} \varepsilon_{ij} u_{kj} + (ij) - \delta_{ij} \varepsilon_{k} u_{kn} u_{lm},
\] (10)

and using the gauge (8) we identify \( \mu_{ij} = -\varepsilon^2 u_{ij} \) and see that (9) yields the well-known elastic energy

\[
Z \propto e^{-\frac{\mu}{4} \int dx \frac{1}{(-\varepsilon^2)^\mu_{ij}}}
\] (11)

between defects (Blin’s law). The interactions have a long range and are of continuous nature (they grow like \( R \)). This has the consequence that disclinations and anti-disclinations are permanently bound together. The lowest bound state is a dislocation line. The binding is relaxed only during the melting process \([5]\) when dislocations proliferate, cause a Meissner screening of the elastic forces\([6]\), and soften the interaction energy to

\[
\frac{1}{(-\varepsilon^2)^2} \approx (4\pi R)^{-1}.
\]

Consider now the Einstein universe and suppose that it is a “world crystal” in which distances are measured by counting atoms along crystalline directions. Then the metric \( g_{ij} \) is given by the plastic strain tensor of this crystal and the energy momentum tensor is equal to the Einstein curvature tensor which, in turn, is equal to the defect tensor \( \mu_{ij} \). The elastic interaction of defects must coincide with Newton’s law such that \( Z \) must be

\[
Z \propto e^{-\frac{\mu}{4} \int dx \frac{1}{(-\varepsilon^2)^\mu_{ij}}}
\] (12)

Working our way backwards we see that such an interaction would arise from a plastically deformed world crystal with an elastic energy

\[
Z = \int \mathcal{L} u(x) e^{-\frac{\mu}{4} \int dx (\varepsilon^2 u_{ij})} \quad (13)
\]

An elastic interaction which behaves like \((\varepsilon^2 u_{ij})^2\) is present in any Cosserat continuum \([1, 7]\).

Hence, we arrive at the conclusion that it is possible to view the Einstein universe as a Cosserat continuum with defects in which the normal elastic constants are accidentally zero.

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References


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