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Perturbative two-loop quark potential of stiff strings in any dimension

G. Germán and H. Kleinert
Institut für Theorie der Elementarteilchen, Freie Universität Berlin, Arnimallee 14, 1000 Berlin 33, West Germany
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We present a systematic perturbative calculation of the static quark-antiquark potential of stiff strings up to two loops. Our result is valid in any number of dimensions d. In particular we find that the string tension M_0^2 is renormalized as follows: $M^2=M_0^2[1+\frac{1}{2}(d-2)(\alpha_0/4\pi)(1+4\pi L)+\frac{1}{4}(d-2)(d-1)(\alpha_0 L)^2]$, where α_0 is the inverse stiffness of the string and L the logarithmic divergence $L=(1/4\pi)\ln(\Lambda^2/\mu^2)$. At large distances, not only Lüscher's $1/R$ term but also the $1/R^3$ term is found to be independent of the coupling strength.

I. INTRODUCTION

The picture that quarks are held together by tubes of color-electric flux has been idealized by Nambu[1] by formulating a string model of hadronic forces. In spacetime, the string is supposed to follow the Nambu-Goto action[2]

$$A_0=M_0^2\int d^2\xi\sqrt{g}\ , \tag{1.1}$$

where M_0^2 is the bare string tension and $g_{ij}=\partial_i x^\mu \partial_j x^\mu$ $(i,j=0,1;\mu=1,\ldots,d)$ is the intrinsic metric induced on the two-dimensional world sheet, parametrized by $x^\mu(\xi)$. Several of the physical properties of such a world sheet are in agreement with what is expected for a flux tube. Unfortunately, however, the quantum system associated with this action has also many undesirable features such as ghosts, tachyons, and an imaginary static potential between quarks when the distance becomes smaller than some critical distance. In an attempt to avoid some of these problems, Polyakov[3] and one of the authors[4] have added to the action (1.1) an extrinsic curvature term

$$A_1=(1/2\alpha_0)\int d^2\xi\sqrt{g}\ D^2x^\mu D^2x^\mu\ , \tag{1.2}$$

where α_0 is the inverse bending stiffness and D^2 is the Laplacian in the space with metric $g_{ij}(\xi)$. The resulting total action $A=A_0+A_1$ leads to a string theory with several interesting properties: At the classical level A_1 is scale invariant, but quantum corrections generate spontaneously a tension,[3–5] which is the dimensionally transmuted coupling constant of the theory. This contributes to the linear behavior of the static quark-antiquark potential at long distances.[6,7] The nonleading $1/R$ term in the potential has Lüscher's universal size[8–10] $-(d-2)\pi/24R$. The extended action naturally gives rise to objects which can be identified as glueballs, with a mass expected from Monte Carlo simulations of lattice QCD (Ref. 11).

For high stiffness $1/\alpha_0$, the static potential can be calculated at any distance.[9,10] For short distances, the potential has the proper asymptotic-freedom behavior $\propto 1/R$ as expected from QCD. Even the quantitative behavior is apparently correct—the prefactor has twice Lüscher's value in agreement with fits to the potential in the spectrum of the J/ψ family.[9,10,12]

Up to now, this last and important result has been obtained in a calculation which employs *two* limits: that of $d\to\infty$ *and* small coupling. Since it is an experimentally observable quantity, we believe it is desirable to know the answer also for finite d. For this it is necessary to use a different technique than that in Ref. 10. In fact, those authors did not proceed very systematically. Initially, they used a saddle-point approximation, good only for large d. Later, when it became necessary to calculate certain trace logs involving space-dependent gap functions, they could no longer exploit the full power of the large-d expansion and had to resort to an additional perturbation expansion in the coupling strength. Obviously, once a perturbation expansion is used, there is really no need for applying the saddle-point approximation and the large-d limit in the beginning.

The purpose of our paper is, therefore, to avoid a mixed approach and to present a systematic perturbative calculation, in which only the limit of small coupling is required. The advantage of it is that it is valid for any dimension d and can easily be extended, at least in principle, to higher loops.

For the benefit of the reader familiar with the earlier works we use a notation close to Ref. 11 and as close as possible to Ref. 10. The organization of the paper is as follows: In Sec. II we fix the gauge by using the Gauss map or "physical" gauge where only the transverse u fields survive. The action is then expanded up to quartic-order terms in the u fields. Section III recalls first the known renormalization of the coupling constant and the string tension and then goes beyond this by calculating the two-loop renormalized string tension stated in the abstract. Section IV is the main part of this paper and the full static potential is obtained for all distances, in the weak-coupling regime, up to two loops. The renormalized quark potential is given in Sec. V together with its large- and small-distance limits. We also display the finite-size correction of the correlation functions $\langle u_i(r)u_j(r)\rangle$ and their large- and small-distance limits. A brief discussion of our results follows in Sec. VI. Finally four appendixes provide some details of the calculation.

II. THE ACTION

Our starting point is the combined action[3,4]

$$A=M_0^2\int d^2\xi\sqrt{g}+(1/2\alpha_0)\int d^2\xi\sqrt{g}\,(D^2x^\mu)^2 . \quad (2.1)$$

The Laplacian operator D^2 reads explicitly

$$D^2\equiv(1/\sqrt{g}\,)\partial_i(\sqrt{g}\,g^{ij}\partial_j) . \quad (2.2)$$

The reparametrization invariance of the action (2.1) allows us to select a convenient set of dynamical degrees of freedom. In this paper we use the parametrization known as Gauss map and defined by

$$x^\mu(\xi)=(\xi^0,\xi^1,x^2,\dots,x^{d-1})=(\xi^k,\boldsymbol{\mu}) . \quad (2.3)$$

The components of the parameter ξ will also be denoted by $\xi=(t,r)$. The vertical displacement field has $d-2$ components $u^a=u^a(\xi^0,\xi^1)$, $a=2,3,\dots,d-1$. In terms of it, the metric is given by

$$g_{ij}=\delta_{ij}+\mathbf{u}_i\cdot\mathbf{u}_j \quad (2.4a)$$

and its inverse

$$g^{ij}=(1/g)[(1+\mathbf{u}_i^2)\delta^{ij}-\mathbf{u}_i\cdot\mathbf{u}_j] , \quad (2.4b)$$

where $g\equiv\det g_{ij}=1+\mathbf{u}_i^2+\frac{1}{2}\mathbf{u}_i^2\mathbf{u}_j^2-\frac{1}{2}(\mathbf{u}_i\cdot\mathbf{u}_j)^2$. Expanding up to fourth-order terms in $\mathbf{u}$ we obtain

$$\sqrt{g}\approx1+\tfrac{1}{2}\mathbf{u}_i^2+\tfrac{1}{8}\mathbf{u}_i^4-\tfrac{1}{4}(\mathbf{u}_i\cdot\mathbf{u}_j)^2 , \quad (2.5a)$$

$$\sqrt{g}\,(D^2\xi^k)^2\approx(\mathbf{u}_i\cdot\mathbf{u}_{kk})^2 , \quad (2.5b)$$

$$\sqrt{g}\,(D^2\mathbf{u})^2\approx\mathbf{u}_{ii}^2+\tfrac{1}{2}\mathbf{u}_{ii}^2\mathbf{u}_j^2-2(\mathbf{u}_i\cdot\mathbf{u}_{kk})^2-2(\mathbf{u}_i\cdot\mathbf{u}_j)(\mathbf{u}_{ij}\cdot\mathbf{u}_{kk}) . \quad (2.5c)$$

Collecting these terms in the action (2.1) we get

$$A=\int d^2\xi\{M_0^2+(1/2\alpha_0)(\mathbf{u}_{ii}^2+m_0^2\mathbf{u}_i^2)+(M_0^2/8m_0^2)[m_0^2\mathbf{u}_i^4-2m_0^2(\mathbf{u}_i\cdot\mathbf{u}_j)^2+2\mathbf{u}_{ii}^2\mathbf{u}_j^2-4(\mathbf{u}_i\cdot\mathbf{u}_{jj})^2-8(\mathbf{u}_i\cdot\mathbf{u}_j)(\mathbf{u}_{ij}\cdot\mathbf{u}_{kk})]+\cdots\} , \quad (2.6)$$

where $m_0^2\equiv\alpha_0M_0^2$, which is the expression relevant for the two-loop calculations to follow.

III. COUPLING CONSTANT AND STRING TENSION RENORMALIZATIONS

Correlation functions for the u's, can be obtained immediately from the quadratic terms in Eq. (2.6). For the infinite-size system (which we distinguish by a subindex I), they are

$$\langle u^a(\xi)u^b(\xi)\rangle_I=\delta^{ab}\alpha_0\int\frac{d^2k}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot(\xi-\xi')}}{k^2(k^2+m_0^2)} . \quad (3.1)$$

For the derivatives $\partial_iu\equiv u_i$, $\partial_i\partial_ju\equiv u_{ij}$ taken at $\xi=\xi'$ this gives the following logarithmically divergent numbers:

$$\langle u_iu_i\rangle_I=\alpha_0\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+m_0^2}\equiv\alpha_0L_0 , \quad (3.2a)$$

$$\langle u_iu_j\rangle_I=\alpha_0\int\frac{d^2k}{(2\pi)^2}\frac{k_ik_j}{k^2(k^2+m_0^2)}=\tfrac{1}{2}\delta_{ij}\alpha_0L_0 , \quad (3.2b)$$

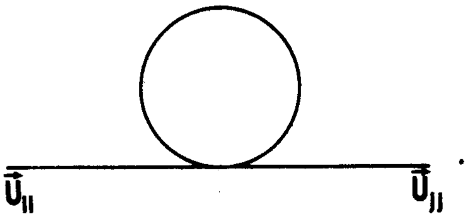
$$\langle u_{ii}u_{jj}\rangle_I=\alpha_0\int\frac{d^2k}{(2\pi)^2}\frac{k^2}{k^2+m_0^2}=-m_0^2\alpha_0L_0 , \quad (3.2c)$$

$$\langle u_{ij}u_{kk}\rangle_I=\alpha_0\int\frac{d^2k}{(2\pi)^2}\frac{k_ik_j}{k^2+m_0^2}=-\tfrac{1}{2}\delta_{ij}m_0^2\alpha_0L_0 , \quad (3.2d)$$

where

$$L_0=\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+m_0^2}=(1/4\pi)\ln(\Lambda^2/m_0^2) \quad (3.3)$$

with Λ an ultraviolet cutoff. We have dropped quadratically divergent unit integrals $\int d^2k/(2\pi)^2$ which vanish in analytic regularization by continuing $\int[d^2k/(2\pi)^2](k^2+m_0^2)^\nu$, to $\nu\to0$. The one-loop contributions to the renormalization of $1/\alpha_0$ are obtained by contracting two u_i's appearing in the interacting part of A, Eq. (2.6). This is equivalent to calculating the Feynman diagram



Using (3.2) we can easily evaluate the relevant terms in (2.6) (see Appendix A). They amount to the replacements

$$\mathbf{u}_i^2\mathbf{u}_{jj}^2\to(d-2)\alpha_0L_0\mathbf{u}_{ii}^2 , \quad (3.4a)$$

$$(\mathbf{u}_i\cdot\mathbf{u}_{jj})^2\to\alpha_0L_0\mathbf{u}_{ii}^2 , \quad (3.4b)$$

$$(\mathbf{u}_i\cdot\mathbf{u}_j)(\mathbf{u}_{ij}\cdot\mathbf{u}_{kk})\to\tfrac{1}{2}(d-2)\alpha_0L_0\mathbf{u}_{ii}^2 . \quad (3.4c)$$

The second term in A goes over into $(1/2\alpha_0)[1-(d/2)\alpha_0L_0]\mathbf{u}_{ii}^2$. From this we can read off the well-known first-order one-loop renormalized inverse stiffness

$$1/\alpha=(1/\alpha_0)[1-(d/2)\alpha_0L_0] . \quad (3.5)$$

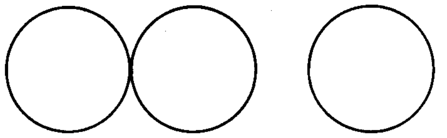
The free energy is obtained by evaluating

$$\langle A\rangle=\int d^2\xi\left[M_0^2+[(d-2)/2]\times\int\frac{d^2k}{(2\pi)^2}\ln[k^2(k^2+m_0^2)]\right]+\langle A_{\rm int}\rangle , \quad (3.6)$$

where $\langle A_{\rm int}\rangle$ is the free field average of the interacting part of the action

$$A_{\rm int}=(M_0^2/8m_0^2)\times\int d^2\xi[m_0^2\mathbf{u}_i^4-2m_0^2(\mathbf{u}_i\cdot\mathbf{u}_j)^2+2\mathbf{u}_i^2\mathbf{u}_{jj}^2-4(\mathbf{u}_i\cdot\mathbf{u}_{jj})^2-8(\mathbf{u}_i\cdot\mathbf{u}_j)(\mathbf{u}_{ij}\cdot\mathbf{u}_{kk})+\cdots]\,. \quad (3.7)$$

The corresponding Feynman diagrams are



As shown in Appendix A, individual terms in $\langle A_{\rm int}\rangle$ are given by

$$\langle \mathbf{u}_i^4\rangle=(d-2)^2\langle u_iu_i\rangle^2+2(d-2)\langle u_iu_j\rangle^2\,, \quad (3.8a)$$

$$\langle(\mathbf{u}_i\cdot\mathbf{u}_i)^2\rangle=(d-2)(d-1)\langle u_iu_j\rangle^2+(d-2)\langle u_iu_i\rangle^2\,, \quad (3.8b)$$

$$\langle \mathbf{u}_i^2\mathbf{u}_{jj}^2\rangle=(d-2)^2\langle u_{ii}u_{jj}\rangle\langle u_ku_k\rangle+2(d-2)\langle u_iu_{kk}\rangle^2\,, \quad (3.8c)$$

$$\langle(\mathbf{u}_i\cdot\mathbf{u}_{jj})^2\rangle=(d-2)(d-1)\langle u_iu_{kk}\rangle^2+(d-2)\langle u_iu_i\rangle\langle u_{jj}u_{kk}\rangle\,, \quad (3.8d)$$

$$\langle(\mathbf{u}_i\cdot\mathbf{u}_j)(\mathbf{u}_{ij}\cdot\mathbf{u}_{kk})\rangle=(d-2)^2\langle u_iu_j\rangle\langle u_{ij}u_{kk}\rangle+2(d-2)\langle u_iu_{ij}\rangle\langle u_ju_{kk}\rangle\,. \quad (3.8e)$$

Making use of Eqs. (3.2) and of the integral

$$f_I\equiv\int[d^2k/(2\pi)^2]\ln[k^2(k^2+m_0^2)]=(m_0^2/4\pi)(1+4\pi L_0) \quad (3.9)$$

with the quadratic infinity discarded, as in an analytic regularization procedure, we obtain

$$V(R)=M^2R\,, \quad (3.10)$$

where M^2 is the two-loop renormalized string tension

$$M^2=M_0^2\left[1+\frac{d-2}{2}\frac{\alpha_0}{4\pi}(1+4\pi L_0)+\frac{(d-2)(d-1)}{4}(\alpha_0L_0)^2\right]. \quad (3.11)$$

This can be rewritten as

$$M^2=M_0^2\left[1+2\bar\alpha_0(1-\bar l_0)+4\bar\alpha_0^2\bar l_0^2+\frac{1}{d-2}4\bar\alpha_0^2\bar l_0^2\right], \quad (3.12)$$

where

$$\bar\alpha_0\equiv\frac{(d-2)\alpha_0}{16\pi}\,, \quad (3.13a)$$

$$\bar l_0\equiv-4\pi L_0\,. \quad (3.13b)$$

In this form, our result can be compared most easily with Eqs. (3.14)–(3.16) of Ref. 10 which uses the renormalized coupling constant e^2 and defines

$$\lambda\equiv\frac{De^2}{16\pi}\,, \quad (3.14a)$$

$$l=\ln\left[\frac{eM_0^2}{\mu^2}\right], \quad (3.14b)$$

where $D=d-2$ is the number of transverse degrees of freedom and μ^2 is an arbitrary renormalization scale. Contact with our result is made by identifying their mass scale μ^2e with our Λ^2. From the renormalization equation for α we find that, at one loop our coupling constant α_0 is related to their e^2 by $\alpha_0=e^2(\mu^2=eM_0^2)$ or

$$\alpha_0=e^2(1+2\lambda+\cdots)|_{\mu^2=eM_0^2e^{-1}} \quad (3.15)$$

(here e^{-1} denotes the inverse Euler number, not the charge), while $\bar l_0$ transforms into their l as follows:

$$\bar l_0=l+\ln(1+2\lambda)\simeq l+2\lambda+\cdots\,; \quad (3.16)$$

this brings Eq. (3.12) to the form

$$M^2=M_0^2\left[1+2\lambda(1-l)-4\lambda^2l(1-l)+\frac{1}{d-2}4\lambda^2l^2+\cdots\right]. \quad (3.17)$$

In the limit of large d, the first three terms reduce to their result and the last one disappears.

IV. ONE- AND TWO-LOOP FINITE-SIZE CONTRIBUTIONS TO $V(R)$

We now turn to the finite-size system and calculate the action

$$\langle A\rangle_F=\int d^2\xi\left[M_0^2+\frac{d-2}{2}f_F\right]+\langle A_{\rm int}\rangle_F \quad (4.1)$$

with the subscript F indicating the finite size. In a strip of spatial width R, the energy (3.9) is to be replaced by

$$f_F=(1/R)\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ln[(\omega^2+k_n^2)(\omega^2+k_n^2+m_0^2)]\,, \quad (4.2)$$

where

$$k_n=\frac{n\pi}{R},\quad n=1,2,\ldots \quad (4.3)$$

are the discrete momenta associated with the wave functions of $u(\xi)$, which vanish at the ends of the string held down by static quarks. Equation (4.2) is regularized by adding and subtracting the regularized infinite-size expression

$$f_F=f_I+(f_F-f_I)=\frac{m_0^2}{4\pi}(1+4\pi L_0)+(1/R)\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right]\times\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ln[(\omega^2+k_n^2)(\omega^2+k_n^2+m_0^2)] \quad (4.4)$$

and doing the ω integration in analytic regularization, according to the formula $\int (d\omega/2\pi)\ln(\omega^2+a^2)=\sqrt{a^2}$. The result is finite since the divergences in the sum and the integral cancel each other. After a few simple manipulations (see Appendix B) the result can be written as

$$f_F=\frac{m_0^2}{4\pi}(1+4\pi L_0)-\frac{\pi}{6R^2}+\frac{m_0^2}{4\pi}\ln\frac{\lambda_{0R}}{4e^{-2\gamma+1}}+\frac{m_0^2}{\pi}S_1 \tag{4.5a}$$

with λ_{0R} being the dimensionless parameter

$$\lambda_{0R}\equiv m_0^2R^2/\pi^2 \ . \tag{4.5b}$$

Thus, up to one-loop level the potential becomes

$$V(R)=M_0^2R[1+\bar{\alpha}_0(-4/3\lambda_{0R}+2\bar{L}_0+8S_1)] \ , \tag{4.6}$$

where

$$\bar{L}_0\equiv 4\pi L_0+\ln\frac{\lambda_{0R}}{4e^{-2\gamma}} \tag{4.7a}$$

and S_1 is the convergent sum

$$S_1\equiv(1/\lambda_{0R})\sum_{n=1}^{\infty}[(n^2+\lambda_{0R})^{1/2}-n-\lambda_{0R}/2n] \tag{4.7b}$$

and $\bar{\alpha}_0$ as defined by Eq. (3.13a). We now calculate the two-loop contribution coming from $\langle A_{\rm int}\rangle_F$. Using Eqs. (3.7) and (3.8) we see that $\langle A_{\rm int}\rangle$ is given by

$$\langle A_{\rm int}\rangle=\frac{(d-2)^2M_0^2}{8m_p^2}\int d^2\xi\{m_0^2\langle u_iu_i\rangle^2-2m_0^2\langle u_iu_j\rangle^2-4\langle u_iu_{kk}\rangle^2+2\langle u_{ii}u_{jj}\rangle\langle u_ku_k\rangle-8\langle u_iu_j\rangle\langle u_{ij}u_{kk}\rangle$$
$$-[1/(d-2)](2m_0^2\langle u_iu_i\rangle^2+4\langle u_{ii}u_{jj}\rangle\langle u_ku_k\rangle+16\langle u_iu_{ij}\rangle\langle u_ju_{kk}\rangle)\} \ . \tag{4.8}$$

Since the u fields vanish at the ends of the string, the correlation functions for the finite-size system depend on the position r on the string and are $(k^2=\omega^2+k_n^2)$

$$\langle u(\xi)u(\xi')\rangle_F=\frac{2\alpha_0}{R}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{k^2(k^2+m_0^2)}\sin\frac{n\pi r}{R}\sin\frac{n\pi r'}{R} \ . \tag{4.9}$$

It follows that

$$\langle u_iu_i\rangle_F=\frac{\alpha_0}{2\pi R}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}d\omega\left[\frac{\omega^2}{k^2(k^2+m_0^2)}\left[1-\cos\frac{2n\pi r}{R}\right]+\frac{k_n^2}{k^2(k^2+m_0^2)}\left[1+\cos\frac{2n\pi r}{R}\right]\right] , \tag{4.10a}$$

$$\langle u_{ii}u_{jj}\rangle=\frac{\alpha_0}{2\pi R}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}d\omega\frac{k^2}{k^2+m_0^2}\left[1-\cos\frac{2n\pi r}{R}\right] . \tag{4.10b}$$

After some calculations, described in Appendix B, we find the following products of correlation functions required in $\langle A_{\rm int}\rangle_F$:

$$\langle u_iu_i\rangle_F^2=\left[\frac{\alpha_0}{4\pi}\right]^2[\bar{L}_0+2S_2-4C_1(r)+2C_2(r)]^2 \ , \tag{4.11a}$$

$$\langle u_iu_j\rangle_F^2=\left[\frac{\alpha_0}{4\pi}\right]^2(\tfrac{1}{16}[2\bar{L}_0-2+8S_1-8C_1(r)]^2+\tfrac{1}{16}\{2\bar{L}_0+2-8(S_1-S_2)-8[C_1(r)-C_2(r)]^2\}) \ , \tag{4.11b}$$

$$\langle u_{ii}u_{jj}\rangle_F\langle u_ku_k\rangle_F=-m_0^2\left[\frac{\alpha_0}{4\pi}\right]^2\{[\bar{L}_0+2S_2-4C_1(r)+2C_2(r)][\bar{L}_0+2S_2-2C_2(r)]\} \ , \tag{4.11c}$$

$$\langle u_{ij}u_{kk}\rangle_F\langle u_iu_j\rangle_F=-m_0^2\left[\frac{\alpha_0}{4\pi}\right]^2\left[-\tfrac{1}{2}\bar{l}_0^2-\bar{l}_0[\bar{L}_0+2S_2-2C_1(r)]+4\left[S_1+\tfrac{1}{4}(L_{0R}-1)-\frac{1}{12}\frac{1}{\lambda_{0R}}-C_1(r)-C_3(r)\right]\right.$$
$$\times[S_1+\tfrac{1}{4}(L_{0R}-1)-C_1(r)]$$
$$+4\left[S_1-S_2-\tfrac{1}{4}(L_{0R}+1)-\frac{1}{12}\frac{1}{\lambda_{0R}}-C_1(r)C_2(r)-C_3(r)\right]$$
$$\left.\times[S_1-S_2-\tfrac{1}{4}(L_{0R}+1)+C_1(r)-C_2(r)]\right] , \tag{4.11d}$$

$$\langle u_iu_{kk}\rangle_F^2=\left[\frac{\alpha_0}{4\pi}\right]^2[C_2'(r)]^2 \ , \tag{4.11e}$$

$$\langle u_i u_{ij} \rangle_F \langle u_j u_{kk} \rangle_F = \left[\frac{\alpha_0}{4\pi} \right]^2 \{ [C_2'(r)]^2 - 2C_1'(r) C_2'(r) \} , \tag{4.11f}$$

where

$$S_2 = \sum_{n=1}^{\infty} \left[\frac{1}{(n^2+\lambda_{0R})^{1/2}} - \frac{1}{n} \right] , \tag{4.12}$$

$$C_1(r) = \frac{1}{\lambda_{0R}} \sum_{n=1}^{\infty} [(n^2+\lambda_{0R})^{1/2} - n] \cos \frac{2n\pi r}{R} , \tag{4.13a}$$

$$C_2(r) = \sum_{n=1}^{\infty} \frac{1}{(n^2+\lambda_{0R})^{1/2}} \cos \frac{2n\pi r}{R} , \tag{4.13b}$$

$$C_3(r) = \frac{1}{\lambda_{0R}} \sum_{n=1}^{\infty} n \cos \frac{2n\pi r}{R} \tag{4.13c}$$

and

$$L_{0R} = \ln \frac{\lambda_{0R}}{4e^{-2\gamma}} . \tag{4.14}$$

Primes in Eqs. (4.11e) and (4.11f) denote derivative with respect to the r variable.

To obtain the final expression for $\langle A_{\rm int} \rangle$ and thus for $V(R)$, we need the integrals (see Appendix C)

$$(1/R) \int_0^R dr [C_1(r)]^2 = \tfrac{1}{2} S_5 , \tag{4.15a}$$

$$(1/R) \int_0^R dr [C_2(r)]^2 = \tfrac{1}{2} S_3 , \tag{4.15b}$$

$$(1/R) \int_0^R dr [C_1(r) C_2(r)] = -\tfrac{1}{2} S_4 , \tag{4.15c}$$

$$(1/R) \int_0^R dr [C_2(r) C_3(r)] = -\tfrac{1}{4}(1/\lambda_{0R}) + \tfrac{1}{2} S_4 , \tag{4.15d}$$

$$(1/R m_0^2) \int_0^R dr [C_2'(r)]^2 = -1/\lambda_{0R} - 2S_3 , \tag{4.15e}$$

$$(1/R m_0^2) \int_0^R dr [C_1'(r) C_2'(r)] = -\tfrac{1}{2}(1/\lambda_{0R}) + 2S_4 + S_5 , \tag{4.15f}$$

where

$$S_3 = \sum_{n=1}^{\infty} \frac{1}{n^2+\lambda_{0R}} , \tag{4.16a}$$

$$S_4 = \frac{1}{\lambda_{0R}} \sum_{n=1}^{\infty} \left[\frac{n}{(n^2+\lambda_{0R})^{1/2}} - 1 \right] , \tag{4.16b}$$

$$S_5 = \frac{1}{\lambda_{0R}^2} \sum_{n=1}^{\infty} [(n^2+\lambda_{0R})^{1/2} - n]^2 . \tag{4.16c}$$

Calculating all terms in Eq. (4.8) we find the static quark potential

$$\begin{aligned} V(R) = {} & M_0^2 R + \bar{\alpha}_0 M_0^2 R \left[-\frac{4}{3} \frac{1}{\lambda_{0R}} + 2\bar{L}_0 + 8S_1 \right] \\ & + \bar{\alpha}_0^2 M_0^2 R \Big[-\tfrac{16}{3}(1+2S_1-S_2) \frac{1}{\lambda_{0R}} + 6 - 48S_1 + 24S_2 + 96S_1^2 - 96S_1S_2 + 40S_2^2 - 12S_3 - 16S_4 + 16S_2\bar{L}_0 + 4\bar{L}_0^2 \\ & + [1/(d-2)](16S_2^2 + 40S_3 + 64S_4 + 32S_5 + 16S_2\bar{L}_0 + 4\bar{L}_0^2) \Big] . \end{aligned} \tag{4.17}$$

A comparison with the result of Ref. 10 is most direct after the finite renormalization (3.15), which implies

$$\bar{\alpha}_0 S_1(\lambda_{0R}) = \lambda S_1(\Lambda^2) + \lambda^2 S_2(\Lambda^2) + O\left[\frac{1}{d}, \lambda^3 \right] , \tag{4.18a}$$

$$\bar{\alpha}_0 \bar{L}_0 = \lambda L + 2\lambda^2 L + O\left[\frac{1}{d}, \lambda^3 \right] . \tag{4.18b}$$

In the large-d limit, the corresponding expression of Ref. 10 [Eq. (4.20)] is reproduced. In contrast, our result is valid for any d.

V. RENORMALIZATION

We now proceed to the renormalization of the entire quark potential Eq. (4.17). For this we invert the formula for the renormalized string tension Eq. (3.12) and the one for the coupling constant Eq. (3.5). The resulting expressions are

$$M_0^2 = M^2 \left[1 - 2\bar{\alpha}(1-l) - 4\bar{\alpha}^2 \bar{l}(1-\bar{l}) + \frac{4\bar{\alpha}^2 \bar{l}^2}{d-2} + O(\bar{\alpha}^3) \right] , \tag{5.1}$$

$$\bar{\alpha}_0 = \bar{\alpha} \left[1 + 2\bar{\alpha}\bar{l} + \frac{4\bar{\alpha}\bar{l}}{d-2} + O(\bar{\alpha}^2) \right] , \tag{5.2}$$

where, as before, the symbols $M^2, \bar{\alpha}$, denote renormalized quantities. The renormalized potential is

$$V(R)=M^2R+\bar{\alpha}M^2R\left[-\frac{4}{3}\frac{1}{\lambda_R}+2(L_R-1)+8S_1\right]$$
$$+\bar{\alpha}^2M^2R\left[-\tfrac{16}{3}(1+2S_1-S_2)\frac{1}{\lambda_R}+6-48S_1+16S_2\right.$$
$$+96S_1^2-96S_1S_2+40S_2^2-12S_3-16S_4+16S_2L_R$$
$$\left.-4L_R+4L_R^2+\frac{1}{d-2}(16S_2^2+40S_3+64S_4+32S_5+16S_2L_R+4L_R^2)\right], \tag{5.3}$$

where $L_R \equiv \ln\lambda_R/4e^{-2\gamma}$.

Defining

$$\tilde{M}^2=\frac{2}{d-2}M^2,\quad \tilde{\alpha}=\frac{d-2}{2}\alpha \tag{5.4}$$

and introducing the quantity $\tilde{V}(R)$,

$$\tilde{V}\equiv\frac{2}{d-2}\frac{1}{\tilde{M}}V(\tilde{M},R), \tag{5.5}$$

we see that $\tilde{V}$ is a dimensionless function of $\tilde{M}R$. It is displayed graphically in Fig. 1 for various values of $\tilde{\alpha}$.

The limits of large and small λ_R have the following expansions.

Large λ_R

We have

$$\tilde{V}(R)=R+\tilde{V}_{\text{vac}}-\frac{\pi}{12}\frac{1}{R}-\frac{1}{2}\left[\frac{\pi}{12}\right]^2\frac{1}{R^3}$$
$$+\frac{9}{d-2}\left[\frac{\pi}{12}\right]^3\frac{1}{\sqrt{\tilde{\alpha}}}\left[\frac{d-10}{5}\frac{1}{R^4}+\frac{3d-22}{42}\left[\frac{\pi}{(\tilde{\alpha})^{1/2}}\right]^2\frac{1}{R^6}+\frac{5d-34}{80}\left[\frac{\pi}{(\tilde{\alpha})^{1/2}}\right]^4\frac{1}{R^8}+O(R^{-10})\right], \tag{5.6}$$

where

$$\tilde{V}_{\text{vac}}=-\frac{(\tilde{\alpha})^{1/2}}{2}\left[1+\frac{\tilde{\alpha}}{8\pi}\left[\frac{3\pi-10}{2}-\frac{15\pi-32}{3(d-2)}\right]\right]. \tag{5.7}$$

As in Refs. 9 and 10 note that in the large $-\sqrt{\lambda_R}$ limit, Lüscher's term receives no correction due to two-loop contributions. We also note that the R^{-3} term in Eq. (5.16) is independent of the coupling constant α. It has, therefore, some chance that its coefficient may be universal as well.

Comparing the first five terms with Refs. 9 and 10, we see that if one replaces in the large-d calculation d by the number $d-2$ of transverse degrees of freedom, then the finite-d terms have an effect on the total size of the potential and on the terms of order larger than $1/R^3$.

Small λ_R

The renormalized potential for small λ_R has the expansion

$$\tilde{V}(R)=-\frac{\pi}{6}\left[1+\frac{\tilde{\alpha}}{2\pi}\right]\frac{1}{R}+\left[1-\frac{\tilde{\alpha}}{4\pi}\left\{1-L_R-\frac{\tilde{\alpha}}{4\pi}\left[-\frac{\zeta(3)}{3}+\frac{3}{2}-\frac{\pi^2}{6}\left[\frac{d-6}{d-2}\right]-L_R+\left[\frac{d-1}{d-2}\right]L_R^2\right]\right\}\right]R$$
$$+\frac{\tilde{\alpha}^2}{8\pi^3}\left[-\zeta(3)+\frac{\tilde{\alpha}}{\pi}\left\{\frac{\zeta(5)}{6}-\frac{\zeta(3)}{4}\left[1+4\left[\frac{d-1}{d-2}\right]L_R\right]+\frac{\pi^4}{360}\left[\frac{3d-16}{d-2}\right]\right\}\right]R^3$$
$$+\frac{\tilde{\alpha}^3}{16\pi^5}\left[\zeta(5)+\frac{\tilde{\alpha}}{\pi}\left\{-\frac{5\zeta(7)}{16}+\frac{3\zeta(5)}{4}\left[1-2\left[\frac{d-1}{d-2}\right]L_R\right]\right.\right.$$
$$\left.\left.+\frac{\zeta^2(3)}{8}\left[\frac{11d-14}{d-2}\right]-\frac{\pi^6}{7560}\left[\frac{14d-73}{d-2}\right]\right\}\right]R^5+O(R^7). \tag{5.8}$$

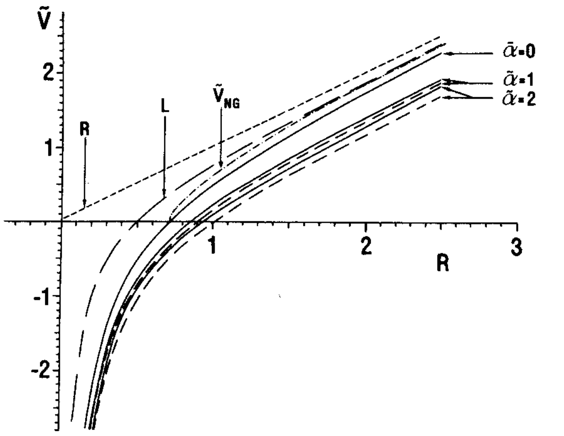


FIG. 1. The quark potential $\tilde{V}$, as defined by Eq. (5.5), is shown as a function of R, for several values of the inverse stiffness $\tilde{\alpha}=0,1,2$ (solid lines) measured in natural units M. The curves are compared with the $d\to\infty$ potential calculation in Ref. 10 (short-dashed lines). The difference vanishes for $\tilde{\alpha}=0$. We also show some other quantities of interest: The linear term (very-short-dashed line) and linear plus Lüscher's term (long-dashed line), the quark potential of the Nambu-Goto model (dash-dotted line).

Hence, at small R, only the leading $1/R$ term in Eq. (5.8) has the same form as in the large-d calculation.

For completeness and future reference we include here the renormalized expressions for the correlation functions $\langle u_t(r)u_t(r)\rangle$ and $\langle u_r(r)u(r)\rangle$. In normal units these are given by (no summation implied)

$$\langle u_i(r)u_i(r)\rangle=\frac{\tilde{\alpha}}{d-2}L_0+\frac{\tilde{\alpha}}{16\pi}\frac{2}{d-2}\widetilde{\langle u_i(r)u_i(r)\rangle}\ , \quad (5.9)$$

where

$$\widetilde{\langle u_t(r)u_t(r)\rangle}=2L_R-2+8S_1-8C_1(r)\ , \quad (5.10a)$$

$$\widetilde{\langle u_r(r)u_r(r)\rangle}=2L_R+2-8(S_1-S_2)-8[C_1(r)-C_2(r)]\ . \quad (5.10b)$$

In Fig. 2 we show these correlation functions for several values of the ordered pair $(R\tilde{M},\tilde{\alpha})$.

For large R, the sums S_1,S_2 and functions $C_1(r)C_2(r)$ have the expansion

$$S_1=\frac{1}{4}-\frac{1}{4}L_R-\frac{1}{2}\frac{1}{\sqrt{\lambda_R}}+\frac{1}{12}\frac{1}{\lambda_R}-\frac{1}{\pi}\frac{1}{\sqrt{\lambda_R}}\sum_{\tilde{n}=1}^{\infty}\frac{K_i(2n\pi\sqrt{\lambda_R})}{\tilde{n}}\ , \quad (5.11a)$$

$$S_2=-\frac{1}{2}L_R-\frac{1}{2}\frac{1}{\sqrt{\lambda_R}}+2\sum_{\tilde{n}=1}^{\infty}K_0(2\pi\tilde{n}\sqrt{\lambda_R})\ , \quad (5.11b)$$

$$C_1=-\frac{1}{2\pi}\frac{1}{\sqrt{\lambda_R}}\sum_{\tilde{n}=1}^{\infty}\left[\frac{K_1(2\pi|(\tilde{n}+\phi)|\sqrt{\lambda_R})}{|\tilde{n}+\phi|}+(\phi\to-\phi)\right]-\frac{1}{2}\frac{1}{\sqrt{\lambda_R}}-\frac{1}{2\pi}\frac{1}{\phi\sqrt{\lambda_R}}K_{-1}(2\pi\phi\sqrt{\lambda_R})-\frac{1}{\lambda_R}\sum_{n=1}^{\infty}n\cos 2n\pi\phi\ , \quad (5.11c)$$

$$C_2=\sum_{\tilde{n}=1}^{\infty}[K_0(2\pi(\tilde{n}+\phi)\sqrt{\lambda_R})+(\phi\to-\phi)]-\frac{1}{2}\frac{1}{\sqrt{\lambda_R}}+K_0(2\pi\phi\sqrt{\lambda_R})\ , \quad (5.11d)$$

where $\phi\equiv r/R$ and $K_{0,1}(z)$ are modified Bessel functions which decay exponentially for large values of the argument. The last sum in Eq. (5.11c) is proportional to $C_3(r)$ of Eq. (4.13c) which for $r=0$, R takes the value $-1/12\lambda_R$. Thus from Eqs. (5.10) and (5.11) we can deduce the large-R expansion of the finite-size correction to the correlation functions.

The short-distance behavior is given by

$$\widetilde{\langle u_t(r)u_t(r)\rangle}=2\ln\left[\frac{\tilde{\alpha}}{\pi^2 e^{-2\gamma+1}}R^2\sin^2\frac{\pi r}{R}\right]-\frac{\tilde{\alpha}}{\pi^2}[\zeta(3)-C(3,2\pi r/R)]R^2+\frac{1}{2}\left[\frac{\tilde{\alpha}}{\pi^2}\right][\zeta(5)-C(5,2\pi r/R)]R^4\ , \quad (5.12a)$$

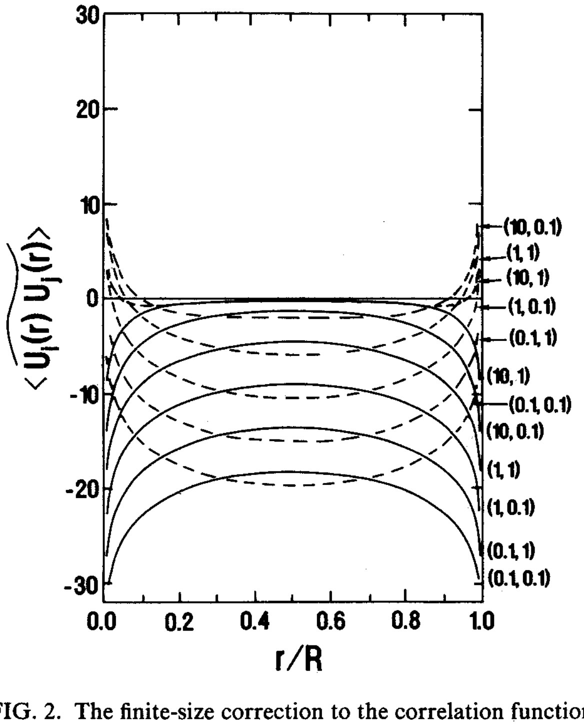


FIG. 2. The finite-size correction to the correlation functions $\langle u_i(r)u_j(r)\rangle$ as functions of r/R. The solid lines correspond to $\widetilde{\langle u_t(r)u_t(r)\rangle}$ for the values of $(R,\tilde{\alpha})$ indicated in the figure, the dashed lines correspond to $\widetilde{\langle u_r(r)u_r(r)\rangle}$.

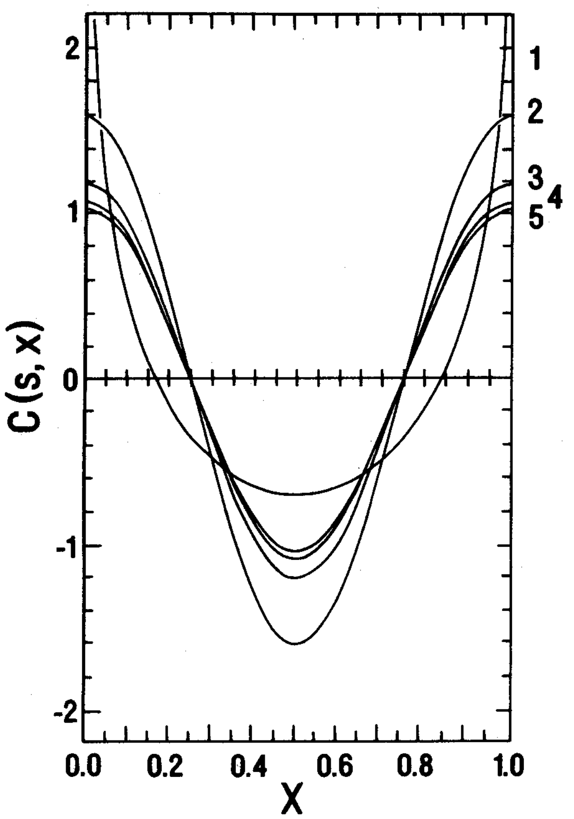


FIG. 3. The Dirichlet-type series $C(s,x)=\sum_{n=1}^{\infty} n^{-s}\cos nx$ for several values of the parameter s. For $s=1$ it is equal to $-\ln(2\sin x/2)$.

$$\widetilde{\langle u_r(r)u_r(r)\rangle}=2\ln\left[\frac{\tilde{\alpha}}{16\pi e^{-2\gamma-1}}\frac{R^2}{\sin^2\frac{\pi r}{R}}\right]-3\left[\frac{\tilde{\alpha}}{\pi^2}\right][\zeta(3)+C(3,2\pi r/R)]R^2+\frac{5}{2}\left[\frac{\tilde{\alpha}}{\pi^2}\right][\zeta(5)+C(5,2\pi r/R)]R^4+\cdots, \quad (5.12b)$$

where

$$C(s,x)\equiv\sum_{n=1}^{\infty} n^{-s}\cos nx . \quad (5.13)$$

For $s=1$ the sum is the well-known Fourier series of $-\ln(2\sin x/2)$, appearing in the first terms of (5.12). For $s>1$ these functions are distorted smoothly into $\cos x\,\zeta(s)$ at $r=0,R$ and $-(1-2^{1-s})\zeta(s)$ at $r=R/2$ and are shown in Fig. 3. For even s (not needed here) the sums can easily be done and yield the Bernoulli polynomials $\frac{1}{2}(-)^{s/2-1}[(2\pi)^s/s!]B_s(x/2\pi)$. The sums for odd s must be done numerically.

VI. CONCLUSION

We have presented a systematic perturbative analysis of a string with curvature stiffness up to two loops in any number of dimensions d. The advantage of our approach is that it can be extended, in principle, to higher-loop calculations. We have calculated the renormalized coupling constant at one-loop and the string tension and the static quark potential at the two-loop level. The potential can be displayed best by plotting it in the reduced dimensional form $2V/(d-2)\tilde{M}$ as a function of the reduced distance $\tilde{M}R$, where $\tilde{M}^2\equiv 2M^2/(d-2)$ and M^2 is the string tension.

Then there is a finite $d\to\infty$ limit which agrees with previous calculations. The finite-d corrections emerging in our work affect the large-distance behavior of the potential primarily in the constant background term, which should be observable in quark-antiquark bound states. At short distances, it enters all powers in $1/R$ except for the leading Coulomb-type $1/R$ term itself. At large distances, the first three terms, R, $1/R$, and $1/R^3$, are all independent of the bending stiffness $\tilde{\alpha}^{-1}$. This opens up the interesting possibility, that not only Lüscher's $1/R$ term, but also the $1/R^3$ term is universal.

ACKNOWLEDGMENTS

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APPENDIX A: WICK CONTRACTIONS FOR COUPLING CONSTANT AND STRING TENSION

We work out Eq. (3.4c) as an example:

$$(\mathbf{u}_i\cdot\mathbf{u}_j)(\mathbf{u}_{ij}\cdot\mathbf{u}_{kk})=\sum_{\alpha=1}^{d-2} u_i^\alpha u_j^\alpha(\mathbf{u}_{ij}\cdot\mathbf{u}_{kk})\to(d-2)\langle u_iu_j\rangle(\mathbf{u}_{ij}\cdot\mathbf{u}_{kk})=\tfrac{1}{2}(d-2)\alpha_0L_0\mathbf{u}_{ii}^2 . \quad (A1)$$

The other results in (3.4) follow just as easily. To obtain Eqs. (3.8) we observe that there are three contractions of two pairs of u's. We work out Eq. (3.8b):

$$\langle(u_i\cdot u_j)^2\rangle=\left\langle\sum_{\alpha,\beta=1}^{d-2} u_i^\alpha u_j^\alpha u_i^\beta u_j^\beta\right\rangle=(d-2)^2\langle u_iu_j\rangle^2+(d-2)\langle u_iu_i\rangle^2+(d-2)\langle u_iu_j\rangle^2=(d-2)(d-1)\langle u_iu_j\rangle^2+(d-2)\langle u_iu_i\rangle^2 . \quad (A2a)$$

Using Eqs. (3.2) we get

$$\langle(\mathbf{u}_i\cdot\mathbf{u}_j)^2\rangle=\tfrac{1}{2}(d-2)(d+1)(\alpha_0 L_0)^2 . \quad \text{(A2b)}$$

In this very simple way we can obtain the free energy of the infinite system and thus the static potential as given by Eqs. (3.10) and (3.11).

APPENDIX B: CALCULATING ONE-LOOP-$V(R)$ AND FINITE-SIZE CORRELATION FUNCTIONS

Here we work out in some detail two examples illustrating the way that the various quantities in Sec. IV are regulated.

First we regulate f_F as given by Eq. (4.4). We write Eq. (4.4) in the form

$$f_F=f_I+\Delta f^R , \quad \text{(B1)}$$

where

$$f_I=\frac{m_0^2}{4\pi}(1+4\pi L_0) , \quad \text{(B2a)}$$

$$\Delta f^R=\frac{1}{R}\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right] \times\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ln[(\omega^2+k_n^2)(\omega^2+k_n^2+m_0^2)] . \quad \text{(B2b)}$$

The integral over ω gives

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}[\ln(\omega^2+k_n^2)+\ln(\omega^2+k_n^2+m_0^2)] =(k_n^2)^{1/2}+(k_n^2+m_0^2)^{1/2} , \quad \text{(B3)}$$

where $k_n^2=n^2\pi^2/R^2$. The divergences in Δf^R cancel each other. To see this we write Δf^R as

$$\Delta f^R=\frac{\pi}{R^2}\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right]\{[2n+(n^2+\lambda_{0R})^{1/2}-n]\}. \quad \text{(B4)}$$

The first term gives

$$\frac{2\pi}{R^2}\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right]n=\frac{2\pi}{R}\left[\sum_{n=1}^{\infty}\frac{1}{n^{\nu}}\right]_{\nu=-1} =\frac{2\pi}{R}\zeta(-1)=-\frac{\pi}{6R^2} , \quad \text{(B5)}$$

where we have used the fact that $\int_0^\infty dn\ n$ vanishes in analytic regularization and $\sum_{n=1}^{\infty}n=\zeta(-1)$ follows directly by doing the analytic continuation via Riemann's zeta function.

The second term in (B4) is evaluated by carrying the integral up to some large but finite value $n=N$:

$$\int_0^N dn[(n^2+\lambda_R)^{1/2}-n]=\frac{\lambda_{0R}}{2}\ln N -\frac{\lambda_{0R}}{4}\ln\frac{\lambda_{0R}}{4e^1}+O\left[\frac{1}{N}\right] . \quad \text{(B6)}$$

We notice that

$$\sum_{n=1}^{N}\frac{1}{n}=\ln N+\gamma+O\left[\frac{1}{N}\right] \quad \text{(B7)}$$

combining these two results into the second term of (B4) gives

$$\frac{\pi}{R^2}\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right][(n^2+\lambda_{0R})^{1/2}-n] =\frac{\pi}{R^2}\sum^{\infty}\left[(n^2+\lambda_{0R})^{1/2}-\frac{\lambda_{0R}}{2n}\right]+\frac{m_0^2}{4\pi}\ln\frac{\lambda_{0R}}{4e^{-2\gamma+1}} . \quad \text{(B8)}$$

Thus Eq. (B1) becomes

$$f_F=\frac{m_0^2}{4\pi}(1+4\pi L_0)-\frac{\pi}{6R^2}+\frac{m_0^2}{\pi}S_1+\frac{m_0^2}{4\pi}\ln\frac{\lambda_{0R}}{4e^{-2\gamma+1}} =\frac{m_0^2}{8\pi}\left[-\frac{4}{3}\frac{1}{\lambda_{0R}}+2\bar{L}_0+8S_1\right] \quad \text{(B9)}$$

and the one-loop contribution to the static potential Eq. (4.6) follows. For the two-loop result we now calculate Eq. (4.10a):

$$\langle u_i u_i\rangle_F^2=(\langle u_i u_i\rangle_I+\langle u_i u_i\rangle_F-\langle u_i u_i\rangle_I)^2 , \quad \text{(B10)}$$

where

$$\langle u_i u_i\rangle_I=\alpha_0 L_0 , \quad \text{(B11a)}$$

$$\langle u_i u_i\rangle_F=\frac{\alpha_0}{2\pi R}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}d\omega\left[\frac{\omega^2}{k^2(k^2+m_0^2)}\left[1-\cos\frac{2n\pi r}{R}\right]+\frac{k_n^2}{k^2(k^2+m_0^2)}\left[1+\cos\frac{2n\pi r}{R}\right]\right] \quad \text{(B11b)}$$

doing the ω integration Eq. (B9) takes the form

$$\langle u_i u_i\rangle_F^2=\left[\frac{\alpha_0}{4\pi}\right]^2\left[4\pi L_0-4C_1(r)+2C_2(r)+2\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right]\frac{1}{(n^2+\lambda_{0R})^{1/2}}\right]^2 , \quad \text{(B12)}$$

where the $C_i(r)$ are defined in Eqs. (4.13). We evaluate now the $(\sum-\int)$ term. Using Eq. (B7),

$$\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right]\frac{1}{(n^2+\lambda_{0R})^{1/2}}=\lim_{N\to\infty}\left[\left[\sum_{n=1}^{N}-\int_0^{N}dn\right]\frac{1}{(n^2+\lambda_{0R})^{1/2}}\right]$$

$$=\lim_{N\to\infty}\left[\sum_{n=1}^{N}\left[\frac{1}{(n^2+\lambda_{0R})^{1/2}}-\frac{1}{n}\right]+\gamma+\frac{1}{2}\ln\frac{\lambda_{0R}}{4}+O\left[\frac{1}{N}\right]\right]$$

$$=S_2+\frac{1}{2}\ln\frac{\lambda_{0R}}{4e^{-2\gamma}}\ , \tag{B13}$$

where S_2 is the convergent sum defined by

$$S_2=\sum_{n=1}^{\infty}\left[\frac{1}{(n^2+\lambda_{0R})^{1/2}}-\frac{1}{n}\right] \tag{B14}$$

substituting Eq. (B13) into (B12) gives the result stated in Eq. (4.10a). The other results are obtained in a similar way.

APPENDIX C: REGULARIZING SINGULAR INTEGRALS VIA RIEMANN'S ZETA FUNCTION

The Fourier series $C_i(r)$ and its derivatives are somewhat pathological due to convergence problems. In Ref. 10 the regularized form of the solutions to the saddle-point equations with $\beta\neq 0$ were used, and the analytic continuation to $\beta=0$ was taken after the integral over r. Here, something similar is done, although the results are obtained with somewhat less effort. We evaluate Eqs. (4.15d) and (4.15e), as examples:

$$\frac{1}{R}\int_0^R dr[C_2(r)C_3(r)]=\frac{1}{R}\int_0^R dr\frac{1}{\lambda_{0R}}\sum_{n,m=1}^{\infty}\frac{m}{(n^2+\lambda_{0R})^{1/2}}\cos\frac{2n\pi r}{R}\cos\frac{2m\pi r}{R}$$

$$=\frac{1}{2}\frac{1}{\lambda_{0R}}\sum_{n=1}^{\infty}\frac{n}{(n^2+\lambda_{0R})^{1/2}}$$

$$=\frac{1}{2\lambda_{0R}}\sum_{n=1}^{\infty}\left[\frac{n}{(n^2+\lambda_{0R})^{1/2}}-1\right]+\frac{1}{2\lambda_{0R}}\sum_{n=1}^{\infty}1$$

$$=\tfrac{1}{2}S_4+\tfrac{1}{2}\zeta(0)\frac{1}{\lambda_{0R}}\ . \tag{C1}$$

$\sum_{n=1}^{\infty}1$ is calculated by doing the analytic continuation via Riemann's zeta function:

$$\sum_{n=1}^{\infty}1=\left[\sum_{n=1}^{\infty}\frac{1}{n^{\nu}}\right]_{\nu=0}=\zeta(0)\ . \tag{C2}$$

In a similar way,

$$\frac{1}{Rm_0^2}\int_0^R dr[C_2'(r)]^2=\left[\frac{1}{Rm_0^2}\right]\left[\frac{2\pi}{R}\right]^2\int_0^R dr\sum_{n,m=1}^{\infty}\frac{n}{(n^2+\lambda_{0R})^{1/2}}\frac{m}{(m^2+\lambda_{0R})^{1/2}}\sin\frac{2n\pi r}{R}\sin\frac{2m\pi r}{R}$$

$$=\frac{2}{\lambda_{0R}}\sum_{n=1}^{\infty}\left[1-\frac{\lambda_{0R}}{n^2+\lambda_{0R}}\right]=2\zeta(0)\frac{1}{\lambda_{0R}}-2S_3\ , \tag{C3}$$

where $\zeta(0)=-\frac{1}{2}$.

APPENDIX D: LARGE- AND SMALL-λ_R EXPANSIONS OF THE SUMS S_i AND FUNCTION $C_j(r)$

Here we give in some detail the large- and small-λ_R limits of the sums S_i and the functions $C_i(r)$ used in the text.

To study the large-distance behavior we consider the divergent sums introduced in Appendix B:

$$\tilde{S}_1\equiv\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right](n^2+\lambda_R)^{1/2}\ , \tag{D1a}$$

$$\tilde{S}_2\equiv\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right]\frac{1}{(n^2+\lambda_R)^{1/2}}\ ; \tag{D1b}$$

we proceed in two stages. First, we calculate $\tilde{S}_i$ as in Appendix B with the results

$$\tilde{S}_1=\lambda_R S_1+\frac{\lambda_R}{4}(L_R-1)-\tfrac{1}{12}\ , \tag{D2a}$$

$$\tilde{S}_2=S_2+\tfrac{1}{2}L_R\ . \tag{D2b}$$

Second, we continue $\tilde{S}_i$ analytically to some $\tilde{S}_i(\nu)$ defined by

$$\tilde{S}_i(\nu)\equiv\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right][(n^2+\lambda_n)^{1/2}]^{-2\nu}$$
$$=-\tfrac{1}{2}(\sqrt{\lambda_R})^{-2\nu}+\frac{2^{-\nu-1}}{\Gamma(\nu)}\int_0^{\infty}\frac{d\tau}{\tau}\tau^{\nu}\left[\sum_{n=-\infty}^{\infty}-\int_{-\infty}^{\infty}dn\right]\times\exp[(-\tau/2)(n^2+\lambda_R)]\,. \quad \text{(D3)}$$

Resumming the $\tilde{S}_i(\nu)$ via Poisson sum formula

$$\sum_{n=-\infty}^{\infty}f(an)=\sum_{\tilde{n}=-\infty}^{\infty}\int_{-\infty}^{\infty}f(an)\exp(2\pi in\tilde{n})dn \quad \text{(D4)}$$

and introducing the modified Bessel functions with the help of their integral representation

$$K_\nu(xz)=\frac{z^\nu}{2}\int_0^{\infty}\frac{d\tau}{\tau}\tau^{-\nu}\exp[(-x/2)(\tau+z^2/\tau)] \quad \text{(D5)}$$

we get

$$\tilde{S}_i(\nu)=\left[\sum_{n=1}^{\infty}-\int_0^{\infty}dn\right][(n^2+\lambda_R)^{1/2}]^{-2\nu}$$
$$=-\frac{1}{2}(\sqrt{\lambda_R})^{-2\nu}+\frac{2\pi^\nu}{\Gamma(\nu)}(\sqrt{\lambda_R})^{-\nu+1/2}\times\sum_{\tilde{n}=1}^{\infty}\frac{K_{-\nu+1/2}(2\pi\tilde{n}\sqrt{\lambda_R})}{\tilde{n}^{-\nu+1/2}}\,. \quad \text{(D6)}$$

Thus

$$\tilde{S}_1=-\frac{1}{2}\sqrt{\lambda_R}-\frac{1}{\pi}\sqrt{\lambda_R}\sum_{\tilde{n}=1}^{\infty}\frac{K_1(2\pi\tilde{n}\sqrt{\lambda_R})}{\tilde{n}}\,, \quad \text{(D7a)}$$

$$\tilde{S}_2=-\frac{1}{2}\frac{1}{\sqrt{\lambda_R}}+2\sum_{\tilde{n}=1}^{\infty}K_0(2\pi\tilde{n}\sqrt{\lambda_R})\,. \quad \text{(D7b)}$$

Comparing with Eqs. (D2) it follows that

$$S_1=\tfrac{1}{4}-\tfrac{1}{4}L_R-\frac{1}{2}\frac{1}{\sqrt{\lambda_R}}+\frac{1}{12}\frac{1}{\lambda_R}-\frac{1}{\pi}\frac{1}{\sqrt{\lambda_R}}\sum_{\tilde{n}=1}^{\infty}\frac{K_1(2\pi\tilde{n}\sqrt{\lambda_R})}{\tilde{n}}\,, \quad \text{(D8a)}$$

$$S_2=-\tfrac{1}{2}L_R-\frac{1}{2}\frac{1}{\sqrt{\lambda_R}}+2\sum_{\tilde{n}=1}^{\infty}K_0(2\pi\tilde{n}\sqrt{\lambda_R})\,. \quad \text{(D8b)}$$

In the same way it can be shown that

$$\sum_{n=1}^{\infty}[(n^2+\lambda_R)^{1/2}]^{-2\nu}=-\tfrac{1}{2}(\sqrt{\lambda_R})^{-2\nu}+\frac{1}{2}\frac{\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\sqrt{\pi}(\sqrt{\lambda_R})^{-2\nu+1}+\frac{2\pi^\nu}{\Gamma(\nu)}(\sqrt{\lambda_R})^{-\nu+1/2}\times\sum_{\tilde{n}=1}^{\infty}\frac{K_{-\nu+1/2}(2\pi\tilde{n}\sqrt{\lambda_R})}{\tilde{n}^{-\nu+1/2}}\,. \quad \text{(D9)}$$

Thus, we recover, for $\nu=1$, the sum S_3 of Eq. (4.16a):

$$S_3=\frac{1}{2}\frac{\pi}{\sqrt{\lambda_R}}-\frac{1}{2}\frac{1}{\lambda_R}+\frac{2\pi}{\lambda_R^{1/4}}\sum_{\tilde{n}=1}^{\infty}\frac{K_{-1/2}(2\pi\tilde{n}\sqrt{\lambda_R})}{\tilde{n}^{-1/2}}$$
$$=\frac{1}{2}\frac{\pi}{\sqrt{\lambda_R}}-\frac{1}{2}\frac{1}{\lambda_R}+\frac{\pi}{\sqrt{\lambda_R}}\sum_{\tilde{n}=1}^{\infty}\exp(-2\pi\tilde{n}\sqrt{\lambda_R})\,. \quad \text{(D8c)}$$

The sums S_4,S_5 require some more work. We introduce a Feynman parametrization for the product $(n^2)^{-\alpha}(n^2+\lambda_R)^{-\beta}$ and then use Eq. (D9); thus,

$$\sum_{n=1}^{\infty}(n^2)^{-\alpha}(n^2+\lambda_R)^{-\beta}=\frac{1}{2}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dx(1-x)^{\alpha-1}x^{\beta-1}\left\{\sqrt{\pi}\Gamma(\nu-\tfrac{1}{2})(\sqrt{\lambda_R x})^{-2\nu+1}+4\pi^\nu(\sqrt{\lambda_R}x)^{-\nu+1/2}\times\sum_{\tilde{n}=1}^{\infty}\frac{K_{-\nu+1/2}(2\pi\tilde{n}\sqrt{\lambda_R x})}{\tilde{n}^{-\nu+1/2}}\right\}. \quad \text{(D10)}$$

It is now easy to work out the corresponding expressions for S_4 and S_5. The results are

$$S_4=-\frac{1}{\sqrt{\lambda_R}}-\zeta(0)\frac{1}{\lambda_R}-\frac{1}{\pi}\frac{1}{\lambda_R}\int_0^1 dx\frac{(\lambda_R x)^{1/4}}{\sqrt{x(1-x)^3}}\sum_{\tilde{n}=1}^{\infty}\frac{K_{1/2}(2\pi\tilde{n}\sqrt{\lambda_R x})}{\tilde{n}^{1/2}}$$
$$=-\frac{1}{\sqrt{\lambda_R}}-\zeta(0)\frac{1}{\lambda_R}-\frac{1}{2\pi}\frac{1}{\lambda_R}\int_0^1\frac{dx}{\sqrt{x(1-x)^3}}\sum_{\tilde{n}=1}^{\infty}\frac{e^{-2\pi\tilde{n}\sqrt{\lambda_R x}}}{\tilde{n}}\,, \quad \text{(D8d)}$$

$$S_5=\zeta(0)\frac{1}{\lambda_R}-\frac{1}{4\pi}\frac{1}{\lambda_R^2}\int_0^1\frac{dx}{\sqrt{x^3(1-x)^3}}\left[\tfrac{4}{3}\pi(\lambda_R x)^{3/2}+\frac{4}{\pi}(\lambda_R x)^{3/4}\sum_{n=1}^{\infty}\frac{K_{3/2}(2\pi\tilde{n}\sqrt{\lambda_R x})}{\tilde{n}^{3/2}}\right]$$
$$=\zeta(0)\frac{1}{\lambda_R}-\frac{1}{4\pi}\frac{1}{\lambda_R^2}\int_0^1\frac{dx}{\sqrt{x^3(1-x)^3}}\left[\tfrac{4}{3}\pi(\lambda_R x)^{3/2}+\frac{2}{\pi}\sqrt{\lambda_R x}\sum_{\tilde{n}=1}^{\infty}\frac{e^{-2\pi\tilde{n}\sqrt{\lambda_R x}}}{\tilde{n}^2}\left[1+\frac{1}{2\pi\tilde{n}\sqrt{\lambda_R x}}\right]\right]. \quad \text{(D8e)}$$

The large-λ_R limit of the functions $C_i(r)$ can be obtained by following the steps leading to Eq. (D9) above. The analogous result is

$$\sum_{n=1}^{\infty}[(n^2+\lambda_R)^{1/2}]^{-2\nu}\cos 2n\pi\phi=-\tfrac{1}{2}(\sqrt{\lambda_R})^{-2\nu}+\frac{\pi^\nu}{\Gamma(\nu)}\left[\frac{\sqrt{\lambda_R}}{\phi}\right]^{-\nu+1/2}K_{\nu-1/2}(2\pi\phi\sqrt{\lambda_R})$$
$$+\frac{\pi^\nu}{\Gamma(\nu)}(\sqrt{\lambda_R})^{-\nu+1/2}\sum_{\tilde n=1}^{\infty}\left[\frac{K_{-\nu+1/2}(2\pi(\tilde n+\phi)\sqrt{\lambda_R})}{(\tilde n+\phi)^{-\nu+1/2}}+(\phi\to\phi)\right]. \tag{D11}$$

It follows that, in terms of Bessel functions, the $C_i(r)$ are given by

$$C_1(r)=-\frac{1}{2}\frac{1}{\sqrt{\lambda_R}}-\frac{1}{2\pi}\frac{1}{\phi\sqrt{\lambda_R}}K_{-1}(2\pi\phi\sqrt{\lambda_R})$$
$$-\frac{1}{2\pi}\frac{1}{\sqrt{\lambda_R}}\times\sum_{\tilde n=1}^{\infty}\left[\frac{K_1(2\pi(\tilde n+\phi)\sqrt{\lambda_R})}{\tilde n+\phi}+(\phi\to-\phi)\right]$$
$$+\frac{1}{4\pi^2}\frac{1}{\lambda_R}\left[\zeta(2,\phi)-\frac{1}{\phi^2}-(\phi\to-\phi)\right], \tag{D12a}$$

$$C_2(r)=-\frac{1}{2}\frac{1}{\sqrt{\lambda_R}}+K_0(2\pi\phi\sqrt{\lambda_R})$$
$$+\sum_{\tilde n=1}^{\infty}[K_0(2\pi(\tilde n+\phi)\sqrt{\lambda_R})+(\phi\to-\phi)], \tag{D12b}$$

$$C_3(r)=-\frac{1}{4\pi^2}\frac{1}{\lambda_R}\left[\zeta(2,\phi)-\frac{1}{\phi^2}+(\phi\to-\phi)\right], \tag{D12c}$$

where $\phi=r/R$ and

$$\zeta(z,q)=\sum_{\tilde n=0}^{\infty}(\tilde n+q)^{-z}\quad(\mathrm{Re}z>1) \tag{D13}$$

is the generalized Riemann ζ function (also called the Hurwitz function).

The short-distance behavior is obtained after a straightforward application of the binomial expansion. The results are

$$S_1=-\tfrac{1}{8}\zeta(3)\lambda_R+\tfrac{1}{16}\zeta(5)\lambda_R^2-\tfrac{5}{128}\zeta(7)\lambda_R^3+\cdots, \tag{D14a}$$
$$S_2=-\tfrac{1}{2}\zeta(3)\lambda_R+\tfrac{3}{8}\zeta(5)\lambda_R^2-\tfrac{5}{16}\zeta(7)\lambda_R^3+\cdots, \tag{D14b}$$
$$S_3=\zeta(2)-\zeta(4)\lambda_R+\zeta(6)\lambda_R^2-\zeta(8)\lambda_R^3+\cdots, \tag{D14c}$$
$$S_4=-\tfrac{1}{2}\zeta(2)+\tfrac{3}{8}\zeta(4)\lambda_R-\tfrac{5}{16}\zeta(6)\lambda_R^2+\tfrac{35}{128}\zeta(8)\lambda_R^3-\cdots, \tag{D14d}$$
$$S_5=\tfrac{1}{4}\zeta(2)-\tfrac{1}{8}\zeta(4)\lambda_R+\tfrac{5}{64}\zeta(6)\lambda_R^2-\tfrac{7}{128}\zeta(8)\lambda_R^3+\cdots. \tag{D14e}$$

The functions $C_i(r)$ have, for small R, the equation

$$C_1(r)=\tfrac{1}{2}C(1,nx)-\tfrac{1}{8}C(3,nx)\lambda_R+\tfrac{1}{16}C(5,nx)\lambda_R^2-\tfrac{5}{128}C(7,nx)\lambda_R^3+\cdots, \tag{D15a}$$

$$C_2(r)=C(1,nx)-\tfrac{1}{2}C(3,nx)\lambda_R+\tfrac{3}{8}C(5,nx)\lambda_R^2-\tfrac{5}{16}C(7,nx)\lambda_R^3+\cdots, \tag{D15b}$$

$$C_3(r)=C(-1,nx)\frac{1}{\lambda_R}, \tag{D15c}$$

where $C(s,x)\equiv\sum_{n=1}^{\infty}n^{-s}\cos nx$ and $x\equiv 2\pi r/R$.

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