Exact temperature behavior of strings with extrinsic curvature stiffness in the limit of infinite dimensions. Thermal deconfinement transition

H. Kleinert
Institut für Theorie der Elementarteilchen, Freie Universität Berlin, Arnimallee 14, 1000 Berlin 33, West Germany
(Received 23 March 1987; revised manuscript received 14 November 1988)

We present the exact solution for the temperature dependence of the tension of a string with extrinsic curvature stiffness in the limit of infinite dimensions. The solution possesses an anisotropic gap parameter and exhibits a deconfinement transition at a temperature T^d that depends only very weakly on the stiffness.

I. INTRODUCTION

The inclusion of an extrinsic curvature term into the string action by Polyakov[1] and the author[2] has led to an interesting generalization of bosonic strings. The new strings seem to be a much better surface representation of the string between quarks in QCD than the former Nambu-Goto string. The most attractive features are asymptotic freedom,[1,2] spontaneous generation of string tension,[3] a realistic quark potential $V(R)$ (Refs. 4–6) [the experimental short-distance behavior $V(R) \propto -0.52/R$ is predicted to be $-\pi/6R \approx -0.5236/R$!) (Refs. 6 and 7), and glueballs.[8] Some time ago we showed in a Letter[9] that, in the limit of infinite dimensions, a *purely spontaneous string* (which is a string that starts life without any Nambu-Goto tension at all) has a thermal deconfinement transition[10] at a temperature roughly where it is expected from Monte Carlo simulations of lattice QCD (Ref. 11). The purpose of this paper is to give a detailed derivation of this result and to extend the calculation to arbitrary combinations of initial tension and an extrinsic curvature term. We shall derive the exact equations for the temperature dependence of the string tension and present numerical solutions as well as analytic approximations.

A consequence of our equations will be that, whatever the mixture of the two terms, the power-series expansions in the temperature do not differ from the pure Nambu-Goto case (there are only exponentially small differences which decrease like $e^{-M/T}$ for small T). In particular, the first nonleading linear term in T has the well-known universal form that corresponds to a two-dimensional blackbody radiation in a box with periodic boundary conditions. It is independent of the extrinsic curvature stiffness, just as the corresponding term in the string tension[12] and in the entropy of large spherical surfaces.[13]

II. THE THERMAL DECONFINEMENT TRANSITION

In a thermal environment, above a certain temperature T^d called the deconfinement temperature, the vacuum of a non-Abelian theory loses its quark-confining property. In SU(3), the transition is of first order as long as the quarks are assumed to be infinitely massive. If the gauge group were SU(2), the deconfinement transition would be continuous. Physically, the distinction is easy to understand. At larger temperatures T the world sheets swept out by the strings acquire a finite temporal width and their Fourier frequencies $\omega_m = 2\pi m T$ are discrete. The Fourier components associated with ω_0 are strings with *no* fluctuations along the time direction; i.e., they are equivalent to polymers in three dimensions. In statistical mechanics it is well known that an ensemble of such objects is dually equivalent to a three-dimensional spin model. In color-SU(N) models, the strings can branch off each other in multiples of N. Correspondingly, the spin model should have as many components as there are colors in SU(N). The diagrams in the high-temperature expansion of the spin model describe the random strings that are relevant at finite temperatures in the gauge theory. In SU(3), where the strings have branch points at which three lines meet, the field-theoretic formulation of the corresponding spin model contains a cubic term in the disorder field. This is responsible for the first order of the transition. In SU(2), where such triple branch points are absent, the transition remains of second order.

Quarks of finite mass allow for the thermal creation of strings with open ends. In the analogous spin system this corresponds to adding a finite external magnetic field. It is well known that such a field wipes out the transition if it is continuous, or decreases its strength, if it is of first order, until the transition disappears. In the absence of dynamical quarks, the thermal deconfinement transition is the result of their strings between quarks becoming infinitely long, due to their overwhelming configurational entropy. This transition happens already at the level of a single string. The fact that every string is embedded in a grand canonical ensemble of closed strings is expected to lead mainly to a slight change of the critical temperature and the critical indices, with respect to those of a single string. This is the experience in polymer physics where a single polymer is described by an O(n) spin model with $n=0$ while an ensemble has $n=1$.

It is the purpose of this paper to study the thermal deconfinement transition as it arises already in a single string, thereby ignoring the effect of the remaining ensemble. We shall see a signal for the transition in the vanishing of the tension at a certain temperature. Since the string tension depends only very weakly on the stiffness parameter, so does the thermal deconfinement temperature, when expressed as a fraction of the string tension.

III. ACTION AND FREE ENERGY AT FINITE T

Consider a string in the form of an infinitely long planar world sheet disturbed by small undulations. In the presence of tension and an extrinsic curvature stiffness, the action is

$$\mathcal{A}=M_{\rm NG}^2\int d^2\xi\sqrt{g}+\frac{1}{2\alpha}\int d^2\xi\sqrt{g}\,[(D^2x^a)^2+\lambda^{ij}(\partial_i x^a\partial_j x^a-g_{ij})]\ , \tag{1}$$

where $x^a(\xi^1,\xi^2)$, $a=1,\ldots,d$ describes the world sheet of the string, the Lagrange multiplier λ^{ij} ensures the correct intrinsic metric $g_{ij}=\partial_i x^a\partial_j x^a$, and D is the covariant derivative. The constants $M_{\rm NG}^2$ and $1/\alpha$ are the Nambu-Goto string tension and the stiffness, respectively. In the limit $d\to\infty$, the x^a variables can be integrated out and the action is given by the extremum of the resulting effective action in g^{ij}, λ^{ij}. At a finite temperature $T_{\rm ext}$, it has a finite extent in the imaginary-time direction, $\tau\in(\beta_{\rm ext}=1/T_{\rm ext})$, with periodic boundary conditions. The subscript ext indicates that a quantity is defined in the extrinsic space x^a. Since space and time directions no longer appear on equal footing, we expect the extremum to have an anisotropic metric

$$g_{ij}=\begin{bmatrix}\rho_0 & 0\\ 0 & \rho_1\end{bmatrix} \tag{2a}$$

with $\rho_0\neq\rho_1$. Correspondingly, we shall parametrize the extremal matrix λ^{ij} as

$$\lambda^{ij}=\begin{bmatrix}\lambda_0/\rho_0 & 0\\ 0 & \lambda_1/\rho_1\end{bmatrix}\ . \tag{2b}$$

Because of the periodic boundary conditions, we expect a saddle point with constant $\rho_0,\rho_1,\lambda_0,\lambda_1$. The periodicity implies that the timelike momenta are discrete. The spectrum of the operator

$$-D^2=-\left[\frac{1}{\rho_0}\partial_0^2+\frac{1}{\rho_1}\partial_1^2\right] \tag{3a}$$

is

$$k_0^2/\rho_0+k_1^2/\rho_1 \tag{3b}$$

with discrete frequencies $k_0=(k_0)_m$

$$(k_0)_m=2\pi T_{\rm ext}m=(2\pi/\beta_{\rm ext})m\ ,\quad m=0,\pm1,\pm2,\ldots, \tag{4}$$

and continuous k_1. The effective action following from (1) after integrating out the x^a fluctuation can therefore be written as

$$\mathcal{A}_{\rm fl}=\frac{d-2}{2}R_{\rm ext}\sum_{m=-\infty}^{\infty}\int\frac{dk_1}{2\pi}\ln\{[(k_0)_m^2/\rho_0+k_1^2/\rho_1]^2+(\lambda_0/\rho_0)(k_0)_m^2+(\lambda_1/\rho_1)k_1^2\}+\int d^2\xi\sqrt{\rho_0\rho_1}\left[-\frac{1}{2\tilde{\alpha}}(\lambda_0+\lambda_1-\lambda_0/\rho_0-\lambda_1/\rho_1)\right]\ . \tag{5}$$

where we have omitted, for a moment, the first term in (1), the Nambu-Goto term. We shall abbreviate (5) as

$$\mathcal{A}_{\rm fl}=\frac{d-2}{2}R_{\rm ext}\beta_{\rm ext}\sqrt{\rho_0\rho_1}f \tag{6}$$

with a "free energy density"

$$f=\frac{1}{\sqrt{\rho_0\rho_1}}T_{\rm ext}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\ln\{[(k_0)_m^2/\rho_0+k_1^2/\rho_1]^2+(\lambda_0/\rho_0)(k_0)_m^2+(\lambda_1/\rho_1)k_1^2\}-\frac{\lambda_0+\lambda_1}{2\tilde{\alpha}}+\frac{1}{2\tilde{\alpha}}\left[\frac{\lambda_0}{\rho_0}+\frac{\lambda_1}{\rho_1}\right] \tag{7}$$

We now introduce the *intrinsic* frequencies and momenta

$$\omega_m=(k_0)_m/\sqrt{\rho_0}=2\pi Tm,\quad m=0,\pm1,\pm2,\ldots,\quad q_1=k_1/\sqrt{\rho_1}\ . \tag{8}$$

The quantity

$$\beta=\frac{1}{T}=\frac{1}{T_{\rm ext}}\sqrt{\rho_0}=\beta_{\rm ext}\sqrt{\rho_0} \tag{9}$$

measures the intrinsic size of the world sheet in the time direction.

Adding the Nambu-Goto term, we are then faced with the action (6) with f replaced by

$$f_{\rm tot}=\tilde{M}_{\rm NG}^2+f=\tilde{M}_{\rm NG}^2+T\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\ln[(\omega_m^2+q_1^2)^2+\lambda_0\omega_m^2+\lambda_1 q_1^2]-\frac{\lambda_0+\lambda_1}{2\tilde{\alpha}}+\frac{1}{2\tilde{\alpha}}\left[\frac{\lambda_0}{\rho_0}+\frac{\lambda_1}{\rho_1}\right]\ , \tag{10}$$

where we have introduced the abbreviation $\tilde{M}^2_{\rm NG} \equiv 2M^2_{\rm NG}/(d-2)$. The expression for $f_{\rm tot}$ can be conveniently split into the terms

$$f_{\rm tot} = \tilde{M}^2_{\rm NG} + \int \frac{d^2q}{(2\pi)^2} \ln(q^4 + \tilde{\lambda} q^2) - \frac{\tilde{\lambda}}{\tilde{\alpha}} + \frac{1}{2\tilde{\alpha}} \left[\frac{\lambda_0}{\rho_0} + \frac{\lambda_1}{\rho_1} \right] + \Delta f^T + \Delta f^\delta , \tag{11}$$

where

$$\tilde{\lambda} = (\lambda_0 + \lambda_1)/2 \tag{12}$$

is the average gap,

$$\Delta f^T = \left[T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \right] \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \ln[(\omega_m^2 + q_1^2)^2 + \tilde{\lambda}(\omega_m^2 + q_1^2)] \tag{13}$$

is the finite-size correction at an isotropic average gap $\tilde{\lambda}$, and

$$\Delta f^\delta = T \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \{ \ln[(\omega_m^2 + q_1^2)^2 + \lambda_0 \omega_m^2 + \lambda_1 q_1^2] - \ln[(\omega_m^2 + q_1^2)^2 + \tilde{\lambda}(\omega_m^2 + q_1^2)] \} \tag{14}$$

is the correction due to the anisotropy of the gap. We shall introduce the gap anisotropy as

$$\delta \equiv (\lambda_1 - \lambda_0)/2\tilde{\lambda} . \tag{15}$$

The sums and integrals diverge and have to be regularized. We do this within the infinite system where Δf^T and Δf^δ are absent and we have to deal only with the first four terms in (11). They can immediately be expressed in terms of a dimensionally transmuted coupling constant $\bar{\lambda}$ as

$$\begin{aligned} f^0_{\rm tot} &= \tilde{M}^2_{\rm NG} + f_0(\tilde{\lambda}) - \frac{\tilde{\lambda}}{4\pi} \\ &= \tilde{M}^2_{\rm NG} - \frac{\tilde{\lambda}}{4\pi}[\ln(\lambda/\bar{\lambda}) - 1] - \frac{\tilde{\lambda}}{4\pi} . \end{aligned} \tag{16}$$

The total free energy density is then given by the finite expression

$$\begin{aligned} f^T_{\rm tot} &= \tilde{M}^2_{\rm NG} + f_0(\tilde{\lambda}) - \frac{\tilde{\lambda}}{4\pi} + \Delta f^T + \Delta f^\delta \\ &\quad + \frac{1}{2\tilde{\alpha}}(\lambda_0/\rho_0 + \lambda_1/\rho_1) . \end{aligned} \tag{17}$$

IV. CALCULATION OF FINITE T TERMS $\Delta f^T, \Delta f^\delta$

The term Δf^T is a finite-size correction for a system with a hypothetical isotropic gap $\lambda^0 = \lambda^1 = \tilde{\lambda}$. We make use of the integral, valid after analytic regularization,

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \ln(q^2 + a^2) = \sqrt{a^2} . \tag{18}$$

There is no divergent part if we follow the standard rules of evaluating Feynman integrals via the proper-time formalism level: i.e.,

$$\begin{aligned} \int \frac{dq}{2\pi} \ln(q^2 + a^2) &= -\int \frac{d\omega}{2\pi} \int_0^\infty \frac{d\tau}{\tau} e^{-(\tau/2)(q^2 + a^2)} \\ &= -\frac{1}{\sqrt{2\pi}} \int \frac{d\tau}{\tau} \tau^{-1/2} e^{-(\tau/2)a^2} \\ &= -\frac{1}{\sqrt{2\pi}} \sqrt{a^2/2}\,\Gamma(-\tfrac{1}{2}) \\ &= \sqrt{a^2} . \end{aligned}$$

We therefore rewrite Δf^T as

$$\begin{aligned} \Delta f^T(\lambda, T) &= \left[T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right] \\ &\quad \times [(\omega_m^2 + \tilde{\lambda})^{1/2} + (\omega_m^2)^{1/2}] \\ &= \left[T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right] \\ &\quad \times [2(\omega_m^2)^{1/2} + (\omega_m^2 + \tilde{\lambda})^{1/2} - (\omega_m^2)^{1/2}] . \end{aligned} \tag{19}$$

The first expression

$$\left[T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right] 2(\omega_m^2)^{1/2} \tag{20}$$

is again treated by analytic regularization. It is equal to

$$2\frac{2^{-\nu}}{\Gamma(\nu)} \int_0^\infty \frac{d\tau}{\tau} \tau^\nu \left[T \sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \right] e^{-(\tau/2)\omega_m^2} \tag{21}$$

with ν continued to $-\frac{1}{2}$. Using the duality transformation

$$T \sum_{m=-\infty}^{\infty} e^{-(\tau/2)\omega_m^2} = \frac{1}{\sqrt{2\pi\tau}} \sum_{\tilde{m}=-\infty}^{\infty} e^{-(\tilde{m}^2/2\tau T^2)} \tag{22}$$

we can write

$$\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right]e^{-(\tau/2)\omega_m^2}$$
$$=\frac{2}{\sqrt{2\pi\tau}}\sum_{\tilde{m}=-\infty}^{\infty}e^{-(\tilde{m}^2/2\tau T^2)} \quad (23)$$

and the expression (20) becomes

$$2\frac{2^{-\nu}}{\Gamma(\nu)}\frac{2}{\sqrt{2\pi}}\int_0^\infty\frac{d\tau}{\tau}\tau^{\nu-(1/2)}\sum_{\tilde{m}=1}^{\infty}e^{-(\tilde{m}^2/2\tau T^2)}$$
$$=2\times2^{-\nu}\frac{\Gamma(\frac{1}{2}-\nu)}{\Gamma(\nu)}\frac{2}{\sqrt{2\pi}}\sum_{\tilde{m}=1}^{\infty}(T^2\tilde{m}^2/2)^{\nu-(1/2)}$$
$$\underset{\nu\to-1/2}{\sim}-\frac{4T^2}{\pi}\sum_{\tilde{m}=1}^{\infty}\frac{1}{\tilde{m}^2}=-\tfrac{2}{3}\pi T^2 . \quad (24)$$

This result could have been obtained directly by performing an analytic continuation of Riemann's ζ function

$$\zeta(\nu)=\sum_{m=1}^{\infty}\frac{1}{m^\nu} \quad (25)$$

in terms of which

$$2\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right]\omega_m$$
$$=8\pi T^2\left[\sum_{m=1}^{\infty}-\int_0^\infty\frac{d\omega_m}{2\pi}\right]m$$
$$=8\pi T^2\zeta(-1)=-\tfrac{2}{3}\pi T^2 . \quad (26)$$

Let us now turn to the remaining piece in (19). The sum and the integral over $(\omega_m^2+\tilde{\lambda})^{1/2}$ and $(\omega_m^2)^{1/2}$ both diverge logarithmically, but the difference is finite. The subtraction is made explicit by carrying the sum up to some very large but finite value $m=N$ and adding the identity

$$0=-2T\sum_{m=1}^{N}\frac{\tilde{\lambda}}{2\omega_m}+\frac{\tilde{\lambda}}{2\pi}\sum_{m=1}^{N}\frac{1}{m}$$
$$\approx-2T\sum_{m=1}^{N}\frac{\tilde{\lambda}}{2\omega_m}+\frac{\tilde{\lambda}}{2\pi}(\ln N+\gamma) . \quad (27a)$$

When taking the integral up to the same large value

$$2\int_0^{\omega_N}\frac{d\omega_m}{2\pi}[(\omega_m^2+\tilde{\lambda})^{1/2}-(\omega_m^2)^{1/2}]$$
$$=\frac{1}{2\pi}[\omega_N(\omega_N^2+\tilde{\lambda})^{1/2}-\omega_N^2+\tilde{\lambda}\,\text{arcsinh}(\omega_N/\sqrt{\tilde{\lambda}})]$$
$$\approx\frac{\tilde{\lambda}}{4\pi}+\frac{\tilde{\lambda}}{2\pi}[\ln N+\ln(4\pi T/\sqrt{\tilde{\lambda}})] \quad (27b)$$

the $\ln N$ terms cancel and we obtain

$$\Delta f^T(\tilde{\lambda},T)=-\tfrac{2}{3}\pi T^2+T\sqrt{\tilde{\lambda}}$$
$$-\frac{\tilde{\lambda}}{4\pi}[\ln(16\pi^2 e^{2\gamma}T^2/\tilde{\lambda})+1]+\frac{\tilde{\lambda}}{\pi}S_1 , \quad (28a)$$

where we have introduced the sum

$$S_1\equiv\frac{2\pi T}{\tilde{\lambda}}\sum_{m=1}^{\infty}\left[(\omega_m^2+\tilde{\lambda})^{1/2}-\omega_m-\frac{\tilde{\lambda}}{2\omega_m}\right]$$
$$=\frac{1}{\tilde{\lambda}_T}\sum_{m=1}^{\infty}[(m^2+\tilde{\lambda}_T)^{1/2}-m-\tilde{\lambda}_T/2m] . \quad (28b)$$

It is a function of the dimensionless quantity

$$\tilde{\lambda}_T\equiv\tilde{\lambda}/4\pi^2T^2 . \quad (28c)$$

Inserting Δf^T into (17) and ignoring Δf^δ gives the renormalized free energy density for an isotropic gap $\tilde{\lambda}$:

$$f_{\text{iso}}^T(\tilde{\lambda},T)=-\frac{2\pi}{3}T^2+T\sqrt{\tilde{\lambda}}+\frac{\tilde{\lambda}}{\pi}S_1$$
$$-\frac{\tilde{\lambda}}{4\pi}\ln(T^2/\bar{T}^2) , \quad (29)$$

where we have introduced the natural temperature scale of the system

$$\bar{T}\equiv\sqrt{\tilde{\lambda}}/4\pi e^{-\gamma} . \quad (30)$$

The sum over m in S_1 converges rapidly for small $\tilde{\lambda}_T$. If $\tilde{\lambda}_T$ becomes large it is better to derive another representation for $\Delta f^T(\tilde{\lambda}_T)$, starting out directly from

$$\Delta f^T(\tilde{\lambda},T)=\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right]$$
$$\times[(\omega_m^2+\tilde{\lambda})^{1/2}+(\tilde{\lambda}=0)] . \quad (31)$$

For this we treat $(\omega_m^2+\tilde{\lambda})^{1/2}$ in the same way as $(\omega_m^2)^{1/2}$ in (20). We write

$$\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right](\omega_m^2+\tilde{\lambda})^{1/2}$$
$$=\lim_{\nu\to-1/2}\frac{2^{-\nu}}{\Gamma(\nu)}\frac{2}{\sqrt{2\pi}}\int\frac{d\tau}{\tau}\tau^{\nu-(1/2)}$$
$$\times\sum_{\tilde{m}=1}^{\infty}\exp\left[\frac{\tilde{m}^2}{2\tau T^2}-\frac{\tau}{2}\tilde{\lambda}\right] . \quad (32)$$

Then we use the formula

$$\int_0^\infty\frac{d\tau}{\tau}\tau^{-1}\exp\left[\frac{\gamma}{2\tau}-\frac{\tilde{\lambda}}{2}\tau\right]=2\sqrt{\tilde{\lambda}/\gamma}K_1(\sqrt{\gamma\tilde{\lambda}}) , \quad (33)$$

where $K_1(z)$ is the modified Bessel function, and we have

$$\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right]\sqrt{\omega_m^2+\tilde{\lambda}}$$
$$=-\frac{\tilde{\lambda}}{4\pi}8K_1(2\pi\tilde{m}\sqrt{\tilde{\lambda}_T})/2\pi\tilde{m}\sqrt{\tilde{\lambda}_T} . \quad (34)$$

It is convenient to introduce the dimensionless expression

$$\tilde{S}_1(\tilde{\lambda}_T)=\frac{\pi}{\tilde{\lambda}}\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right](\omega_m^2+\tilde{\lambda})^{1/2}$$

$$=\frac{1}{2\tilde{\lambda}_T}\left[\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}dm\right](m^2+\tilde{\lambda}_T)^{1/2}\ . \quad (35)$$

To obtain the $\tilde{\lambda}=0$ term in (31) we take the limit

$$K_1(\tilde{m}\sqrt{\tilde{\lambda}}/T)\rightarrow\frac{T}{\sqrt{\tilde{\lambda}}}\frac{1}{\tilde{m}}$$

and the right-hand side of (34) reduces to

$$-\frac{2T^2}{\pi}\sum_{\tilde{m}=1}^{\infty}\frac{1}{\tilde{m}^2}=-\frac{\pi}{3}T^2\ .$$

Then (34) amounts to

$$\tilde{S}_1(\tilde{\lambda}_T)=-2\sum_{\tilde{m}=1}^{\infty}K_1(2\pi\tilde{m}\sqrt{\tilde{\lambda}_T})/(2\pi\tilde{m}\sqrt{\tilde{\lambda}_T}) \quad (36)$$

and we have the alternative expansion, from (31),

$$\Delta f^T(\tilde{\lambda},T)=-\frac{\pi}{3}T^2+\frac{\tilde{\lambda}}{\pi}\tilde{S}_1(\tilde{\lambda}_T)\ . \quad (37)$$

This representation allows us to calculate the behavior for small T. Since $K_1(z)$ decreases exponentially fast in z, this limit reads

$$\Delta f^T(\tilde{\lambda},T)\underset{T\rightarrow 0}{\sim}-\frac{\pi}{3}T^2+O(e^{-\pi\sqrt{\tilde{\lambda}}/T})\ . \quad (38)$$

There exists yet another useful representation for $\Delta f^T(\tilde{\lambda},T)$. By representing $K_1(z)$ as an integral

$$K_1(z)=\int_0^{\infty}ds\,e^{-\sqrt{s^2+1}\,z}$$

$$=\frac{\pi}{\sqrt{\tilde{\lambda}}}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}e^{-(q_1^2+\tilde{\lambda})^{1/2}z/\tilde{\lambda}} \quad (39)$$

we can rewrite the $\tilde{S}_1(\tilde{\lambda}_T)$ of (36) also as

$$-(2\pi T/\tilde{\lambda})\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\sum_{\tilde{m}=1}^{\infty}\frac{1}{\tilde{m}}e^{-(q_1^2+\tilde{\lambda})^{1/2}\tilde{m}/T}$$

in which case the sum can be performed with the result

$$\tilde{S}_1(\tilde{\lambda}_T)=(2\pi T/\tilde{\lambda})\int_{\infty}^{\infty}\frac{dq_1}{2\pi}\ln(1-e^{-(q_1^2+\tilde{\lambda})^{1/2}/T})$$

$$=\frac{1}{2\pi\sqrt{\tilde{\lambda}_T}}\int_{-\infty}^{\infty}ds\,\ln[1-\exp(-\sqrt{s^2+1}\,2\pi\tilde{\lambda}_T)]\ . \quad (40)$$

Hence we have also

$$\Delta f^T(\tilde{\lambda},T)=-\frac{\pi}{3}T^2+2T\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\ln(1-e^{-(q_1^2+\tilde{\lambda})^{1/2}/T})\ . \quad (41)$$

This result could have been found directly from the original expression

$$\Delta f^T(\tilde{\lambda},T)=\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right]$$

$$\times\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}[\ln(\omega_m^2+q_1^2+\tilde{\lambda})+(\tilde{\lambda}\rightarrow 0)] \quad (42)$$

using the well-known formula

$$\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right]\ln(\omega_m^2+a^2)=2T\ln(1-e^{-a/T}), \quad (43)$$

which can be verified by differentiation with respect to a^2,

$$\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right]\frac{1}{\omega_m^2+a^2}=\frac{1}{a}[\coth(a/2T)-1], \quad (44)$$

and making sure that there is no extra constant of integration by (treating the limit $T\rightarrow\infty$ via the Euler-McLaurin formula.) Then

$$\Delta f^T(\tilde{\lambda},T)=2T\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}(\ln\{1-\exp[-(q_1^2+\tilde{\lambda})^{1/2}/T]\}$$

$$+(\tilde{\lambda}=0))\ . \quad (45)$$

The integration of the $\tilde{\lambda}=0$ term gives $-(\pi/3)T^2$ and we arrive back at (41).

The calculation of the anisotropy correction Δf^{δ} Eq. (14), is more tedious. As it will turn out at the end, for not too extreme values of $\tilde{M}^2_{\rm NG}$, the thermal deconfinement transition takes place at a relatively low temperature (high $\beta_{\rm ext}$) where the gap anisotropy is very small. It is therefore useful to study the transition first within the approximation of an isotropic gap and take a look at the modification by $\delta\neq 0$ later.

V. ISOTROPIC GAP APPROXIMATION

We set $\delta=0$, $\tilde{\lambda}\equiv\lambda$, and consider the action

$$\mathcal{A}_{\rm iso}=\frac{d-2}{2}R_{\rm ext}\beta_{\rm ext}\sqrt{\rho_0\rho_1}f_{\rm iso} \quad (46)$$

with

$$f_{\rm iso}=\tilde{M}^2_{\rm NG}+f_0(\lambda)-\frac{\lambda}{4\pi}+\Delta f^T(\lambda,T)+\frac{\lambda}{2\tilde{\alpha}}\left[\frac{1}{\rho_0}+\frac{1}{\rho_1}\right]\ . \quad (47)$$

We also shall first look at the purely spontaneous string, i.e., $\tilde{M}^2_{\rm NG}=0$ ($\nu=0$).

When extremizing this simplified action it is useful to observe that $\Delta f^T(\lambda,T)$ has the functional form

$$\Delta f^T(\lambda,T)=\lambda g(\lambda_T)\ , \quad (48)$$

where λ_T is the dimensionless variable

$$\lambda_T\equiv\lambda/4\pi^2T^2\ . \quad (49a)$$

We must make sure that the extremization of the total action $\mathcal{A}_{\rm iso}$ with respect to ρ_0,ρ_1,λ has to be carried out at fixed coordinates, i.e., at fixed ξ^0,ξ^1. Hence $T_{\rm ext}$ is fixed and λ_T depends on ρ_0, via

$$\lambda_T=\tilde{\lambda}\rho_0/4\pi^2T_{\rm ext}^2 \; . \tag{49b}$$

Therefore, the extremization of $\mathcal{A}_{\rm iso}$ gives from the derivative $\sqrt{\rho_0}\,\partial/\partial\sqrt{\rho_0}$, the equation

$$f_0(\lambda)-\frac{\lambda}{4\pi}+\lambda g(\lambda_T)+2\lambda g'(\lambda_T)\lambda_T+\frac{\lambda}{2\tilde{\alpha}}\left[-\frac{1}{\rho_0}+\frac{1}{\rho_1}\right]=0\;, \tag{50}$$

from $\sqrt{\rho_1}\,\partial/\partial\sqrt{\rho_1}$,

$$f_0(\lambda)-\frac{\lambda}{4\pi}+\lambda g(\lambda_T)+\frac{\lambda}{2\tilde{\alpha}}\left[\frac{1}{\rho_0}-\frac{1}{\rho_1}\right]=0\;, \tag{51}$$

and from $\partial/\partial\lambda$, the gap equation

$$\frac{\partial}{\partial\lambda}f_0(\lambda)-\frac{1}{4\pi}+g(\lambda_T)+g'(\lambda_T)\lambda_T+\frac{1}{2\tilde{\alpha}}\left[\frac{1}{\rho_0}+\frac{1}{\rho_1}\right]=0\;, \tag{52}$$

where the prime denotes the derivative with respect to λ_T. Adding and subtracting Eqs. (50) and (51) gives

$$f_0(\lambda)-\frac{\lambda}{4\pi}+\lambda g(\lambda_T)+\lambda g'(\lambda_T)\lambda_T=0\;, \tag{53}$$

$$\frac{\lambda}{2\tilde{\alpha}}\left[\frac{1}{\rho_0}-\frac{1}{\rho_1}\right]=\lambda g'(\lambda_T)\lambda_T\;. \tag{54}$$

The second equation determines the difference between the extremal ρ_0 and ρ_1. Using the obvious relation, $f_0(\lambda)=\lambda(\partial/\partial\lambda)f_0+(\lambda/4\pi)$, the first of the equations can also be rewritten as

$$\lambda\left[\frac{\partial}{\partial\lambda}f_0(\lambda)+g(\lambda_T)+g'(\lambda_T)\lambda_T\right]=0\;. \tag{55}$$

Inserting this into the gap equation (52) shows that the minimum lies at

$$\frac{1}{2\tilde{\alpha}}\left[\frac{1}{\rho_0}+\frac{1}{\rho_1}\right]=\frac{1}{4\pi}\;. \tag{56}$$

When putting this back into the gap equation this reads

$$\frac{\partial}{\partial\lambda}[f_0(\lambda)+\lambda g(\lambda_T)]=\frac{\partial}{\partial\lambda}[f_0(\lambda)+\Delta f^T(\lambda,T)]=0 \tag{57}$$

and the total action is simply (with $A\equiv R_{\rm ext}\beta_{\rm ext}\sqrt{\rho_0\rho_1}\equiv R\beta$)

$$\mathcal{A}_{\rm iso}=\frac{d-2}{2}A[f_0(\lambda)+\Delta f^T(\lambda,T)] \equiv\frac{d-2}{2}Af^T_{\rm iso}(\lambda,T)\;. \tag{58}$$

The metric has disappeared when going to intrinsic quantities. In the final expression, only $f^T_{\rm iso}(\lambda,T)$ has to be extremized in λ. Depending on which of the three forms of $f_{\rm iso}$ [(29), (37), or (41)] we prefer, the gap equation reads

$$-\frac{1}{4\pi}\ln(T^2/\bar{T}^2)+\frac{T}{2\sqrt{\lambda}}+T\sum_{m=1}^{\infty}\left[\frac{1}{(\omega_m^2+\lambda)^{1/2}}-\frac{1}{(\omega_m^2)^{1/2}}\right]=0\;, \tag{59a}$$

$$-\frac{1}{4\pi}\ln(\lambda/\bar{\lambda})+\frac{1}{\pi}\sum_{\tilde{m}=1}^{\infty}K_0(\tilde{m}\sqrt{\lambda}/T)=0\;, \tag{59b}$$

$$-\frac{1}{4\pi}\ln(\lambda/\bar{\lambda})+\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\frac{1}{2(q_1^2+\lambda)^{1/2}}\times\{\coth[(q_1^2+\lambda)^{1/2}/2T]-1\}\;. \tag{59c}$$

In deriving (59b) we have used the identity $-K_1'(z)-(1/z)K_1(z)=K_0(z)$. The last equation (59c) is identical with (59b) via the expansion $\coth[(q_1^2+\lambda)^{1/2}/2T]-1=2\sum_{\tilde{m}=1}^{\infty}\exp[-(q_1^2+\lambda)^{1/2}\tilde{m}/T]$ and the integral representation for $K_0(z)$:

$$K_0(z)=\int_1^{\infty}dt\frac{1}{\sqrt{t^2-1}}e^{-tz} =\pi\sqrt{\lambda}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\frac{1}{(q_1^2+\lambda)^{1/2}}\times\exp[-(q_1^2+\lambda)^{1/2}z/\sqrt{\lambda}]\;. \tag{60}$$

It is useful to introduce the sum

$$S_2(\lambda_T)\equiv2\pi T\sum_{m=1}^{\infty}\left[\frac{1}{\sqrt{\omega_m^2+\lambda}}-\frac{1}{\omega_m}\right] =\sum_{m=1}^{\infty}\left[\frac{1}{\sqrt{m^2+\lambda_T}}-\frac{1}{m}\right] \tag{61a}$$

in analogy to S_1 of (28b) and

$$\tilde{S}_2(\lambda_T)=\pi\left[T\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}d\omega_m\right]\frac{1}{(\omega_m^2+\lambda)^{1/2}} =\frac{1}{2}\left[\sum_{m=-\infty}^{\infty}-\int_{-\infty}^{\infty}dm\right]\frac{1}{(m^2+\lambda_T)^{1/2}} \tag{61b}$$

in analogy to (35). Just as $\tilde{S}_1$ in (36) and (40) we can represent $\tilde{S}_2$ in two alternative ways:

$$\tilde{S}_2(\lambda_T)=2\frac{\partial}{\partial\lambda_T}[\lambda_T\tilde{S}_1(\lambda_T)] =2\sum_{\tilde{m}=1}^{\infty}K_0(2\pi\tilde{m}\sqrt{\lambda_T}) =\int_0^{\infty}\frac{ds}{\sqrt{s^2+1}}[\coth(\sqrt{s^2+1}\,2\pi\sqrt{\lambda_T})-1]\;. \tag{61c}$$

Then the gap equations read

$$-\frac{1}{4\pi}\ln(T^2/\bar{T}^2)+\frac{T}{2\sqrt{\lambda}}+\frac{1}{2\pi}S_2(\lambda_T)=0\;, \tag{62a}$$

$$-\frac{1}{4\pi}\ln(\lambda/\bar{\lambda})+\frac{1}{2\pi}\tilde{S}_2(\lambda_T)=0\;. \tag{62b}$$

The action (58) can now be used to calculate the string tension as a function of the temperature T_{ext}. Since a state of energy E has a Boltzmann factor $\exp(-\beta_{ext}E)$, and the tension is defined by $M_{tot}^2=E/R_{ext}$, it is obtained by dropping the extrinsic area factor in the action (58):

$$\mathcal{A}_{fl}=\frac{d-2}{2}R_{ext}\beta_{ext}\sqrt{\rho_0\rho_1}f_{iso}^T(\lambda,T) . \qquad (63a)$$

Hence,

$$M_{tot}^2(T_{ext})=\frac{d-2}{2}\sqrt{\rho_0\rho_1}f_{iso}^T(\lambda,T) . \qquad (63b)$$

It is useful to write $f_{iso}^T(\lambda,T)$ as $f_0(\lambda)+\lambda g(\lambda_T)$, thereby exhibiting its functional dependence on λ,λ_T:

$$M_{tot}^2(T_{ext})=\frac{d-2}{2}\sqrt{\rho_0\rho_1}[f_0(\lambda)+\lambda g(\lambda_T)] . \qquad (63c)$$

Let us also express the right-hand side in terms of extrinsically renormalized quantities, using the dimensionally transmuted coupling constant $\bar\lambda$ defined in (16) and going to extrinsic space by a metric factor $\bar\rho$:

$$\bar\lambda_{ext}=\bar\rho\,\bar\lambda . \qquad (64)$$

In the present T-dependent calculation, the quantity $\bar\rho$ is the common limit of ρ_0,ρ_1 for $T\to 0$. Let us introduce the notation

$$\hat\rho_0\equiv\rho_0/\bar\rho , \qquad (65)$$

$$\hat\rho_1\equiv\rho_1/\bar\rho . \qquad (66)$$

Then we write

$$\sqrt{\rho_0\rho_1}\lambda=\sqrt{\hat\rho_0\hat\rho_1}\lambda_{ext},\quad \sqrt{\rho_0\rho_1}\bar\lambda=\sqrt{\hat\rho_0\hat\rho_1}\bar\lambda_{ext} \qquad (67)$$

and the string tension becomes

$$\begin{aligned}M_{tot}^2(T_{ext})&=\frac{d-2}{2}\sqrt{\hat\rho_0\hat\rho_1}[f_{0,ext}(\lambda_{ext})+\lambda_{ext}g(\lambda_T)]\\ &=\frac{d-2}{2}\frac{\bar\lambda_{ext}}{4\pi}\left[\sqrt{\hat\rho_0\hat\rho_1}4\pi\left[\frac{1}{\bar\lambda_{ext}}f_{0,ext}(\lambda_{ext})+\frac{\lambda_{ext}}{\bar\lambda_{ext}}g(\lambda_T)\right]\right]\\ &=M_{tot}^2(T_{ext}=0)\hat M_{tot}^2(T_{ext}) ,\end{aligned} \qquad (68)$$

where $\hat M_{tot}^2$ is defined as the expression in the large square brackets. It is the ratio of string tension at $T_{ext}=0$ versus the tension at zero temperature. There $f_{0,ext}(\lambda_{ext})$ is given by (16), except with $\lambda,\bar\lambda$ replaced by $\lambda_{ext},\bar\lambda_{ext}$. Now $(1/\lambda_{ext})f_{0,ext}(\lambda_{ext})$ depends only on the ratio $\lambda_{ext}/\bar\lambda_{ext}$ and the bracket is a function of $\lambda_{ext}/\bar\lambda_{ext}$ and λ_T. In fact, it depends on these variables in the same way as the intrinsically renormalized quantity $(1/\lambda)f_0(\lambda)+(\lambda/\bar\lambda)g(\lambda_T)$ depends on $\lambda/\bar\lambda$ and λ_T. For various λ_T we calculate, from (62b),

$$\begin{aligned}\lambda/\bar\lambda&=\exp[2\tilde S_2(\lambda_T)]\\ &=\exp\left[4\sum_{\tilde m=1}^{\infty}K_0(2\pi\tilde m\sqrt{\lambda_T})\right] .\end{aligned} \qquad (69)$$

The ratio $\lambda/\bar\lambda$ can be rewritten directly as the extrinsically renormalized ratio $\lambda_{ext}/\bar\lambda_{ext}$. This is inserted into (68) to find the numerical values for quantities in large square brackets. It remains only to calculate the prefactor $\sqrt{\hat\rho_0\hat\rho_1}$. For this we use Eqs. (54) and (56), according to which

$$\frac{1}{\tilde\alpha\rho_0}=\frac{1}{4\pi}+g'(\lambda_T)\lambda_T , \qquad (70)$$

$$\frac{1}{\tilde\alpha\rho_1}=\frac{1}{4\pi}-g'(\lambda_T)\lambda_T . \qquad (71)$$

Moreover, because of Eq. (53), we can reexpress $g'(\lambda_T)\lambda_T$ as

$$\begin{aligned}g'(\lambda_T)\lambda_T&=-\frac{1}{\lambda}\left[f_0(\lambda)-\frac{\lambda}{4\pi}+\lambda g(\lambda_T)\right]\\ &=-\frac{1}{\lambda}f_{iso}^T(\lambda,T)+\frac{1}{4\pi}\end{aligned} \qquad (72)$$

and obtain the simple equations

$$\frac{1}{\tilde\alpha\rho_0}=\frac{1}{2\pi}-\frac{1}{\lambda}f_{iso}^T(\lambda,T) , \qquad (73)$$

$$\frac{1}{\tilde\alpha\rho_1}=\frac{1}{\lambda}f_{iso}^T(\lambda,T) . \qquad (74)$$

They determine the ratio

$$\rho_1/\rho_0=\left[\frac{\lambda}{2\pi}-f_{iso}^T(\lambda,T)\right]\Big/f^T(\lambda,T) \qquad (75)$$

and, since $(1/\lambda)f_{iso}^T(\lambda,T)\to 1/4\pi$ for $T\to 0$,

$$\frac{1}{\hat\rho_0}=2-\frac{4\pi}{\lambda}f_{iso}^T(\lambda,T) , \qquad (76)$$

$$\frac{1}{\hat\rho_1}=\frac{4\pi}{\lambda}f_{iso}^T(\lambda,T) . \qquad (77)$$

The expressions (75)–(77) are all renormalized in extrinsic space. This follows from the fact that $(1/\lambda)f_{iso}^T(\lambda,T)$ can be expressed entirely in terms of $\lambda_{ext}/\bar\lambda_{ext}$ and λ_T, and this may be written as

$$\lambda_T=\lambda/4\pi^2T^2=\lambda^{ext}\hat\rho_1/4\pi^2T_{ext}^2 \qquad (78)$$

with (77) appearing once more on the right-hand side.

It remains to the plot $\hat M_{tot}^2(T_{ext})$ as a function of T_{ext}. For this we introduce the extrinsic temperature scale

$$\bar T_{ext}=\sqrt{\bar\rho_0}\bar T=\sqrt{\bar\lambda_{ext}}/4\pi e^{-\gamma} . \qquad (79)$$

It is directly related to the string tension at zero temperature M_{tot}:

$$\bar T_{ext}=e^{\gamma}\sqrt{1/(d-2)2\pi}M_{tot} . \qquad (80)$$

Moreover, since

$$\begin{aligned}\lambda_T&=\lambda/4\pi^2T^2\\ &=(\lambda/\bar\lambda)(\bar T^2/T^2)\left[\frac{4\pi e^{-\gamma}}{2\pi}\right]^2\\ &=(\lambda_{ext}/\bar\lambda_{ext})(\bar T_{ext}^2/T_{ext}^2)\hat\rho_0 4e^{-2\gamma} ,\end{aligned} \qquad (81)$$

we have

$$T_{\rm ext}/\bar{T}_{\rm ext}=\sqrt{\hat{\rho}_0}\bar{T}/T=\sqrt{\hat{\rho}_0}\frac{1}{\sqrt{\lambda_T}}\sqrt{\lambda/\bar{\lambda}^2}e^{-\gamma} \tag{82}$$

with all quantities on the right-hand side being known once the gap equation (69) is solved as a function of λ_T. For a numerical treatment it is more convenient to use the gap equation in the form (59a) and (62a):

$$T/\bar{T}=\exp\left[\frac{1}{2\sqrt{\lambda_T}}+S_2(\lambda_T)\right] =\exp\left[\frac{1}{2\sqrt{\lambda_T}}+\sum_{m=1}^{\infty}\left[\frac{1}{(m^2+\lambda_T)^{1/2}}-\frac{1}{m}\right]\right] . \tag{83}$$

From this we calculate $T/\bar{T}$ as a function of λ_T. Then we use the relation

$$\frac{\lambda}{\bar{\lambda}}=\frac{e^{2\gamma}}{4}\lambda_T\left[\frac{T}{\bar{T}}\right]^2 \tag{84}$$

to find $\lambda/\bar{\lambda}$ and thus the expression in large square brackets, $\hat{M}^2_{\rm tot}$ in (68). Finally, we obtain $T_{\rm ext}/\bar{T}_{\rm ext}$ from (82).

The solution of the gap equation is displayed in Fig. 1. We have taken $f^T_{\rm iso}$ as the sum (29) and brought the large square brackets in (68) to the form

$$\hat{M}^2_{\rm tot}(T_{\rm ext})=\sqrt{\hat{\rho}_0\hat{\rho}_1}\frac{4\pi}{\lambda}f^T_{\rm iso}(\lambda,T) =\frac{\lambda}{\bar{\lambda}}(f^T_{\rm iso}(\lambda,T)/[(2\lambda/4\pi)-f^T_{\rm iso}(\lambda,T)])^{1/2} . \tag{85}$$

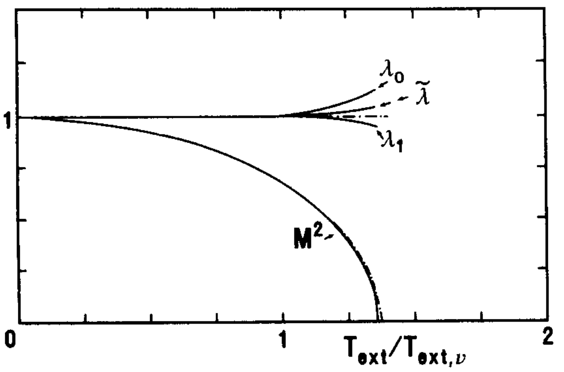

FIG. 1. The string tension as a function of the reduced temperature $T_{\rm ext}/\bar{T}_{{\rm ext},\nu}$ where

$$\bar{T}_{{\rm ext},\nu}=\frac{e^{\gamma}}{\sqrt{2\pi(d-2)}}M_{\rm tot}\approx 0.502M_{\rm tot} ,$$

where $M_{\rm tot}$ is the string at zero temperature. The curves are for a purely spontaneous string ($\nu=0$), but there is little dependence on ν. The deconfinement temperature is seen to lie at $T_{\rm ext}\approx 0.68M_{\rm tot}$. The dashed-dotted curves represent the simple analytic approximation explained in the text. The upper three curves show the corresponding gaps $\lambda_0,\lambda_1,\tilde{\lambda}=(\lambda_1+\lambda_0)/2$ as a function of temperature as well as the analytic approximation (dashed-dotted). For moderate values of ν, the figure has almost no ν dependence.

This quantity is plotted as a function of $T_{\rm ext}/\bar{T}_{\rm ext}$ in Fig. 1. By definition, it tends to unity for $T_{\rm ext}\to 0$. The transition occurs at a temperature

$$T^d_{\rm ext}/\bar{T}_{\rm ext}\approx 1.35 \tag{86}$$

with

$$\bar{T}_{\rm ext}\approx\frac{e^{\gamma}}{2}\sqrt{2/(d-2)\pi}M_{\rm tot}\approx 0.502M_{\rm tot} . \tag{87}$$

This implies a ratio between deconfinement temperature and string tension of about

$$T^d_{\rm ext}/M_{\rm tot}\approx 0.68 . \tag{88}$$

This value is not far from what is found in Monte Carlo simulations of lattice gauge models. These give, for SU(2) and SU(3) (Ref. 11),

$$T^d_{\rm ext}\approx\begin{cases}41\Lambda_E^{\rm latt} & \text{in SU(2)} ,\\ 80\Lambda_E^{\rm latt} & \text{in SU(3)} ,\end{cases} \tag{89}$$

or, using simulation relations between $\Lambda_E^{\rm latt}$ and $M_{\rm tot}$,

$$T^d_{\rm ext}\approx\begin{cases}0.35\pm 0.05M_{\rm tot} & \text{in SU(2)} ,\\ 0.75\pm 0.25M_{\rm tot} & \text{in SU(3)} .\end{cases} \tag{90}$$

When comparing these numbers with the result of our calculation we have to remember that this result is valid to leading order in d only.

VI. ANALYTIC APPROXIMATION

It should be noted that the above deconfinement temperature is very close to what can be obtained analytically by means of a very simple approximation to $\lambda(T)/\bar{\lambda}$, valid for small to moderate temperatures. We observe that for $T\to 0$, the argument of the Bessel function becomes large and, since $K_0(z)\to\sqrt{\pi/2z}\,e^{-z}$ decrease exponentially for large z, the gap equation (59b) gives

$$\lambda/\bar{\lambda}-1\approx 4\sqrt{\pi T/2\sqrt{\lambda}}e^{-\sqrt{\lambda}/T} , \tag{91}$$

i.e., $\lambda\approx\bar{\lambda}$ with only exponentially small corrections. This makes it easy to estimate the deconfinement temperature T^d quite accurately. In the limit of small T, also $K_1(z)$ is exponentially small and, with the gap λ being close to $\bar{\lambda}$, we find right away the approximation

$$f^T(\lambda,T)\approx\frac{\bar{\lambda}}{4\pi}-\frac{\pi}{3}T^2 . \tag{92}$$

Therefore, if T exceeds the value

$$T^d_0=\sqrt{3/4\pi^2}\sqrt{\bar{\lambda}}=0.276\sqrt{\bar{\lambda}}\approx\sqrt{12}e^{-\gamma}\bar{T}\approx 1.95\bar{T} \tag{93}$$

or

$$\sqrt{\bar{\rho}}T^d_0=\sqrt{6/\pi(d-2)}M_{\rm tot}\approx 0.977M_{\rm tot} \tag{94}$$

the string tension turns negative and the confinement is gone. The subscript of T^d_0 records the fact that we are dealing with the purely spontaneously string, $\nu=0$. In order to compare this number with experiment, we have to go over to an extrinsically renormalized quantity. For this we calculate

$$T^d_{\text{ext},0}=\hat{\rho}_0^{1/2}|_{T_0^d}T_0^d \ . \tag{95}$$

The quantity $\hat{\rho}_0^{1/2}|_{T_0^d}$ is found from (76):

$$\frac{1}{\hat{\rho}_0}=2-\frac{4\pi}{\lambda}f^T_{\text{iso}}(\lambda,T)\approx 2-\frac{4\pi}{\bar{\lambda}}\left[\frac{\bar{\lambda}}{4\pi}-\frac{\pi}{3}T^2+\cdots\right] \tag{96}$$

with only exponentially small corrections. At T_0^d this gives

$$\left.\frac{1}{\hat{\rho}_0}\right|_{T_0^d}\approx 2 \tag{97}$$

and hence

$$T^d_{\text{ext},0}/\bar{T}_{\text{ext}}=\sqrt{\hat{\rho}_0}T_0^d/\bar{T}\approx\sqrt{6}e^{-\gamma}\approx 1.38 \ . \tag{98}$$

We therefore find the approximate deconfinement temperature

$$T^d_{\text{ext},0}=\sqrt{3/\pi(d-2)}M_{\text{tot}}\approx 0.691M_{\text{tot}} \ , \tag{99}$$

which is very close to the precise numerical result (88).

Let us also give the full temperature dependence of the string tension. The approximation of neglecting the Bessel function in the gap equation in $f^T_{\text{iso}}(\lambda,T)$ using (85) and (76) we obtain

$$\hat{M}^{2\text{app}}_{\text{tot}}(T_{\text{ext}})=\sqrt{[1-(4\pi^2/3)T^2/\bar{\lambda}]/[1+(4\pi^2/3)T^2/\bar{\lambda}]} \ , \tag{100}$$

where T is related to the true extrinsic temperature by

$$T^2_{\text{ext}}=T^2\bar{\rho}\hat{\rho}_0=T^2\frac{\bar{\rho}}{1+\frac{4\pi^2}{3}T^2/\bar{\lambda}} \ . \tag{101}$$

Inserting this into (100) we find the simple expression

$$\hat{M}^{2\text{app}}_{\text{tot}}(T_{\text{ext}})=[1-(8\pi^2/3)T^2_{\text{ext}}/\bar{\lambda}]^{1/2} \ . \tag{102}$$

It is plotted in Fig. 1. Later we shall see that this approximation is so good that it can hardly be distinguished from the full numerical calculation.

It is interesting to realize that, in this approximation, the purely spontaneous string tension coincides precisely with the Nambu-Goto tension as a function of temperature. This will be shown in the next section. The difference is exponentially small, $\propto e^{-\sqrt{\bar{\lambda}}/T}$. For $\nu=0$, it is, at T^d, of the order of $\exp(-\sqrt{4\pi^2/3})\approx 2.7\%$, as we can see from (92)–(94).

VII. THE NAMBU-GOTO CASE

For completeness, let us go through the well-known calculation of the thermal deconfinement transition in the Nambu-Goto case.[14] We shall write the action in a form convenient for the $d\to\infty$ limit as

$$\mathcal{A}_{\text{NG}}=\int d^2\xi\sqrt{g}\left[M^2_{\text{NG}}+\frac{\lambda^{ij}}{2\tilde{\alpha}}(\partial_i u\partial_j u-g_{ij}+\delta_{ij})\right] , \tag{103}$$

where, $\tilde{\alpha}$ is here only a dummy parameter, $\tilde{\alpha}\neq 0$, introduced merely to make the following equations look as similar as possible to the previous ones. The free energy density can be written down right away as [see Eqs. (6) and (7)]

$$\begin{aligned} f_{\text{NG}}(\lambda,\rho_0,\rho_1)&=\tilde{M}^2_{\text{NG}}+\frac{T_{\text{ext}}}{\sqrt{\rho_0\rho_1}}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\ln[\lambda_0(k_0)^2_m/\rho_0+\lambda_1k_1^2/\rho_1]-\bar{\lambda}/\tilde{\alpha}+(1/2\tilde{\alpha})(\lambda_0/\rho_0+\lambda_1/\rho_1)\\ &=\tilde{M}^2_{\text{NG}}-(\pi T^2/3)\sqrt{\lambda_0/\lambda_1}-\bar{\lambda}/\tilde{\alpha}+(1/2\tilde{\alpha})(\lambda_0/\rho_0+\lambda_1/\rho_1) \ . \end{aligned} \tag{104}$$

Let us introduce the parameter

$$\gamma\equiv\pi T^2\rho_0/3=\pi T^2_{\text{ext}}/3 \ . \tag{105}$$

Extremization of $\mathcal{A}_{\text{NG}}$ gives the equations

$$\tilde{M}^2_{\text{NG}}+\frac{\gamma}{\rho_0}\left[\frac{\lambda_0}{\lambda_1}\right]^{1/2}-\frac{\bar{\lambda}}{\tilde{\alpha}}+\frac{1}{2\tilde{\alpha}}(-\lambda_0/\rho_0+\lambda_1/\rho_1)=0 \ , \tag{106a}$$

$$\tilde{M}^2_{\text{NG}}-\frac{\gamma}{\rho_0}\left[\frac{\lambda_0}{\lambda_1}\right]^{1/2}-\frac{\bar{\lambda}}{\tilde{\alpha}}+\frac{1}{2\tilde{\alpha}}(\lambda_0/\rho_0-\lambda_1/\rho_1)=0 \ , \tag{106b}$$

$$-\frac{1}{\tilde{\alpha}}+\frac{1}{2\tilde{\alpha}\bar{\lambda}}\left[\frac{\lambda_0}{\rho_0}+\frac{\lambda_1}{\rho_1}\right]=0 \ , \tag{106c}$$

$$-\frac{\gamma}{\rho_0}\frac{1}{\sqrt{1-\delta^2}}\frac{1}{1+\delta}+\frac{\bar{\lambda}}{2\tilde{\alpha}}\left[\frac{1}{\rho_0}-\frac{1}{\rho_1}\right]=0 \ , \tag{106d}$$

From these we find

$$\frac{\bar{\lambda}}{\tilde{\alpha}}=\tilde{M}^2_{\text{NG}} \ , \tag{107a}$$

$$\frac{1}{2\tilde{\alpha}}\left[\frac{\lambda_0}{\rho_0}-\frac{\lambda_1}{\rho_1}\right]=\frac{\gamma}{\rho_0}\sqrt{(1-\delta)/(1+\delta)} \ , \tag{107b}$$

$$\frac{\lambda_{0,1}}{\tilde{\alpha}\rho_{0,1}}=\tilde{M}^2_{\text{NG}}\pm\frac{\gamma}{\rho_0}\sqrt{(1-\delta)/(1+\delta)} \ , \tag{107c}$$

and at the extremum

$$f_{\text{NG}}=\tilde{M}^2_{\text{NG}}\left[1-\frac{\gamma}{\tilde{M}^2_{\text{NG}}\rho_0}\right] \ . \tag{108}$$

Due to (107c), the gap equation (106d) for δ becomes

$$\gamma \frac{1}{\sqrt{1-\delta^2}} \frac{1}{1+\delta} = \tilde{M}_{\rm NG}^2 \frac{1}{1-\delta^2} \left[\delta + \frac{\gamma}{\tilde{M}_{\rm NG}^2} \sqrt{(1-\delta)/(1+\delta)} \right] . \quad (107d)$$

This has only the isotropic solution $\delta=0$; i.e., the gap matrix is $\lambda^{ij}=\lambda g^{ij}$. Then, Eq. (107c) shows that $\rho_{0,1}\to 1$ for $T\to 0$ so that $\hat{\rho}_{0,1}=\rho_{0,1}$ and we have

$$1/\hat{\rho}_{0,1} = 1 \pm \gamma / \tilde{M}_{\rm NG}^2 \hat{\rho}_0 \quad (108a)$$

and hence

$$\hat{\rho}_0 = 1 - \gamma / \tilde{M}_{\rm Ng}^2 , \quad (108b)$$

$$\hat{\rho}_1 = (1-\gamma/\tilde{M}_{\rm NG}^2)/(1-2\gamma/\tilde{M}_{\rm NG}^2) . \quad (109)$$

Therefore,

$$\begin{aligned} \tilde{M}_{\rm tot}^2 &= \sqrt{\hat{\rho}_0\hat{\rho}_1} f_{\rm NG} = \tilde{M}_{\rm NG}^2 \sqrt{\hat{\rho}_0\hat{\rho}_1} (1-\gamma/\tilde{M}_{\rm NG}\hat{\rho}_0) \\ &= \tilde{M}_{\rm NG}^2 (1-2\gamma/\tilde{M}_{\rm NG})^{1/2} \\ &= \tilde{M}_{\rm NG}^2 [1-(d-2)T_{\rm ext}^2/3M_{\rm tot}^2]^{1/2} \\ &= \tilde{M}_{\rm NG}^2 \hat{M}^2(T_{\rm ext}) . \end{aligned} \quad (110)$$

This is indeed the same as (102).

VIII. INCLUDING THE GAP ANISOTROPY

We shall now study what happens if we include the effects of gap anisotropy. We work with the full action, including the additional Nambu-Goto term. It will turn out that due to some accidental cancellations, the anisotropy remains exponentially small at small T. For this reason, including δ it will hardly change the above calculated deconfinement temperature. Also the Nambu-Goto term will be seen to have little effect on this temperature if it is expressed in terms of the total tension.

$$f_{\rm tot}^T = \tilde{M}_{\rm NG}^2 + f_0(\tilde{\lambda}) - \frac{\tilde{\lambda}}{4\pi} + \Delta f^T(\tilde{\lambda},T) + \Delta f^\delta(\tilde{\lambda},T,\delta) + \frac{1}{2\tilde{\alpha}} \left[\frac{\lambda_0}{\rho_0} + \frac{\lambda_1}{\rho_1} \right] . \quad (111)$$

Let me calculate the anisotropy correction $\Delta f^\delta(\tilde{\lambda},T,\delta)$. We shall do this first at zero temperature where it can be rewritten as

$$\begin{aligned} \Delta f^\delta &= \int \frac{d^2q}{(2\pi)^2} [\ln(q^4+\lambda_0 q_0^2+\lambda_1 q_1^2) - \ln(q^4+\tilde{\lambda}q^2)] \\ &= \frac{1}{4\pi}\int_0^\infty dq^2 \int_0^{2\pi} \frac{d\phi}{2\pi} \left[\ln\left[q^2+\tilde{\lambda}-\frac{\Delta\lambda}{2}\cos(2\phi)\right] - \ln(q^2+\tilde{\lambda}) \right] , \end{aligned} \quad (112)$$

where $\Delta\tilde{\lambda}\equiv\lambda_1-\lambda_0$, $\arctan(q_1/q_0)$ and, as before, $\tilde{\lambda}=(\lambda_0+\lambda_1)/2$. Regularizing the integrals via the dimensionally transmuted coupling constant $\tilde{\lambda}$,

$$\Delta f^\delta = \int_0^{2\pi} \frac{d\phi}{2\pi} \left\{ -\frac{1}{4\pi}[\tilde{\lambda}-(\Delta\lambda/2)\cos(2\varphi)] \times \ln[\tilde{\lambda}-(\Delta\lambda/2)\cos 2\varphi]/\tilde{\lambda} - (\Delta\lambda=0) \right\} . \quad (113)$$

This has the functional form

$$\Delta f^\delta(\tilde{\lambda},T,\delta) = \tilde{\lambda} h(\delta) . \quad (114)$$

By expanding the logarithm into a power series in $\delta=\Delta\lambda/2\tilde{\lambda}$ and doing the integrations $\int_0^{2\pi}(d\phi/2\pi)\cos^{2m}\phi = (2m-1)!!/(2m)!!$ we obtain

$$\begin{aligned} h(\delta) &= -\frac{1}{4\pi}\sum_{m=1}^\infty \frac{1}{2m(2m-1)} \frac{(2m-1)!!}{(2m)!!}\delta^{2m} \\ &= -\frac{1}{16\pi}\left[\delta^2 + \frac{\frac{1}{2}}{2}\frac{\delta^4}{2!} + \frac{\frac{1}{2}\times\frac{3}{2}}{3}\frac{\delta^6}{3!} + \frac{\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}}{4}\frac{\delta^8}{4!} + \cdots \right] . \end{aligned} \quad (115)$$

Alternatively, these series could have been found by writing the first term in (112) as

$$\begin{aligned} -\int_0^{2\pi}\frac{d\phi}{2\pi}\int\frac{d^2q}{(2\pi)^2}\int_0^\infty \frac{d\tau}{\tau}\exp\left[-\frac{\tau}{2}\left[q^2+\tilde{\lambda}-\frac{\Delta\lambda}{2}\cos(2\phi)\right]\right] &= -\frac{1}{2\pi}\int_0^\infty \frac{d\tau}{\tau}\tau^{-(2+\epsilon)/2}\exp\left[-\frac{\tau}{2}\tilde{\lambda}\right] I_0(\tau\Delta\lambda/4) \\ &= -\frac{1}{2\pi}\Gamma\left[-1-\frac{\epsilon}{2}\right]\left[\frac{\tilde{\lambda}}{2}\right]^{1+\epsilon/2} F\left[-\frac{1}{2}-\frac{\epsilon}{4},-\frac{\epsilon}{4},1,\delta^2\right] \\ &= -\frac{\mu_1^\epsilon}{2\pi\epsilon}\tilde{\lambda} - \frac{\tilde{\lambda}}{4\pi}\ln(\tilde{\lambda}/\mu^2) \\ &\quad -\frac{\tilde{\lambda}}{4\pi}\frac{2}{\epsilon}\left[F\left[-\frac{1}{2}-\frac{\epsilon}{4},-\frac{\epsilon}{4},1,\delta^2\right]-1\right] \end{aligned} \quad (116)$$

so that

$$h(\delta) \underset{\epsilon\to 0}{\sim} -\frac{1}{4\pi}\frac{2}{\epsilon}\left[F\left[-\frac{1}{2}-\frac{\epsilon}{4},-\frac{\epsilon}{4},1,\delta^2\right]-1\right], \tag{117}$$

where $F(a,b,c,z)$ is the hypergeometric function.

We now turn to the finite-temperature expansion

$$\Delta f^{\delta}(\tilde\lambda,\Delta\lambda,T)=T\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\left[\ln\left[(\omega_m^2+q_1^2)^2+\tilde\lambda(\omega_m^2+q_1^2)-\frac{\Delta\lambda}{2}(\omega_m^2-q_1^2)\right]-\ln(\omega_m^2+q_1^2+\tilde\lambda)-\ln(\omega_m^2+q_1^2)\right]. \tag{118}$$

By integrating over q_1 we obtain the sum

$$\Delta f^{\delta}=T\sum_{m=-\infty}^{\infty}[A_m^+ + A_m^- -(\omega_m^2+\tilde\lambda)^{1/2}-(\omega_m^2)^{1/2}], \tag{119}$$

where

$$A_m^{\pm}\equiv[\omega_m^2+(\lambda_1/2)(1\pm B_m)]^{1/2}$$
$$=[\omega_m^2+(\tilde\lambda/2)(1+\delta)(1\pm B_m)]^{1/2}, \tag{120}$$
$$B_m\equiv[1+4(\Delta\lambda/\lambda_1^2)\omega_m^2]^{1/2}$$
$$=[1+8\delta\omega_m^2/\tilde\lambda(1+\delta)^2]^{1/2}. \tag{121}$$

This expression is not quite correct, though. In the limit $T\to 0$, the sum (119) contains a term linear in $\Delta\lambda$ which we know to be absent in Δf^{δ}, due to Eqs. (114) and (115). The mistake arises from the fact that formula (119) is obtained by doing the q_1 integration first at finite T, and then the limit $T\to 0$. The limit and the integral over q_1 do not commute. Indeed, if we expand at small $\delta=\Delta\lambda/2\tilde\lambda$

$$A_m^+=(\omega_m^2+\tilde\lambda)^{1/2}\left[1+\delta-\frac{\delta}{2}\frac{\tilde\lambda}{\omega_m^2+\tilde\lambda}+\cdots\right], \tag{122}$$

$$A_m^-=(\omega_m^2)^{1/2}(1-\delta+\cdots), \tag{123}$$

$$B_m=1+4\delta\omega_m^2/\tilde\lambda+\cdots, \tag{124}$$

we find the expansion

$$\Delta f^{\delta}=\frac{\Delta\lambda}{2\tilde\lambda}\sum_{m=-\infty}^{\infty}\left[\frac{2\omega_m^2+\tilde\lambda}{(\omega_m^2+\tilde\lambda)^{1/2}}-\frac{2\omega_m^2}{(\omega_m^2)^{1/2}}\right]+\cdots. \tag{125}$$

In the limit $T\to 0$, this does not vanish but goes against

$$\frac{\Delta\lambda}{4\tilde\lambda}\frac{1}{\pi}[\omega_m(\omega_m^2+\tilde\lambda)^{1/2}-\omega_m^2]_0^{\infty}=\frac{\Delta\lambda}{8\pi}. \tag{126}$$

This can also be seen directly from the expansion of Δf^{δ} in (118) in powers of $\Delta\lambda$:

$$\Delta f^{\delta}=-\frac{\Delta\lambda}{2}T\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\frac{\omega_m^2-q_1^2}{(\omega_m^2+q_1^2)(\omega_m^2+q_1^2+\tilde\lambda)}-\frac{(\Delta\lambda)^2}{8}T\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{(\omega_m^2-q_1^2)^2}{(\omega_m^2+q_1^2)^2(\omega_m^2+q_1^2+\tilde\lambda)^2}+\cdots. \tag{127}$$

In the limit $T\to 0$, we can set $\omega_m^2-q_1^2=(\omega_m^2+q^2)\cos(2\phi)$ and integrate $\int(d\omega_m/2\pi)\int dq_1/2\pi$ as

$$\frac{1}{2\pi^2}\int_0^{\infty}d(\omega_m^2+q_1^2)\int_0^{2\pi}d\phi$$

which shows that the term linear in $\Delta\lambda$ must vanish. In order to correct for the mistake in (119) we have to subtract from the sum the superfluous linear term in $\Delta\lambda$ and replace (119) by

$$\Delta f^{\delta}=T\sum_{m=-\infty}^{\infty}[A_m^+ + A_m^- -(\omega_m^2+\tilde\lambda)^{1/2}-(\omega_m^2)^{1/2}]$$
$$-\frac{\Delta\lambda}{8\pi}. \tag{128}$$

It can now be verified that in the limit $T\to 0$, f^{δ} tends properly to $\tilde\lambda h(\delta^2)$, as it should. A more systematic discussion of the term linear in $\Delta\lambda$ via dimensional regularization is given in Appendix A.

When trying to evaluate the expansion (128) numerically, we see that it is well defined only as long as $\Delta\lambda\geq 0$, or, if $\Delta\lambda<0$, for low $\omega_m^2<-\lambda_1^2/(4\Delta\lambda)=-\tilde\lambda(1+\delta)^2/8\delta$. If $-\Delta\lambda$ exceeds this value, the inner square root becomes imaginary and we have to calculate $A_m^+ + A_m^-$ as

$$A_m^+ + A_m^- = 2|A_m|\cos(\gamma_m/2), \tag{129}$$

where

$$|A_m|=[\omega_m^2(\omega_m^2+\lambda_0)]^{1/4}$$
$$=[\omega_m^2(\omega_m^2+\tilde\lambda-\Delta\lambda/2)]^{1/2} \tag{130}$$

and

$$\gamma_m=\arctan[(-\Delta\lambda\omega_m^2-\lambda_1^2/4)^{1/2}/(\omega_m^2+\lambda_1/2)]$$
$$=\arctan[|B_m|\lambda_1/2(\omega_m^2+\lambda_1/2)]. \tag{131}$$

Hence

$$|A_m|\begin{Bmatrix}\cos(\gamma_m/2)\\ \sin(\gamma_m/2)\end{Bmatrix}=\frac{1}{\sqrt{2}}[|A_m|^2\pm(\omega_m^2+\lambda_1/2)]^{1/2} \tag{132}$$

and the terms in the sum (128) for $\omega_m^2>-\lambda_1^2/4\Delta\lambda$ have to be taken as

$$\Delta f^\delta=T\sum_{m=-\infty}^{\infty}\left\{\sqrt{2}\left[[\omega_m^2(\omega_m^2+\lambda_0)]^{1/2}+\left[\omega_m^2+\frac{\lambda_1}{2}\right]\right]^{1/2}-(\omega_m^2+\tilde\lambda)-(\omega_m^2)^{1/2}\right\}-\frac{\Delta\lambda}{8\pi} . \tag{133}$$

The variables $\tilde\lambda$, T enters (128) and (133) in the dimensionless combinations $\tilde\lambda_T=\tilde\lambda/4\pi^2T^2$, and Δf^δ has the functional form

$$\Delta f^\delta(\tilde\lambda,\Delta\lambda,T)=\tilde\lambda h^T(\tilde\lambda_T,\delta) . \tag{134}$$

We are now in the position to extremize the total action $\mathcal{A}_{\rm tot}=[(d-2)/2]R_{\rm ext}\beta_{\rm ext}\sqrt{\rho_0\rho_1}f_{\rm tot}^T$. When differentiating with respect to $\sqrt{\rho_0},\sqrt{\rho_1},\tilde\lambda,\delta$ at fixed $T_{\rm ext}$ we have to remember that $\tilde\lambda_T\equiv\tilde\lambda/4\pi^2T$ contains ρ_0 as a factor. Then, using once more the function $g(\tilde\lambda_T)$ defined in (48), we obtain, as a generalization of (50)–(52) the four equations

$$\tilde M_{\rm NG}^2+f_0(\tilde\lambda)-\frac{\tilde\lambda}{4\pi}+\tilde\lambda[g(\tilde\lambda_T)+h^T(\tilde\lambda_T,\delta)]+2\tilde\lambda[g'(\tilde\lambda_T)+h^{T\prime}(\tilde\lambda_T,\delta)]\tilde\lambda_T+\frac{1}{2\tilde\alpha}\left[\frac{-\lambda_0}{\rho_0}+\frac{\lambda_1}{\rho_1}\right]=0 , \tag{135}$$

$$\tilde M_{\rm NG}^2+f_0(\tilde\lambda)-\frac{\tilde\lambda}{4\pi}+\tilde\lambda[g(\tilde\lambda_T)+h^T(\tilde\lambda_T,\delta)]+\frac{1}{2\tilde\alpha}\left[\frac{\lambda_0}{\rho_0}-\frac{\lambda_1}{\rho_1}\right]=0 , \tag{136}$$

$$\frac{\partial}{\partial\tilde\lambda}f_0(\tilde\lambda)-\frac{1}{4\pi}+g(\tilde\lambda_T)+h^T(\tilde\lambda_T,\delta)+[g'(\tilde\lambda_T)+h^{T\prime}(\tilde\lambda_T,\delta)]\tilde\lambda_T+\frac{1}{2\tilde\alpha\tilde\lambda}\left[\frac{\lambda_0}{\rho_0}+\frac{\lambda_1}{\rho_1}\right]=0 , \tag{137}$$

$$\tilde\lambda\dot h^T(\tilde\lambda_T,\delta)-\frac{\tilde\lambda}{2\tilde\alpha}\left[\frac{1}{\rho_0}-\frac{1}{\rho_1}\right]=0 , \tag{138}$$

where the overdots and primes denote the derivatives with respect to $\delta,\tilde\lambda_T$, respectively. Adding and subtracting the first two equations gives [compare (53) and (54)]

$$\tilde M_{\rm NG}^2+f_0(\tilde\lambda)-\frac{\tilde\lambda}{4\pi}+\tilde\lambda(g+h^T)+\tilde\lambda(g'+h^{T\prime})\tilde\lambda_T=0 , \tag{139}$$

$$\frac{1}{\tilde\alpha}\left[\frac{\lambda_0}{\rho_0}-\frac{\lambda_1}{\rho_1}\right]=2\tilde\lambda(g'+h^{T\prime})\tilde\lambda_T . \tag{140}$$

Inserting into (139) the identity given above Eq. (55) and the gap equation (137) we find [extending (56)]

$$\frac{1}{\tilde\alpha}\left[\frac{\lambda_0}{\rho_0}+\frac{\lambda_1}{\rho_1}\right]=\frac{\tilde\lambda}{2\pi}+2\tilde M_{\rm NG}^2 \tag{141a}$$

or

$$\frac{\tilde\lambda}{\tilde\alpha}\left[\frac{1}{\rho_0}+\frac{1}{\rho_1}\right]-\frac{\tilde\lambda\delta}{\tilde\alpha}\left[\frac{1}{\rho_1}-\frac{1}{\rho_1}\right]=\frac{\tilde\lambda}{2\pi}+2\tilde M_{\rm NG}^2 . \tag{141b}$$

Using the last equation of (138) amounts to

$$\frac{1}{\tilde\alpha}\left[\frac{1}{\rho_0}+\frac{1}{\rho_1}\right]=\frac{1}{2\pi}+2\delta\dot h^T(\tilde\lambda_T,\delta)+2\tilde M_{\rm NG}^2/\tilde\lambda . \tag{142}$$

Equation (141) implies [just as before Eq. (56)] that the total free energy density reduces simply to

$$f_{\rm tot}^T(\tilde\lambda,\Delta\lambda,T)=2\tilde M_{\rm NG}^2+f_0(\tilde\lambda)+\Delta f^T(\tilde\lambda,T)+\Delta f^\delta(\tilde\lambda,\Delta\lambda,T)=2\tilde M_{\rm NG}^2+f_0(\tilde\lambda)+\tilde\lambda[g(\tilde\lambda_T)+h^T(\tilde\lambda_T,\delta)] . \tag{143}$$

In order to simplify the gap equations we insert (141) into (137) and find

$$\frac{\tilde M_{\rm NG}^2}{\tilde\lambda}+\frac{\partial}{\partial\tilde\lambda}f_0(\tilde\lambda)+g+h^T+(g'+h^{T\prime})\tilde\lambda_T=\frac{\tilde M_{\rm NG}^2}{\tilde\lambda}+\frac{\partial}{\partial\tilde\lambda}f_{\rm tot}^T=0 . \tag{144}$$

Then we combine (141a) with (140) and obtain [compare (70) and (71)]

$$\frac{1}{\tilde\alpha}\frac{\lambda_0}{\rho_0}=\tilde M_{\rm NG}^2+\frac{\tilde\lambda}{4\pi}+\tilde\lambda(g'+h^{T\prime})\tilde\lambda_T ,\qquad \frac{1}{\tilde\alpha}\frac{\lambda_1}{\rho_1}=\tilde M_{\rm NG}^2+\frac{\tilde\lambda}{4\pi}-\tilde\lambda(g'+h^{T\prime})\tilde\lambda_T . \tag{145}$$

Using Eq. (139) this can be rewritten as

$$\frac{1}{\tilde{\alpha}}\frac{\lambda_0}{\rho_0}=\tilde{M}^2_{\rm NG}+\frac{\tilde{\lambda}}{4\pi}-\left[f^T_{\rm tot}-\tilde{M}^2_{\rm NG}-\frac{\tilde{\lambda}}{4\pi}\right]$$
$$=2\tilde{M}^2_{\rm NG}+\frac{\tilde{\lambda}}{2\pi}-f^T_{\rm tot}\ ,$$
$$\frac{1}{\tilde{\alpha}}\frac{\lambda_1}{\rho_1}=\tilde{M}^2_{\rm NG}+\frac{\tilde{\lambda}}{4\pi}+\left[f^T_{\rm tot}-\tilde{M}^2_{\rm NG}-\frac{\tilde{\lambda}}{4\pi}\right]$$
$$=f^T_{\rm tot} \tag{146}$$

so that

$$\rho_1/\rho_0=[(1+\delta)/(1-\delta)]\left[2\tilde{M}^2_{\rm NG}+\frac{\tilde{\lambda}}{2\pi}-f^T_{\rm tot}\right]/f^T_{\rm tot}\ . \tag{147}$$

The normalized quantities $\hat{\rho}_0,\hat{\rho}_1$, are obtained from (146) by removing the $T\to 0$ limit

$$f^T_{\rm tot}\to\bar{\lambda}_\nu(1+\nu)/4\pi,\quad \frac{1}{\tilde{\alpha}\rho_{0,1}}\to\frac{1+\nu}{4\pi}$$

so that [compare with (76) and (77)]

$$\frac{1}{\hat{\rho}_0}=\frac{1}{1+\nu}\frac{1}{1-\delta}\left[2-\frac{4\pi}{\tilde{\lambda}}(f^T_{\rm tot}-2\tilde{M}^2_{\rm NG})\right],$$
$$\frac{1}{\hat{\rho}_1}=\frac{1}{1+\nu}\frac{1}{1+\delta}\frac{4p}{\tilde{\lambda}}f^T_{\rm tot}\ . \tag{148}$$

When combining (146) with (142), finally, we obtain the following gap equation for δ:

$$h^T(\bar{\lambda}_T,\delta)=2\frac{\partial\Delta f}{\partial\Delta\lambda}$$
$$=\frac{1}{4\pi}\frac{1}{1-\delta^2}\{\delta[1+(4\pi/\tilde{\lambda})\tilde{M}^2_{\rm NG}]$$
$$-[(4\pi/\tilde{\lambda})(f^T_{\rm tot}-\tilde{M}^2_{\rm NG})-1]\}\ . \tag{149}$$

We now evaluate these equations. In (144) and (149), we need the derivatives of h with respect to $\tilde{\lambda}$ at fixed δ and with respect to $\Delta\lambda$ at fixed $\tilde{\lambda}$. These are obtained from

$$\left.\frac{\partial A^\pm_m}{\partial\tilde{\lambda}}\right|_\delta=\frac{1+\delta}{4A^\pm_m}\left[1\pm\frac{1}{2}\left[B_m+\frac{1}{B_m}\right]\right]$$
$$=\frac{1+\delta}{4A^\pm_m}\left[1\pm\left[1+\frac{1}{2B_m}(B_m-1)^2\right]\right],$$
$$\left.\frac{\partial A^\pm_m}{\partial\Delta\lambda}\right|=\frac{1}{8A^\pm_m}\left[1\pm B_m\pm\frac{4\omega^2_m}{\tilde{\lambda}B_m}\frac{1-\delta}{(1+\delta)^2}\right]$$
$$=\pm\frac{1}{4\lambda_1}\frac{1}{B_m}(A^\pm_m+\omega^2_m/A^\pm_m) \tag{150}$$

so that the gap equation for $\tilde{\lambda}$ and $\Delta\lambda$ involve the expressions

$$a_m=\frac{1+\delta}{4}\left[1/A^+_m+1/A^-_m\right.$$
$$\left.+\frac{1}{2}\left[B_m+\frac{1}{B_m}\right](1/A^+_m-1/A^-_m)\right], \tag{151a}$$
$$b_m=\frac{1}{4\lambda_1 B_m}[A^+_m-A^-_m+\omega^2_m(1/A^+_m-1/A^-_m)]\ .$$

For $\omega^2_m>\lambda^2_1/4\Delta\lambda$, we have to use

$$a_m=\frac{1+\delta}{2|A_m|}\{\cos(\gamma_m/2)$$
$$-\sin(\gamma_m/2)[1+4\delta\omega^2_m/(1+\delta)^2\tilde{\lambda}]/|B_m|\}, \tag{151b}$$
$$b_m=-\frac{1}{2\lambda_1|B_m|}|A_m|\sin(\gamma_m/2)(1-\omega^2_m/|A_m|^2)\ .$$

The gap equations read

$$\frac{1}{4\pi}\frac{\bar{\lambda}_\nu}{\tilde{\lambda}}\nu-\frac{1}{4\pi}\ln(T^2/\bar{T}^2)+\frac{T}{2\sqrt{\tilde{\lambda}}}+\frac{1}{2\pi}S_2$$
$$+T(a_0-1/2\sqrt{\tilde{\lambda}})$$
$$+T\sum_{m=1}^{\infty}[2a_m-1/(\omega^2_m+\tilde{\lambda})^{1/2}]-\delta/4\pi=0 \tag{152}$$

and

$$2Tb_0+4T\sum_{m=1}^{\infty}b_m-\frac{1}{4\pi}$$
$$=\frac{1}{4\pi}\frac{1}{1-\delta^2}\{\delta[1+(4\pi/\tilde{\lambda})\tilde{M}^2_{\rm NG}]$$
$$-[(4\pi/\tilde{\lambda})(f^T_{\rm tot}-\tilde{M}^2_{\rm NG})-1]\}, \tag{153}$$

where

$$a_0-1/2\sqrt{\tilde{\lambda}}=(\sqrt{1+\delta}-1)/2\sqrt{\tilde{\lambda}},$$
$$b_0=1/4\sqrt{\tilde{\lambda}(1+\delta)}\ . \tag{154}$$

In Appendix B we have written out these equations explicitly in a form which is most convenient for numerical calculations. These proceed as follows. For each $\bar{\lambda}_T=\tilde{\lambda}/4\pi^2T^2$, we calculate from (83) [or (B4)] a lowest approximation for T assuming $\delta=0$. For that T we find from (153) a new δ which we insert into (152) to obtain an improved value of T. After a few iterations, this procedure converges. In this way we find T and δ as a function of $\bar{\lambda}_T$ and thus also $\tilde{\lambda}=\bar{\lambda}_T 4\pi^2T^2$. These values are inserted into (143) to get the free energy density $f^T_{\rm tot}$ and into (148) to get $\hat{\rho}_{0,1}$. At the end we plot

$$\hat{M}^2_{\rm tot}=\hat{M}^2_{\rm tot}(T_{\rm ext})/\tilde{M}^2_{\rm tot}(0)$$
$$=\sqrt{\hat{\rho}_0\hat{\rho}_1}(4\pi/\tilde{\lambda})f^T_{\rm tot} \tag{155}$$

as a function of $T_{\rm ext}/\bar{T}_{{\rm ext},\nu}$ where

$$\bar{T}_{\text{ext},\nu} \equiv \sqrt{\bar{\rho}\,\bar{\lambda}_\nu(1+\nu)}/4\pi e^{-\gamma} = \sqrt{1/2\pi(d-2)}\,e^{\gamma} M_{\text{tot}} \; . \tag{156}$$

Let us study the limit of small T analytically. In this limit, the gap becomes nearly isotropic so that it is sufficient to keep the anisotropic terms only up to second order in δ and T. In particular there will be terms $\delta^2, T\delta$.

The first is known from Eq. (115), $\Delta f^\delta = -(\tilde{\lambda}/16\pi)\delta^2 + \cdots$. The $T\delta$ term is found by applying the Euler-McLaurin formula to the first term in Eq. (125). It gives

$$\Delta f^\delta \underset{\substack{\Delta\lambda\approx 0\\ T\approx 0}}{\sim} \frac{\Delta\lambda}{4\tilde{\lambda}}\left[T\sum_{m=-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d\omega_m}{2\pi}\right] \times \left[\frac{2\omega_m^2+\tilde{\lambda}}{(\omega_m^2+\tilde{\lambda})^{1/2}} - \frac{2\omega_m^2}{(\omega_m^2)^{1/2}}\right] + O((\Delta\lambda)^2) \; , \tag{157}$$

$$\underset{\substack{\Delta\lambda\approx 0\\ T\approx 0}}{\sim} \frac{\Delta\lambda}{4\sqrt{\tilde{\lambda}}}\,T[1-2B_1(1)] \; ,$$

where $B_1(1)$ is the Bernoulli number $B_1(0) = -\zeta(0) = \frac{1}{2}$. Hence the linear term in $T\delta$ cancels and we have the approximation, correct to order $\delta^2, T\delta$,

$$\Delta f^\delta = \tilde{\lambda} h^T = \frac{\tilde{\lambda}}{4\pi}(-\delta^2/4) + \cdots \; . \tag{158}$$

Notice that there are terms of order δ to higher order in T. Indeed, we know *all* terms of order δ from the calculation in Appendix B. Using Eq. (A11) and ignoring only the Bessel functions which are exponentially small for small T we have

$$\Delta f_1^\delta = \tilde{\lambda} h_1^T = \frac{\tilde{\lambda}}{4\pi}\frac{\delta}{3\tilde{\lambda}_T} = \frac{\delta\pi}{3}T^2 \tag{159}$$

plus $O(\delta^2)$. Hence we can rewrite

$$\Delta f^\delta = \tilde{\lambda} h^T = \frac{\tilde{\lambda}}{4\pi}\left[\frac{\delta}{3\tilde{\lambda}_T} - \delta^2/4\right] + \cdots \; , \tag{160}$$

whose δ term is correct to all order in T while the δ^2 term will have corrections of order T. With this approximation, the total free energy at the extremum (143) is simplified to

$$f_{\text{tot}}^T = 2\frac{\bar{\lambda}_\nu \nu}{4\pi} + \frac{\tilde{\lambda}}{4\pi}\left[-\ln(\tilde{\lambda}/\bar{\lambda}_\nu) - \nu + 1 - \frac{1}{3\tilde{\lambda}_T} + \frac{\delta}{3\tilde{\lambda}_T} - \frac{\delta^2}{4}\right] + \cdots \tag{161}$$

and the gap equations for $\tilde{\lambda}$ and δ reduce to

$$\frac{1}{4\pi}[(\bar{\lambda}_\nu/\tilde{\lambda} - 1)\nu - \ln(\tilde{\lambda}/\bar{\lambda}_\nu) - \delta^2/4] = 0 + \cdots \; , \tag{162}$$

$$\frac{1}{4\pi}\left[\frac{1}{3\tilde{\lambda}_T} - \frac{\delta}{2}\right] = \frac{1}{4\pi}\frac{1}{1-\delta^2}\left[\delta[1+(\bar{\lambda}_\nu/\tilde{\lambda})\nu] - \left[(\bar{\lambda}_\nu/\tilde{\lambda} - 1)\nu - \ln(\tilde{\lambda}/\bar{\lambda}_\nu) - (1-\delta)/3\tilde{\lambda}_T - \frac{\delta^2}{4}\right]\right] + \cdots \; . \tag{163}$$

For $\delta = 0$, the first equation is solved by $\tilde{\lambda} = \bar{\lambda}_\nu$. For $\delta \neq 0$, we expand $\tilde{\lambda} = \bar{\lambda}_\nu(1+l)$ and find, to lowest order in l,

$$l \approx -\frac{1}{1+\nu}\delta^2/4 \; . \tag{164}$$

In the second equation we observe an important cancellation: The term $1/3\tilde{\lambda}_T$ drops out on both sides. Thus, for small l, the gap equation becomes

$$0 = \frac{1}{1-\delta^2}\left[\frac{\delta}{2}(1-\delta^2) + \delta(1+\nu-l\nu) + (1+\nu)l + \frac{\delta^2}{4} - \delta(1-\delta)/3\tilde{\lambda}_T\right] + \cdots \; . \tag{165}$$

Since l vanishes at $\delta = 0$ like δ^2, this equation is solved by $\delta = 0$. Only exponentially small corrections are expected from the omitted Bessel functions.

It is then easy to write down a very accurate approximate solution for the string tension M_{tot}^2 as a function of temperature. Since $\tilde{\lambda} = \bar{\lambda}_\nu$ up to exponentially small corrections, the total free energy density becomes simply

$$f_{\text{tot}}^T = \frac{\bar{\lambda}\nu}{4\pi}(1+\nu-1/3\tilde{\lambda}_T) + \cdots \; . \tag{166}$$

Thus the thermal deconfinement transition lies at

$$\tilde{\lambda}_T = \tilde{\lambda}/4\pi^2 T^2 = \bar{\lambda}_\nu/4\pi^2 T^2 = 1/3(1+\nu) \; . \tag{167}$$

This implies

$$\sqrt{\bar{\rho}}\,T_\nu^d = \sqrt{3/4\pi^2}\sqrt{\bar{\rho}\,\bar{\lambda}_\nu(1+\nu)} = \sqrt{12}\,e^{-\gamma}\bar{T}_{\text{ext},\nu} = \sqrt{6/\pi(d-2)}\,M_{\text{tot}} \approx 0.977 M_{\text{tot}} \; . \tag{168}$$

Thus, when expressed in terms of the total string tension M_{tot}, $\sqrt{\bar{\rho}}\,T_\nu^d$ is independent of ν, in particular, it is the same as for $\nu = 0$, in (94).

In order to translate $\sqrt{\bar{\rho}}\,T_\nu^d$ into physical extrinsic temperature we have to calculate $\hat{\rho}_0$: From (76) and (166), we find

$$\frac{1}{\hat{\rho}_{0,1}} = \frac{1}{1+\nu}(1+\nu \pm 1/3\tilde{\lambda}_T) + \cdots \; . \tag{169}$$

At the deconfinement transition $\hat{\rho}_0^{-1}=2$ so that the extrinsic temperature is lowered with respect to (168) by a factor $\sqrt{2}$; i.e., the physical deconfinement temperature lies for any normality at

$$T^d_{\text{ext},\nu} \approx 0.691 M_{\text{tot}} \tag{170}$$

just as in the case $\nu=0$. Certainly, this result holds only if ν is not too large. Otherwise $\bar{\lambda}$ will be much smaller than $\bar{\lambda}(1+\nu)$ and the Bessel functions behave like powers in $\bar{\lambda}_T$ rather than being exponentially small.

The quantity $\hat{M}^2_{\text{tot}}=M^2_{\text{tot}}(T_{\text{ext}})/M^2_{\text{tot}}(0)$ has, for all ν, the approximate form

$$\hat{M}^2 \approx \left[\frac{1-1/3\bar{\lambda}_T(1+\nu)}{1+1/3\bar{\lambda}_T(1+\nu)}\right]^{1/2} \approx \left[\frac{1-4\pi^2T^3/3\bar{\lambda}_\nu(1+\nu)}{1+4\pi^2T^2/3\bar{\lambda}_\nu(1+\nu)}\right]^{1/2} . \tag{171}$$

This implies the small-T expansion

$$\hat{M}^2 \approx 1-\frac{d-2}{2}\frac{\pi}{3}T^2\bar{\rho}\hat{\rho}_0/M^2_{\text{tot}} . \tag{172}$$

Inserting $\hat{\rho}_0$,

$$\hat{\rho}_0 \approx [1+4\pi^2T^2/3\bar{\lambda}_\nu(1+\nu)]^{-1} \tag{173}$$

we find from (171) once more the pure Nambu-Goto tension, up to exponentially small corrections,

$$\hat{M}^2 \approx [1-(d-2)\pi T^2_{\text{ext}}/3M^2_{\text{tot}}]^{1/2} \tag{174}$$

just as in (102b), but now valid for $\nu\neq 0$. In particular, the T^2 correction to the total string tension ΔM^2_{tot} is the same for all ν, having the universal form

$$\Delta M^2_{\text{tot}} = -\frac{d-2}{2}\frac{\pi}{3}T^2_{\text{ext}} . \tag{175}$$

In Fig. 1 we have plotted the string tension as a function of temperature including the effect of the Bessel functions *and* gap anisotropy and see that the result is indistinguishable from our simple approximation (172) and (174). Indeed, the gaps $\lambda_0,\lambda_1,\bar{\lambda}$ remain almost constant and *identical*, up to the deconfinement transition.

The solution of the equations for various values of ν will be presented elsewhere[16] where we shall calculate, in particular, the deconfinement temperature as a function of ν. In the limits $\nu \to -1$ and $\nu \to \infty$ the deconfinement temperature coincides with the Nambu-Goto results.

IX. CONCLUSION

The temperature behavior of the tension of a string with extrinsic curvature stiffness is exactly calculable. When expanded in powers of T, the result is independent of the stiffness, i.e., of the normality ν. For the region of moderate ν we have been able to find an analytic approximation to the solution which is extremely accurate up to the deconfinement temperature. It coincides exactly with the corresponding result of a pure Nambu-Goto string. The ν dependence resides all in exponentially small terms of the order $e^{-\sqrt{\bar{\lambda}_\nu}/T}$. These terms become important only if the spontaneously generated part of the tension, $\bar{\lambda}_\nu$, is much smaller than the normal part $\bar{\lambda}_\nu\nu$. Then the exponentially small terms turn into powers and require special consideration. In the limit of large ν our solution approaches the perturbative expansion in powers of $\bar{\alpha} \propto 1/(1+\nu)$ which will be discussed in detail in Ref. 15.

An important aspect of the exact solution is the singularity at the transition, just as in the Nambu-Goto string. A similar singularity is expected to occur also in the not exactly soluble problem of the quark potential, at some R_c. Thus, the string with extrinsic curvature stiffness has at present similar problems in explaining real physical phenomena as the Nambu-Goto string. These singularity aspects will be studied in more detail in Ref. 16.

The only way to circumvent such unpleasant singularities seems to lie in the existence of a nontrivial infrared stable fixed point in the β function. Apparently, this can exist only if there is some kind of self-avoidance constraint in the ensemble of surfaces. This will probably have to be imposed via some negative weights of intertwined surfaces.[1] Recent simulations of surfaces[17–19] without such constraints have, until now, not helped to clarify the issue. In simulations, the difficulty is to avoid breaking reparametrization invariance as in Ref. 19. Otherwise the surface possesses in-plane elasticity, which they are not intended to have and which can produce additional unphysical phase transitions.

Note added in proof. After this manuscript was submitted, several papers have appeared dealing with the same subject but not solving the problem completely as in this work: S.-M. Tse, Phys. Rev. D **37**, 2337 (1988); Z. Xiaoan and K. S. Viswanathan, Mod. Phys. Lett. A **4**, 99 (1989); Z. Xiaoan, Simon Fraser University report, 1989 (unpublished); G. Germán and H. Kleinert, Ref. 15. See also the most recent work in Ref. 16 which evaluates the consequences of the equations derived here.

ACKNOWLEDGMENTS

The author is grateful to Dr. G. Germán for many useful discussions. This work was supported in part by Deutsche Forschungsgemeinschaft under Grant No. K1256.

APPENDIX A: EVALUATION OF THE LINEAR TERM $\propto \Delta\lambda \ln \Delta f^\delta$

Let us evaluate the linear term in the anisotropic energy (127),

$$\Delta f_1^\delta = -\frac{\Delta\lambda}{2}T\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\frac{\omega_m^2-q_1^2}{(\omega_m^2+q_1^2)(\omega_m^2+q_1^2+\bar{\lambda})} , \tag{A1}$$

systematically instead of the nemistic derivation in the text. The sum over all m can be converted into an infinite sum over integrals:

$$\sum_{m=-\infty}^{\infty} = \sum_{\tilde{m}=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega_m}{2\pi} \exp(i\omega_m \tilde{m}/T) \ . \quad \text{(A2)}$$

The Poisson summation formula

$$\sum_{\tilde{m}=-\infty}^{\infty} \exp(2\pi i \mu \tilde{m}) = \sum_{m=-\infty}^{\infty} \delta(\mu - m) \quad \text{(A3)}$$

ensures the equality on both sides, modulo divergencies which vanish in dimensional regularization. The full sum over m in (A1) becomes, therefore,

$$\Delta f_1^\delta = -\frac{\Delta\lambda}{2} \sum_{\tilde{m}=-\infty}^{\infty} \int \frac{d\omega_m}{2\pi} \int \frac{dq_1}{2\pi} \exp(i\omega_m \tilde{m}/T) \frac{\omega_m^2 - q_1^2}{(\omega_m^2+q_1^2)(\omega_m^2+q_1^2+\tilde{\lambda})} \ . \quad \text{(A4)}$$

We now go to polar coordinates in the ω_m, q_1 plane and set

$$\omega_m = q\cos\phi, \quad q_1 = q\sin\phi \quad \text{(A5)}$$

so that

$$\Delta f_1^\delta = -\frac{\Delta\lambda}{2} \sum_{\tilde{m}=-\infty}^{\infty} \int_0^\infty \frac{dq\, q}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \exp(iq\tilde{m}\cos\phi/T) \frac{\cos 2\phi}{q^2+\tilde{\lambda}}$$

$$= \frac{\Delta\lambda}{2} \frac{2}{2\pi} \sum_{\tilde{m}=-1}^{\infty} \int_0^\infty dq\, q \frac{1}{q^2+\tilde{\lambda}} J_2(q\tilde{m}/T) \ , \quad \text{(A6)}$$

where $J_2(z)$ is a Bessel function. We now make use of the identity

$$J_2(qa) = -J_0(qa) + (2/qa) J_1(qa)$$
$$= -J_0(qa) - (2/qa^2) \frac{\partial}{\partial q} J_0(qa) \quad \text{(A7)}$$

and the integral becomes

$$-\int_0^\infty dq\, q \frac{1}{q^2+\tilde{\lambda}} J_0(qa)$$
$$-(2/a^2) \int_0^\infty dq \frac{1}{q^2+\tilde{\lambda}} \frac{\partial}{\partial q} J_0(qa) \ , \quad \text{(A8)}$$

where $a = \tilde{m}/T$. The first integral is well known; using the formula

$$\int_0^\infty dz \frac{J_\nu(za)}{(z^2+b^2)^{\mu+1}} z^{\nu+1} = \frac{b^{\nu-\mu} a^\mu}{2^\mu \Gamma(1+\mu)} K_{\nu-\mu}(ab) \quad \text{(A9)}$$

it becomes $-K_0(\sqrt{\tilde{\lambda}}a)$. The second integral can be brought, by a partial integration, to the form

$$\frac{2}{a^2}\frac{1}{\tilde{\lambda}} - \frac{4}{a^2} \int_0^\infty dq \frac{q}{(q^2+\tilde{\lambda})^2} J_0(qa)$$
$$= \frac{2}{a^2\tilde{\lambda}} + \frac{4}{a^2} \frac{\partial}{\partial\tilde{\lambda}} K_0(\sqrt{\tilde{\lambda}_a}) \quad \text{(A10)}$$

with the identity $K_0(z) + 2K_1(z)/z = K_2(z)$ we obtain

$$\Delta f_1^\delta = \frac{\Delta\lambda}{2\pi} \left[(2T^2/\tilde{\lambda}) \sum_{\tilde{m}=1}^{\infty} (1/\tilde{m}^2) - \sum_{\tilde{m}=1}^{\infty} K_2(2\pi\tilde{m}\sqrt{\tilde{\lambda}_T})/(2\pi\tilde{m}\sqrt{\tilde{\lambda}_T}) \right] \quad \text{(A11)}$$

with the first term in large parentheses being equal to $\pi^2 T^2/3\tilde{\lambda}$ and the second term going to zero exponentially for small T. Thus,

$$\Delta f_1^\delta = \frac{\tilde{\lambda}\delta}{4\pi} \left[1/3\tilde{\lambda}_T - 4 \sum_{\tilde{m}=1}^{\infty} \frac{K_2(2\pi\tilde{m}\sqrt{\tilde{\lambda}_T})}{2\pi\tilde{m}\sqrt{\tilde{\lambda}_T}} \right] . \quad \text{(A12)}$$

If we now remember the alternative representation for Δf^T, once via the Bessel series (36), and once via the ω_m series (28), and the corresponding alternative representation of the gap equation (59b) and (59a), we see that Δf_1^δ can be written as

$$\Delta f_1^\delta = \frac{\Delta\lambda}{2\tilde{\lambda}} \left[\left[\frac{T}{2}\sqrt{\tilde{\lambda}} + 2T \sum_{m=1}^{\infty} (\omega_m^2+\tilde{\lambda})^{1/2} - \omega_m - \frac{\tilde{\lambda}}{2\omega_m} \right] - \tilde{\lambda} T \sum_{m=1}^{\infty} [1/(\omega_m^2+\tilde{\lambda})^{1/2} - 1/\omega_m] \right]$$
$$-\Delta\lambda/8\pi \ . \quad \text{(A13)}$$

The terms in the large square brackets, on the other hand, can be combined to give the expression (125), so that the ω_m sum (128) is indeed a corrected representation of (118).

APPENDIX B: EXPLICIT FORM OF EQUATIONS FOR THERMAL DECONFINEMENT

The total free energy density (143) is

$$f^T_{\rm tot}=2\tilde M^2_{\rm NG}+f_0(\tilde\lambda)+\tilde\lambda[g(\tilde\lambda_T)+h^T(\tilde\lambda_T,\delta)]$$
$$=2\frac{\bar\lambda v}{4\pi}v+\frac{\tilde\lambda}{4\pi}\left[-\ln(\tilde\lambda/\bar\lambda_v)-v+1-1/3\tilde\lambda_T-8\sum_{\tilde m=1}^{\infty}K_1(2\pi\tilde m\sqrt{\tilde\lambda_T})/\tilde m\right]$$
$$+\frac{\tilde\lambda}{4\pi}\left[2(\sqrt{1+\delta}-1)/\sqrt{\tilde\lambda_T}+(4/\sqrt{\tilde\lambda_T})\sum_{m=1}^{\infty}[\hat A^+_m+\hat A^-_m-(m^2+\tilde\lambda_T)^{1/2}-m]-\delta\right], \quad \text{(B1)}$$

where

$$\hat A^{\pm}_m\equiv A^{\pm}_m/2\pi T=[m^2+(\tilde\lambda_T(1+\delta)/2)(1\pm B_m)]^{1/2},\quad B_m=[1+8\delta m^2/\tilde\lambda_T(1+\delta)^2]^{1/2}\,. \quad \text{(B2)}$$

For small $\tilde\lambda_T$ ($\lesssim 3$ say) the first two rows in (B1) are taken as

$$2\frac{\bar\lambda_v}{4\pi}v+\frac{\tilde\lambda}{4\pi}\left[-\ln(T^2/\bar T^2)-2/3\tilde\lambda_T+2/\sqrt{\tilde\lambda_T}+(4/\tilde\lambda_T)\sum_{m=1}^{\infty}[(m^2+\tilde\lambda_T)^{1/2}-m-\tilde\lambda_T/2m]\right]. \quad \text{(B3)}$$

The gap equation for $\tilde\lambda$, (144) reads

$$\frac{1}{4\pi}\left[\left[(\bar\lambda_v/\tilde\lambda)v-\ln(\tilde\lambda/\bar\lambda_v)-v+4\sum_{\tilde m=1}^{\infty}K_0(2\pi\tilde m\sqrt{\tilde\lambda_T})\right]+(\sqrt{1+\delta}-1)/\sqrt{\tilde\lambda_T}+2\sum_{m=1}^{\infty}[2\hat a_m-1(m^2+\tilde\lambda_T)^{1/2}]-\delta\right]=0\,, \quad \text{(B4)}$$

where

$$\hat a_m\equiv 2\pi T a_m=[(1+\delta)/4]\left[1/\hat A^+_m+1/\hat A^-_m+\frac{1}{2}\left[B_m+\frac{1}{B_m}\right](1/\hat A^+_m-1/\hat A^-_m)\right]. \quad \text{(B5)}$$

For small $\tilde\lambda_T$, the first row is calculated via

$$\frac{1}{4\pi}\left[(\bar\lambda_v/\tilde\lambda)v-\ln(T^2/\bar T^2)+1/\sqrt{\tilde\lambda_T}+2\sum_{m=1}^{\infty}[1/(m^2+\tilde\lambda_T)^{1/2}-1/m]\right]. \quad \text{(B6)}$$

The gap equation for δ, (149), is

$$\frac{1}{4\pi}\left[1/(\sqrt{1+\delta}\sqrt{\tilde\lambda_T})+8\sum_{m=1}^{\infty}\hat b_m-1\right]=\frac{1}{4\pi}\frac{1}{1-\delta^2}\left[\delta[1+(\bar\lambda_v/\tilde\lambda)v]-\left[(4\pi/\tilde\lambda)f^T_{\rm tot}-\frac{\bar\lambda_v}{\tilde\lambda}v-1\right]\right], \quad \text{(B7)}$$

where

$$\hat b_m\equiv 2\pi T b_m=[1/4\tilde\lambda_T(1+\delta)B_m][\hat A^+_m-\hat A^-_m+m^2(1/\hat A^+_m-1/\hat A^-_m)]\,. \quad \text{(B8)}$$

[1]A. M. Polyakov, Nucl. Phys. **B268**, 406 (1986).
[2]H. Kleinert, Phys. Lett. B **174**, 335 (1986).
[3]F. David, Europhys. Lett. **2**, 577 (1986), and the papers in Ref. 28.
[4]H. Kleinert, Phys. Rev. Lett. **58**, 1915 (1987).
[5]P. Olesen and S.-K. Yang, Nucl. Phys. **B283**, 73 (1987); F. Alonso and D. Espriu, *ibid.* **B283**, 393 (1987).
[6]E. Braaten, R. D. Pisarski, and Sze-Man Tse, Phys. Rev. Lett. **58**, 93 (1987).
[7]G. Germán and H. Kleinert, Phys. Rev. D (to be published).
[8]H. Kleinert, Phys. Rev. D **37**, 351 (1987).
[9]H. Kleinert, Phys. Lett. B **189**, 187 (1987).
[10]T. Banks and A. Ukawa, Nucl. Phys. **B225** [FS9],145 (1983); D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981); B. Svetitsky and L. Yaffe, Nucl. Phys. **B210** [FS6],423 (1982); J. Kuti, J. Polonyi, and K. Szlachanyi, Phys. Lett. **110B**, 395 (1982); F. Green and K. Karsch, Nucl. Phys. **B238**, 297 (1984); N. Ogilvie, Phys. Rev. Lett. **52**, 1369 (1984); N. Weiss, Phys. Rev. D **24**, 475 (1981); M. Gross, Phys. Lett. **132B**, 125 (1983).
[11]L. McLerran and B. Svetitsky, Phys. Lett. **98B**, 195 (1981); J. Kuti, J. Polonyi, and K. Szlachanyi, *ibid.* **98B**, 199 (1981); L. McLerran and B. Svetitsky, Phys. Rev. D **24**, 450 (1981); R. Gavai and H. Satz, Phys. Lett. **145B**, 248 (1984); C. Curci and R. Tripiccione, *ibid.* **151B**, 145 (1985); S. Das and J. Kogut, Nucl. Phys. **B265** [FS15], 319 (1986); report, 1986 (unpublished); J. Kogut, M. Stone, H. Wyld, W. Gibbs, J. Shigemitsu, S. Shenker, and D. Sinclair, Phys. Rev. Lett. **50**, 393 (1983).
[12]H. Kleinert, Phys. Lett. B **197**, 351 (1987).
[13]H. Kleinert, Mod. Phys. Lett. A **3**, 531 (1987).
[14]R. Pisarski and O. Alvarez, Phys. Rev. D **26**, 3735 (1982). Calculations in the framework of the dual resonance model are given in *Dual Theory*, edited by M. Jacob (North-Holland,

Amsterdam, 1974).
[15]G. Germán and H. Kleinert, Phys. Lett. B (to be published).
[16]G. Germán and H. Kleinert, Phys. Lett. B **220**, 133 (1989).
[17]J. Ambjorn, B. Durhuus, and T. Jonsson, Phys. Rev. Lett. **58**, 2619 (1987).
[18]M. Baig, D. Espriu, and J. F. Wheater, Nucl. Phys. **B314**, 587 (1989).
[19]J. Kogut and A. Moreo, University of Illinois Report No. ILL-(TH)-88-#33, 1988 (unpublished).