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Nonfactorizing Saturation of Current Algebra

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A general method is devised by which nonfactorizing solutions (i.e., those in which the currents are not proportional to the charges) of the charge-current commutation rules between arbitrary states and the current-current commutation rules between states of equal momenta can be constructed, given any set of factorizing solutions of the current-current rules. A theorem is proved according to which such factorizing solutions can be obtained from a large class of infinite-component wave equations.

THE considerable success of current algebras in deriving sum rules has led to an intensive search for simple algebraic models of currents satisfying the $SU(2)\times SU(2)$ or $SU(3)\times SU(3)$ current-current commutation relations, which imply the following sum rules for form factors of the weak and electromagnetic currents in the infinite-momentum frame[1] (i.e., $p_z\to\infty$):

$$\sum_n [g_{an}{}^{iV}(\mathbf{q}_\perp')g_{nb}{}^{jV}(\mathbf{q}_\perp)-g_{an}{}^{jV}(\mathbf{q}_\perp)g_{nb}{}^{iV}(\mathbf{q}_\perp')] =if_{ijk}g_{ab}{}^{kV}(\mathbf{q}_\perp'+\mathbf{q}_\perp)\,, \quad (1)$$

and two similar equations, one for the commutator of an axial-vector (A) current and a vector (V) current giving an axial-vector current, and another one in which the commutator of two axial-vector currents becomes a vector current in the usual way.[2]

The $g_{nm}{}^i(\mathbf{q}_\perp)$ are the infinite-momentum limits of the current matrix elements between physical single-particle states[3]:

$$g_{nm}{}^i(\mathbf{q}_\perp)=\lim_{\kappa\to\infty}\left(\frac{M_nM_m}{E_nE_m}\right)^{1/2} \times\langle\kappa\,\mathbf{a}+\mathbf{p}_n;n|j_i{}^0|\kappa\,\mathbf{a}+\mathbf{p}_m;m\rangle_{\text{phys}}\,, \quad (2)$$

where $\mathbf{a}$ is a vector pointing along the direction in which the momentum becomes infinite, while $\mathbf{q}_\perp$ is the momentum transfer orthogonal to the momentum $\mathbf{a}$.

The problem we pose is to find a set of functions $g_{nm}{}^i(\mathbf{q}_\perp)$ which satisfy Eq. (1), but such that those $g_{nm}{}^i(\mathbf{q}_\perp)$ which vanish at $\mathbf{q}_\perp=\mathbf{0}$ do not vanish identically. The latter condition means that the solution will be "nonfactorizing." For an important subset of the relations in Eq. (1), we find a solution to this problem. The solution is based upon the use of infinite-component wave equations and is exact in some physically important limiting cases in which the spacelike solutions of such equations "decouple" from the timelike ones.

The procedure for constructing such infinite-component wave equations goes as follows: One introduces a field $\psi_\alpha(p^\mu)$ transforming under Lorentz transformations $\Lambda=(\Lambda_\mu{}^\nu)$ as

$$\psi_\alpha(p)\to\psi_{\alpha'}{}'(p')=D_{\alpha'\alpha}(\Lambda)\psi_\alpha(\Lambda^{-1}p)\,, \quad (3)$$

where $D_{\alpha'\alpha}(\Lambda)$ is some reducible representation of the Lorentz group $O(3,1)$, selects some vector operator $J_{\alpha\beta}{}^\mu(p)$, and a scalar operator $\kappa_{\alpha\beta}$ in the representation space (3) which satisfy

$$\begin{aligned} D_{\alpha\beta}(\Lambda)J_{\beta\gamma}{}^\mu(p)D^{-1}{}_{\gamma\delta}(\Lambda)&=J_{\alpha\delta}{}^\nu(\Lambda p)\Lambda_\nu{}^\mu\,,\\ D_{\alpha\beta}{}^\mu(\Lambda)\kappa_{\beta\gamma}D^{-1}{}_{\gamma\delta}(\Lambda)&=\kappa_{\alpha\delta}\,, \end{aligned} \quad (4)$$

then postulates the field equation

$$[J_{\alpha\beta}{}^\mu(p)p_\mu-\kappa_{\alpha\beta}]\psi_\beta(p)=0\,, \quad (5)$$

and determines the conserved current operator j^μ by finding a Lagrangian for (5) and performing a gauge transformation.[4,5] Our starting point will be a factorizing solution. Therefore, the $SU(3)$ octet of vector currents $j_i{}^\mu$ is introduced as the product of j^μ and the

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[1] For references on this subject see, for example, F. Coester and G. Roeppstorf, Phys. Rev. **155**, 1583 (1967).

[2] The sum in (1) means summation over all internal quantum numbers and spins.

[3] We use the normalization of states:

$$\langle\mathbf{p}'|\mathbf{p}\rangle=(2\pi)^3(P_0/M)\delta^3(\mathbf{p}'-\mathbf{p}).$$

[4] The first such wave equation was discussed by E. Majorana, Nuovo Cimento **9**, 335 (1932). The approach has been revived to find form factors of composite particles by Y. Nambu, Phys. Rev. **160**, 1171 (1960); A. O. Barut and H. Kleinert, *ibid.* **156**, 1546 (1967); and C. Fronsdal, *ibid.* **156**, 1653 (1967). Using such wave equations, saturation models were given by S. Fubini, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energy, 1967*, edited by A. Perlmutter and B. Kursunoglu (W. H. Freeman and Co., San Francisco, 1967); and by M. Gell-Mann, D. Horn, and J. Weyers, in Proceedings of the International Conference on Particle Physics in Heidelberg, 1967 (unpublished) and amplified version October 1967; H. Bebié and H. Leutwyler, Phys. Rev. Letters **19**, 618 (1967).

[5] For fixed spatial momentum $\mathbf{p}$, we define the eigenstates of L^2, L_3 with eigenvalues $s(s+1)$ and s_3 as particles. Possible additional quantum numbers are omitted. Note that s may, in general, also take continuous values $S=-\frac{1}{2}+i\sigma$ corresponding to the little spin group $0(2,1)$ if Eq. (5) has spacelike solutions at that $\mathbf{p}$, which is the case for almost all wave equations discussed until now.

eight generators of $SU(3)$, λ_i,[6] i.e.,

$$j_i^\mu = \lambda_i j^\mu. \quad (6)$$

The content of the model is to define the functions $g_{nm}{}^i(\mathbf{q}_\perp)$ in terms of the "spinor" quantities related to the wave equation as

$$g_{nm}{}^i(\mathbf{q}_\perp) = \lim_{\kappa\to\infty}\left(\frac{M_n M_m}{E_n E_m}\right)^{1/2} \times \langle \kappa\,\mathbf{a}+\mathbf{p}_n;\, n\,|\, j_i{}^0\,|\,\kappa\,\mathbf{a}+\mathbf{p}_m;\, m\rangle_{\text{spinor}}. \quad (2')$$

This equation is the analog of Eq. (2), except that the states are now the timelike solutions of the wave equation which are considered in one-to-one correspondence with physical states; the operator $j_i{}^0$ in Eq. (2′) is defined in Eq. (6). Then the quantities $g_{nm}{}^i(\mathbf{q}_\perp)$ can be shown to satisfy the sum rules (1) for an important class of (algebraic as well as nonalgebraic) current operators.[7]

First note that the commutation rule (1) implies that the matrix elements of j^0 itself,

$$g(\mathbf{q}_\perp) \equiv \lim_{\kappa\to\infty}\left(\frac{M'M}{E'E}\right)^{1/2} \langle \mathbf{p}'s's_3'\,|\,j^0\,|\,\mathbf{p}ss_3\rangle, \quad (7)$$

have to satisfy the product rule

$$\sum_n g_{an}(\mathbf{q}_\perp')g_{nb}(\mathbf{q}_\perp) = g_{ab}(\mathbf{q}_\perp'+\mathbf{q}_\perp). \quad (8)$$

In order to discuss this relation, let us rewrite the limit (7) in the more explicit form (indices being understood)

$$\begin{aligned} g(\mathbf{q}_\perp) &= \lim_{\kappa\to\infty}\left(\frac{M'M}{E'E}\right)^{1/2} \langle 0s's_3'\,|\,e^{-i\zeta'\cdot\mathbf{M}} j^0 e^{i\zeta\cdot\mathbf{M}}\,|\,0ss_3\rangle, \\ &= \left(\frac{M}{M'}\right)^{1/2} \langle 0s's_3'\,|\,(j^0+j^3)e^{i\alpha\cdot\mathbf{M}}\,|\,0ss_3\rangle, \\ &= \left(\frac{M'}{M}\right)^{1/2} \langle 0s's_3'\,|\,e^{i\alpha\cdot\mathbf{M}}(j^0+j^3)\,|\,0ss_3\rangle, \end{aligned} \quad (9)$$

where ζ, ζ' are the rapidities $[=\tanh^{-1}(v/c)]$ of the initial and final particle, and $e^{i\alpha\cdot\mathbf{M}}$ is defined in the $SL(2,C)$ representation of the Lorentz group with $\mathbf{M} = -\frac{1}{2}i\boldsymbol{\sigma}$ by

$$e^{i\alpha\cdot\mathbf{M}} = \frac{1}{\sqrt{(M'M)}}\begin{pmatrix} \sqrt{M'} & 0 \\ -q_\perp{}^\dagger & \sqrt{M} \end{pmatrix}, \quad q_\perp{}^\dagger = q_{\perp 1} + iq_{\perp 2}. \quad (10)$$

But $e^{i\alpha\cdot\mathbf{M}}$ can be factorized[8] as

$$e^{i\alpha\cdot\mathbf{M}} = e^{-i\rho M_3} Q e^{i\rho M_3}, \quad (11)$$

with

$$\rho = \ln\frac{M_0}{M} \quad \text{hence,} \quad e^{\rho\sigma_3} = \begin{pmatrix} \sqrt{(M_0/M)} & 0 \\ 0 & \sqrt{(M/M_0)} \end{pmatrix} \quad (12)$$

and

$$Q = \begin{pmatrix} 1 & 0 \\ -q_\perp{}^\dagger/M_0 & 1 \end{pmatrix}, \quad (M_0>0,\ \text{arbitrary}), \quad (13)$$

and if one commutes $e^{-i\rho M_3}$ through (j^0+j^3) to the left, one obtains

$$g(\mathbf{q}_\perp) = \frac{\sqrt{(M'M)}}{M_0} \langle 0s's_3'\,|\,e^{-i\rho M_3}(j^0+j^3)Qe^{i\rho M_3}\,|\,0ss_3\rangle.$$

At this point it is convenient to introduce a new set of states:

$$|qss_3\} \equiv Q\sqrt{(M/M_0)}e^{i\rho M_3}|0ss_3\rangle, \quad |ss_3\} \equiv |0,ss_3\}. \quad (14)$$

In terms of these, $g(\mathbf{q}_\perp)$ becomes[9]

$$\begin{aligned} 'g(\mathbf{q}_\perp) &= \{ss_3'\,|\,(j^0+j^3)\,|\,qss_3\} \\ &= \{-q,s's_3'\,|\,(j^0+j^3)\,|\,ss_3\}. \end{aligned} \quad (15)$$

Let us now consider the consequences of current conservation. Since $j^\mu q_\mu = 0$, we find for states at equal momenta

$$(p_0'-p_0)\langle \mathbf{p}s's_3'\,|\,j_0\,|\,\mathbf{p}ss_3\rangle = 0, \quad (16)$$

such that suitably normalized states with different mass and spin are orthogonal:

$$\langle \mathbf{p}s's_3'\,|\,j_0\,|\,\mathbf{p}ss_3\rangle = \delta_{s's}\delta_{s_3's_3}\delta_{M'M}(p_0/M). \quad (17)$$

This leads in the infinite-momentum limit to the orthogonality relation

$$\{0,s's_3'\,|\,j^0+j^3\,|\,0,ss_3\} = \delta_{s's}\delta_{s_3's_3}\delta_{M'M}. \quad (18)$$

Because of this property, the completeness of the solutions $|\mathbf{p}ss_3\rangle$ of the wave equation at any fixed momentum $\mathbf{p}$ can be expressed in the infinite-momentum

[6] Note that in such a scheme there is a natural way to introduce nonconserved strangeness changing currents simply by letting $\kappa_{\alpha\beta}$ in Eq. (5) depend on the strangeness. This dependence also causes an $SU(3)$ mass splitting which can be fitted to the observed one (compare Ref. 18).

[7] It will turn out [Eqs. (17)–(19)] that the states in the solution space of the wave equation have in general different completeness properties from those of the physical states. This, however, causes no difficulties, since we only have to show that the functions defined in (2′) in terms of quantities related to a wave equation satisfy the sum rules (1), and correspond to a relativistically covariant physical current. There is certainly no additional requirement that the physical metric and the metric in the solution space of the wave equation must be the same.

[8] See H. Bebié and H. Leutwyler, Ref. 4.

[9] Note that Q and (j_0+j_3) commute.

limit as[10]

$$\sum_{ss_3}(j_0+j_3)|0,ss_3\}\{0,ss_3| =\sum_{ss_3}|0,ss_3\}\{0,ss_3|(j^0+j_3)=1. \quad (19)$$

Note that this relation holds up to now only in the space spanned by the states $|0,ss_3\}$. The crucial point of the proof is that for a large class of currents j^μ, this relation can be extended to hold in the much larger Hilbert space $|qss_3\}$. To see this, let us assume q to point in x direction and apply the expression (19) to the state $|qss_3\}$. We observe that the most general current following from the wave equation (5) may consist of so-called algebraic vectors Γ^μ which have no explicit momentum dependence and nonalgebraic ones $j^\mu(p';p)$ which explicitly use the momenta p'^ν and p^ν of the external states to couple to a vector. Since the momentum of the states $|qss_3\}$ is

$$p^\mu=\left(\frac{(M_0^2+M^2+q^2)}{2M_0},\ -q,\ 0,\ \frac{(M_0^2-M^2-q^2)}{2M_0}\right), \quad (20)$$

the matrix elements of j^{0+3} between $\{0ss_3|$ and $|qss_3\}$ can be written in more detail as

$$\{0s's_3'|j^{0+3}\left(\frac{(M_0^2+M'^2)}{2M_0},\ 0,\ 0,\ \frac{(M_0^2-M'^2)}{2M_0};\ \frac{(M_0^2+M^2+q^2)}{2M_0},\ -q,\ 0,\ \frac{(M_0^2-M^2-q^2)}{2M_0}\right)|qss_3\}, \quad (21)$$

where M' and M are the masses of the particles with spin s' and s, respectively.

Let us now assume that the current contains besides an algebraic part Γ^μ only so-called convective currents of the form[11] $S\times(p'+p)^\mu$ or $S\times(p'-p)^\mu$, where S is an algebraic scalar operator. In this case, the matrix elements (21) are in fact independent of M', M, and q. But then since $|qss_3\}$ can be expanded in states $|0ss_3\}$, and in this space (19) is known to be true, we see that (19) indeed reproduces $|qss_3\}$ for arbitrary q.

From this result the product rule (8) follows immediately:

$$\sum_{s''s_3''}\{-q's's_3'|j^{0+3}|s''s_3''\}\{s''s_3''|j^{0+3}|q,ss_3\} =\{-q's's_3'|j^{0+3}|qss_3\}, \quad (22)$$

but this is the same as (8). Hence, we have proved *Theorem 1*:

Every conserved current combined of algebraic and convective terms saturates the factorized current commutation rules at infinite momentum.[10]

This type of approach to the saturation problem has as yet produced only solutions with unphysical features:

(i) Either the mass spectrum contains an infinitely degenerate mass, or the wave equation has spacelike solutions.

(ii) The currents are proportional to the charges (such that the neutron has vanishing electric and magnetic form factors).

In fact, it has been proved that (i) holds as a theorem if the vector $J^\mu(p)$ is purely algebraic and does not depend on p.[12] But there are also examples with unphysical solutions when J^μ has a convective part.[13]

It is the purpose of this paper to give a solution to the second problem for two important subalgebras of (1):

(a) $q'=0$. In this case Eq. (1) forms the so-called charge-current commutation rules.

(b) $q'=-q$, $s_3'=s_3$, which is the subalgebra of (1) taken between states of equal momenta and spin orientation.[14]

Furthermore, in this paper we shall restrict our attention to saturating only the vector-vector commutation rule (1).[15]

Let j^μ be a conserved current which satisfies the product sum rule (18) with a complete set of solutions of Eq. (5). Let furthermore $T^{\mu\nu}$ be an arbitrary algebraic antisymmetric tensor operator in the Hilbert space defined by the solution of the corresponding wave equations. Such a tensor operator always exists. An example is the set of Lorentz group generators $L^{\mu\nu}$ of $D_{\alpha\beta}(\Lambda)$ in (3).

Then consider the current defined by

$$k_i^\mu=\lambda_i j^\mu+\delta_i T^{\mu\nu}q_\nu, \quad (23)$$

where $q^\mu=p'^\mu-p^\mu$ is the momentum transfer. Since $T^{\mu\nu}$ is antisymmetric, this current is always conserved. In the infinite-momentum limit defined by (2), this current becomes

$$g_i(\mathbf{q}_\perp)=\lambda_i\{s's_3'|(j^0+j^3)Q|ss_3\} +\delta_i\{s's_3'|(T^{0\nu}+T^{3\nu})q_\nu Q|ss_3\}. \quad (24)$$

From (20) and a similar expression for p'^μ, we deduce that

$$q^0+q^3=0. \quad (25)$$

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[10] If the space of solutions contains spacelike momenta, then all steps still hold true if one replaces the particle mass m by $\mu=\sqrt{(-p^2)}=iM$ and interprets the state $|0ss_3\rangle$ as state with momentum $p=(0,0,0,\mu)$.

[11] Any current derivable from a conventional second-order Lagrangian through a gauge transformation is at most linear in the external momenta p_μ and p_μ' and hence of this form. For this reason, our theorem covers all cases discussed in the literature so far.

[12] I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968).

[13] E.g., the model of H. Bebié and H. Leutwyler (see Ref. 4) has spacelike solutions.

[14] Note that for zero momentum transfer, spacelike and timelike solutions of the wave equation cannot couple in the exact $SU(3)$ limit, such that in this limit, case (a) is even free from the difficulty (i).

[15] In a forthcoming paper, the effects of $SU(3)$ breaking will be included; compare Ref. 18.

Therefore

$$(T^{0\nu}+T^{3\nu})q_\nu=(T^{0r}+T^{3r})q_r\,,\quad (r=1,2)$$

so that

$$g_i(\mathbf{q}_\perp)=\lambda_i\{s's_3'|(j_0+j_3)Q|ss_3\}+\delta_i\{s's_3'|(T^{0r}+T^{3r})Q|ss_3\}q_r. \tag{26}$$

In fact, noting that $T^{0r}+T^{3r}$, Q, and Q' all commute with each other, we see that the commutator of $g_i(\mathbf{q}_\perp)$ and $g_j(\mathbf{q}_\perp)$ becomes

$$[g_i(\mathbf{q}_\perp'),g_j(\mathbf{q}_\perp)]=[\lambda_i,\lambda_j]\{s's_3'|(j^0+j^3)Q'Q|ss_3\}+([\delta_i,\lambda_j]q_r'+[\lambda_i,\delta_j]q_r)\{s's_3'|(T^{0r}+T^{3r})Q'Q|ss_3\}$$
$$+q_r'q_s[\delta_i\delta_j\sum_{s''s_3''}\{s's_3'|(T^{0r}+T^{3r})Q'|s''s_3''\}\{s''s_3''|(T^{0s}+T^{3s})Q|ss_3\}$$
$$-\delta_j\delta_i\sum_{s''s_3''}\{s's_3'|(T^{0s}+T^{3s})Q|s''s_3''\}\{s''s_3''|(T^{0r}+T^{3r})Q'|ss_3\}]. \tag{27}$$

This equation shows that we can indeed fulfill two subalgebras of (1):

(a) q' or $q=0$, which is called the charge-current algebra. In this case λ_i and δ_i have to satisfy the commutation rules

$$[\lambda_i,\lambda_j]=if_{ijk}\lambda_k\,,\quad [\lambda_i,\delta_j]=if_{ijk}\delta_k. \tag{28}$$

Note that $[\delta_i,\delta_j]$ is completely arbitrary.[16]

(b) $q'=-q$ and $s_3'=s_3$, and equal $SU(3)$ quantum numbers in initial and final state. Here we have to impose, in addition to (28), the condition[17]

$$[\delta_i,\delta_j]=0. \tag{29}$$

If λ_i are the generators of $SU(3)$, both algebras (28) and (29) allow for a nonvanishing d/f ratio in δ_j.

Thus, neutral particles can have nonvanishing form factors. Notice that the generators λ_i have matrix elements only within the same $SU(3)$ multiplet. Different $SU(3)$ multiplets are connected by the δ_j currents. In case (a), the matrix elements of δ_i can be fixed by postulating a commutation rule for $[\delta_i,\delta_j]$ and specifying a representation in agreement with the observed spectrum of internal quantum numbers. Let us summarize our result in *Theorem 2.*

Given any conserved current $\lambda_i j^\mu$ satisfying the factorized current commutation rules and then using an arbitrary antisymmetric algebraic tensor operator $T^{\mu\nu}$, one can construct a new current

$$j_i^\mu=\lambda_i j^\mu+\delta_i T^{\mu\nu}q_\nu\,, \tag{30}$$

whose infinite momentum limit satisfies two important subalgebras of the current-current commutation rules without factorizing. The matrices δ_i have to be a vector operator with respect to λ_i and are otherwise arbitrary for the charge-current subalgebra while they have to commute for the $q'=-q$ subalgebra.

Currents of such a type have led to good agreement with experimental results in dynamical group calculations.[18] The possibility of using them to describe transitions between different $SU(3)$ multiplets will be exploited in a forthcoming paper.[19]

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[16] In particular, we may choose δ_i to close back in the form $[\delta^i,\delta^j]\propto if_{ijk}\delta^k$, in which case λ^i and δ^i form the so-called noninvariance groups of T. G. Kuriyan and E. C. G. Sudarshan, Phys. Letters **21**, 106 (1966). The compact form of this commutation rule is clearly preferable since only a finite number of isospins has been observed until now.

[17] These commutation rules are the same as those used by T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965) in their strong-coupling theory. They have the disadvantage of containing only multiplets with infinitely many isospins.

[18] A. O. Barut, D. Corrigan, and H. Kleinert, Phys. Rev. Letters **20**, 167 (1968); Phys. Rev. **167**, 1527 (1968); D. Corrigan, B. Hamprecht, and H. Kleinert, Nucl. Phys. B (to be published). For the current of the H atom and further references, see H. Kleinert, Phys. Rev. **168**, 1827 (1968).

[19] D. Corrigan and B. Hamprecht, *Lectures in Theoretical Physics* (Gordon and Breach, Science Publishers, Inc., New York, 1969), Vol. XIB. H. Kleinert, *Karlsruhe Lectures 1968, Springer Tracts in Modern Physics* (Springer-Verlag, Berlin, 1968).