PATH INTEGRALS OVER FLUCTUATING NON-RELATIVISTIC FERMION ORBITS ☆

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We show that non-relativistic fermions can be described by a path integral over fluctuating orbits $x^{(i)}(t)$, $\Pi_i\,[\int \mathscr{D}x^{(i)}(t)]\exp\{i\mathscr{A}[x_i]+i\Sigma_{i<j}\mathscr{A}_f[x^{(ij)}]\}$, where $\mathscr{A}[x^{(i)}]$ is the usual action and $\mathscr{A}_f[x^{(ij)}]$ (with $x^{(ij)}\equiv x^{(i)}-x^{(j)}$) is a topological interaction that accounts for the fermionic properties. In $D=1$ dimensions, $\mathscr{A}_f[x]=\pi\int_{t_a}^{t_b}dt\,\dot{x}(t)\delta(x(t))$ and in $D=2$, $\mathscr{A}_f[x]=\int_{t_a}^{t_b}dt\,(\dot{\boldsymbol{x}}\times\boldsymbol{x})/\boldsymbol{x}^2$.

1. A well-known shortcoming of Feynman's path integral for non-relativistic particles is its inability to describe the quantum fluctuations of particle orbits with Fermi statistics [1]. Although there is no problem at the level of second quantization, i.e. for field fluctuations, where the use of anticommuting Grassmann fields leads to the correct statistics, no satisfactory description is available at the first quantized level, i.e. for fluctuating individual orbits. In this note we want to present such a description. It is based on adding to the action an appropriate topological term which guarantees completely antisymmetric wave functions when *summing* over all paths.

2. Consider two ordinary distinguishable particles with a relative interaction $V(x^{(1)}-x^{(2)})$ described by an orbital action

$$\mathscr{A}=\int_{t_a}^{t_b}dt\left(M^{(1)}\frac{\dot{x}^{(1)2}}{2}+M^{(2)}\frac{\dot{x}^{(2)2}}{2}-V(x^{(1)}-x^{(2)})\right). \tag{1}$$

The standard change of variables to center-of-mass and relative coordinates $X=(M^{(1)}x^{(1)}+M^{(2)}x^{(2)})/(M^{(1)}+M^{(2)})$, $x=(x^{(1)}-x^{(2)})$ separates $\mathscr{A}$ into a free center of mass and a relative action

$$\mathscr{A}=\mathscr{A}_{CM}+\mathscr{A}_r=\int_{t_a}^{t_b}dt\frac{M^{(1)}+M^{(2)}}{2}\dot{X}^2+\int_{t_a}^{t_b}dt\left(\frac{\mu}{2}\dot{x}^2-V(x)\right) \tag{2}$$

with the reduced mass $\mu=M^{(1)}M^{(2)}/(M^{(1)}+M^{(2)})$. The time displacement amplitude to be calculated factorizes into that of an ordinary oscillator $(X_bt_b|X_at_a)$ and a relative amplitude $(x_bt_b|x_at_a)$ with well-known result.

Let us now introduce the fermionic character into the description. First, we take care of the indistinguishability and restrict x to the positive semi-axis $x=r>0$. There the completeness relation of local states reads

$$\int_0^\infty dr\,|r\rangle\langle r|=1\,. \tag{3}$$

On introducing the orthogonality relation we have to specify the bosonic or fermionic nature of the wave func-

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tions. Since these occupy only symmetric or antisymmetric wave functions, we take for $\langle r_b | r_a \rangle$ the corresponding representations of the δ-function

$$\langle r_b | r_a \rangle = \int_0^\infty \frac{dp}{\pi} 2 \begin{Bmatrix} \cos pr_b \cos pr_a \\ \sin pr_b \sin pr_a \end{Bmatrix} = \delta(r_b - r_a) . \tag{4}$$

These may be rewritten as

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \{\exp[ip(r_b - r_a)] \pm \exp[ip(r_b + r_a)]\} = \delta(r_b - r_a) . \tag{5}$$

The infinitesimal time displacement amplitude is then in the canonical formulation

$$(r_n \varepsilon | r_{n-1} 0) = \langle r_n | \exp(-i\varepsilon \hat{H}) | r_{n-1} \rangle$$
$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \{\exp[ip(r_n - r_{n-1})] \pm \exp[ip(r_n + r_{n-1})]\} \exp[-i\varepsilon H(p_n, r_n)] , \tag{6}$$

where $H(p, x)$ is the hamiltonian of relative motion. By chaining up $N+1$ of these factors we find the amplitude

$$(r_b t_b | r_a t_a) = \prod_{n=1}^{N} \left(\int_0^\infty dr_n \right) \prod_{n=1}^{N+1} \left(\int_{-\infty}^{\infty} \frac{dp_n}{2\pi} \right) \left[\exp\left(i \sum_{n=1}^{N+1} p_n(r_n - r_{n-1}) \right) \pm \exp\left(i \sum_{n=1}^{N+1} p_n(r_n + r_{n-1}) \right) \right]$$
$$\times \exp\left(-i\varepsilon \sum_{n=1}^{N+1} H(p_n, r_n) \right) \tag{7}$$

for bosons and fermions, respectively. We remove the $\pm$ term by extending the radial integral over all space and write

$$(r_b t_b | r_a t_a) = \sum_{x_b = \pm r_b} \prod_{n=1}^{N} \left(\int_{-\infty}^{\infty} dx_n \right) \prod_{n=1}^{N+1} \left(\int_{-\infty}^{\infty} \frac{dp_n}{2\pi} \right)$$
$$\times \exp\left(i \sum_{n=1}^{N+1} \{[p_n(x_n - x_{n-1})] - \varepsilon H(p_n, x_n) + \pi[\sigma(x_n) - \sigma(x_{n-1})]\} \right), \tag{8}$$

where for bosons $\sigma \equiv 0$ and for fermions $\sigma(x) = \theta(-x)$, the reflected Heaviside function [i.e. $\sigma(x) = 0$, 1 for $x \gtrless 0$] and $x_b \equiv x_{N+1}$, $x_a \equiv x_0 = r_a$.

It is easy to calculate these path integrals for free particles. In the bosonic case we obtain immediately the symmetrized amplitude

$$(r_b t_b | r_a t_a) = \frac{1}{\sqrt{2\pi i (t_b - t_a)/M}} \left[\exp\left(i \frac{\mu}{2} \frac{(r_b - r_a)^2}{(t_b - t_a)} \right) + (r_b \to -r_b) \right] . \tag{9}$$

The spectral decomposition of this contains only the symmetric wave functions. For fermions we observe that the phases σ cancel successively except for the boundary term

$$\exp\{i\pi[\sigma(x_b) - \sigma(x_a)]\} . \tag{10}$$

On summing over $x_b = \pm r_b$ in (8) this term causes a sign change of the $x_b = -r_b$ term and leads to

$$(r_b t_b | r_a t_a) = \frac{1}{\sqrt{2\pi i (t_b - t_a)/M}} \left[\exp\left(i \frac{\mu}{2} \frac{(r_b - r_a)^2}{(t_b - t_a)} \right) - (r_b \to -r_b) \right] , \tag{11}$$

which contains only the antisymmetric wave functions.

Since x, p cover the entire space and $\sigma(x)$ enters only on the boundary we see that it is possible to add, in the action, any potential $V(x)$ and, as long as the ordinary path integral exists, also the integrals with the additional σ-terms in (8) can be done.

It remains to take the continuum limit of the time sliced action in (8), yielding

$$\mathscr{A}=\int_{t_a}^{t_b} dt\,[p\dot{x}-H+\pi\dot{x}(t)\partial_x\sigma(x(t))]\,. \tag{12}$$

The second term can also be written as

$$\mathscr{A}_f=-\pi\int_{t_a}^{t_b} dt\,\dot{x}(t)\delta(x(t))=-\pi\int_{t_a}^{t_b} dt\,\partial_t\theta(x(t))\,, \tag{13}$$

which shows that it is a purely topological term.

It is easy to see that for many fermion orbits $x^{(i)}(t)$, $i=1, ..., I$, the interaction $\sum_{i<j}\mathscr{A}_f[x^{(ij)}]$ leads to completely antisymmetric wave functions. Indeed, eq. (8) becomes

$$(x_b^{(i)}t_b\,|\,x_a^{(j)}t_a)=\sum_{\substack{P(j)\\ x_b^{(j)}=x_{N+1}^{P(j)}}}\prod_{i=1}^{I}\left[\prod_{n=1}^{N}\left(\int_{-\infty}^{\infty}dx_n^{(i)}\right)\prod_{n=1}^{N+1}\left(\int_{-\infty}^{\infty}\frac{dp_n^{(i)}}{2\pi}\right)\right]$$

$$\times\exp\left[i\sum_{n=1}^{N+1}\left(\sum_i[p_n^{(i)}(x_n^{(i)}-x_{n-1}^{(i)})]-\varepsilon H(p_n^{(i)},x_n^{(i)})+\pi\sum_{i<j}[\sigma(x_n^{(ij)})-\sigma(x_{n-1}^{(ij)})]\right)\right], \tag{14}$$

where $\sum_{P(j)}$ is the sum over all permutations of the orbits. The phases $\exp[i\pi\sigma(x)]$ produce the complete antisymmetry.

3. Consider now two particles in two dimensions. Let the relative motion be described in terms of polar coordinates. The scalar product of localized states is

$$\langle r_b\varphi_b\,|\,r_a\varphi_a\rangle=\int_0^{\infty}dp\,p\sum_{m=-\infty}^{\infty}I_m(pr_b)I_m(pr_a)\exp[im(\varphi_b-\varphi_a)]/2\pi\,. \tag{15}$$

For indistinguishable particles, the angle φ is restricted to a half-space, say $\varphi\in(0,\pi)$. When considering bosons or fermions, the phase factor $\exp[im(\varphi_b-\varphi_a)]$ must be replaced by $\exp[im(\varphi_b-\varphi_a)]\pm\exp[im(\varphi_b+\pi-\varphi_a)]$, respectively. On chaining up $N+1$ of such amplitudes the $\pm$ terms can again be accounted for by completing the half-space in φ to $(-\pi,\pi)$ and adding the field $\sigma(\varphi)$. Inserting a hamiltonian and going back to euclidian coordinates x_1, x_2 we arrive at the relative amplitude

$$\langle \boldsymbol{x}_b t_b\,|\,\boldsymbol{x}_a t_a\rangle=\left(\int\mathscr{D}\boldsymbol{x}\int\frac{\mathscr{D}\boldsymbol{p}}{2\pi}\exp(i\mathscr{A}_r[\boldsymbol{x}]+i\mathscr{A}_f[\boldsymbol{x}])+(\boldsymbol{x}_b\to-\boldsymbol{x}_b)\right) \tag{16}$$

with obvious time-slicing as in (8). In polar coordinates, $\mathscr{A}_f$ looks just like (12) but with x replaced by φ:

$$\mathscr{A}_f=\pi\int_{t_a}^{t_b} dt\,\dot{\varphi}(t)\partial_\varphi\sigma(\varphi(t))\,. \tag{17}$$

For more than two particles the amplitude (16) is generalized to the two-dimensional analogue of eq. (14).

More adapted to the periodicity in φ of the system we may replace $\sigma(\varphi)$ by a step function $\sigma_p(\varphi)$, which jumps by 1 at every integer multiple of π. Since σ_p depends only on φ we can also write

$$\mathscr{A}_f=\pi\int_{t_a}^{t_b} dt\, \dot{x} A(\varphi)\,, \tag{18}$$

where $A\equiv\partial\sigma_p(\varphi)$.

When calculating particle distributions or partition functions, i.e. for periodic boundary conditions, this coupling is invariant under gauge transformations

$$A\to A+\partial\Lambda, \tag{19}$$

with smooth and single-valued functions Λ, i.e.

$$(\partial_i\partial_j-\partial_j\partial_i)\Lambda(x)=0\,. \tag{20}$$

We can therefore replace $\sigma_p(\varphi)$ by any function of x which changes by one unit when going from φ_b to $\varphi_b+\pi$, most conveniently

$$\sigma_p(x)=\frac{1}{\pi}\varphi(x)\equiv\frac{1}{\pi}\operatorname{arctg}\frac{x_2}{x_1}\,. \tag{21}$$

Then the action (18) becomes

$$\mathscr{A}_f=\int_{t_a}^{t_b} dt\, \dot{x}\,\partial\varphi(x)=\int_{t_a}^{t_b} dt\, \varepsilon_{ij}\frac{x_i\dot{x}_j}{x^2}\,, \tag{22}$$

where ϵ_{ij} is the antisymmetric 2×2 tensor. Just as (13), this is again a purely topological interaction. It is the same as the action of an infinitesimally thin magnetic flux tube, of elementary flux $\Phi=\hbar c/e$, which makes the tube invisible with no Bohm–Aharonov scattering [2]. The path integral, however, is different in the present case by the extra paths running to the reflected final point $-x_b$.

4. In summary, we have succeeded in taking care of the identity of the particles and the fermionic nature by the simple interaction terms (13) and (22). These terms cause the wave functions to be antisymmetric when *summing* over all paths from the initial point x_a to the final points x_b *and* the reflected $-x_b$.

References

[1] R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals (McGraw-Hill, New York, 1965), see, in particular, the remark on p. 356.
[2] Y. Aharonov and D. Bohm, Phys. Rev. 115 (1959) 485.