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Isoscalar Nucleon Form Factor from $O(4,2)$ Dynamics*

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The most general theory of isoscalar form factors with $O(4,2)$ as dynamical group and a purely algebraic current operator has been constructed for a fermion representation space. Adjusting one free parameter appropriately, we obtain for the $J^P=\frac{1}{2}^+$ ground state

$$G_M{}^S(t)=\frac{\mu^S}{(1-t/0.71)^2};\quad G_E{}^S(t)=\frac{\frac{1}{2}}{(1-t/0.71)^2}\left[1+\frac{t/0.71}{(1-t/0.71)}\right].$$

μ^S is determined to be $\mu^S=-\frac{1}{6}$. The agreement with experiment is excellent for $G_M{}^S/\mu^S$, moderate for $G_E{}^S$, and bad for μ^S.

I. INTRODUCTION

IN a previous paper,[1] the dynamical features of the two maximally degenerate unitary representations of $O(3,1)$ have been extensively discussed. The electromagnetic form factors have been calculated, and they are found to decrease too slowly as a function of the invariant momentum transfer t. If one uses the other unitary representations of $O(3,1)$, one can improve the shape for small momentum transfers.[2] Indeed, when restricted to this range, the theory has been able to reproduce the regularity in the pionic decay widths of baryon resonances astonishingly well.[3] For larger t, however, the form factors start oscillating, a feature that one does not expect on general grounds.

To obtain better results, mixing of $O(3,1)$ representations is needed. This need is also indicated by the fact that more than one $O(3,1)$ tower seems to exist. There are, for instance, four $I=\frac{1}{2}$, $j=\frac{1}{2}$ resonances in the Rosenfeld tables. If several towers do exist, then the physical particles will, in general, be mixtures of these towers.

Such a representation mixing occurs most naturally if one formulates dynamics in terms of a group larger than $O(3,1)$.[4] A model for such a dynamical structure is given by the group theoretical formulation of the electromagnetic interaction of the H atom. There, the group $O(4,2)$ turns out to be the dynamical group.[5–9]

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For this model, the mechanism of representation mixing and the form of the electromagnetic coupling have been studied in considerable detail.[2] It is therefore quite tempting to try to find other representations of $O(4,2)$ which might have a physical interpretation.

The next simplest representation of $O(4,2)$ which includes parity and permits a dynamical theory constructed in a way similar to the H atom contains the fermions with spin parity

$$J^P=\tfrac{1}{2}^\pm, \tfrac{3}{2}^\pm, \cdots, (n-\tfrac{1}{2})^\pm \text{ for every } n=1, 2, 3, \cdots,$$

There are, in fact, experimental indications for such a level structure in the isospin-$\frac{1}{2}$ baryon resonances. The lowest states $\frac{1}{2}^\pm$; $\frac{3}{2}^\pm$, $\frac{1}{2}^\pm$, $\frac{5}{2}^\pm$, etc. have been found, and the empty places may yet be filled (see Fig. 1).

We therefore postulate $O(4,2)$ dynamics for the isospin-$\frac{1}{2}$ baryon resonances in the sense defined in Ref. 2. We can then construct the most general theory of transition form factors with $O(4,2)$ dynamics. This will be done in this paper. We shall not carry isospin explicitly through the calculations. Therefore, our results are to be applied to the isoscalar properties of the baryons. That this identification is possible can be seen by adding isospin trivially to the states and writing the $O(4,2)$ generators as isoscalars. All calculations are then exactly the same as if one neglects isospin completely.

II. THE REPRESENTATION SPACE

We use the representation of $O(4,2)$ in terms of creation and annihilation operators $a_r^\dagger, a_r, b_r^\dagger, b_r$ $(r=1, 2)$ constructed with the Pauli matrices σ_k and $C=i\sigma_2$[6]:

$$\begin{aligned}
L_{ij}&=\tfrac{1}{2}(a^\dagger\sigma_k a+b^\dagger\sigma_k b)=L_k,\\
L_{i4}&=-\tfrac{1}{2}(a^\dagger\sigma_i a-b^\dagger\sigma_i b),\\
L_{i5}&=-\tfrac{1}{2}(a^\dagger\sigma_i Cb^\dagger-aC\sigma_i b),\\
L_{45}&=\frac{1}{2i}(a^\dagger Cb^\dagger-aCb),\\
L_{i6}&=\frac{1}{2i}(a^\dagger\sigma_i Cb^\dagger+aC\sigma_i b),\\
L_{46}&=\tfrac{1}{2}(a^\dagger Cb^\dagger+aCb),\\
L_{56}&=\tfrac{1}{2}(a^\dagger a+b^\dagger b+2),
\end{aligned}\tag{2.1}$$

with the commutation rules

$$[L_{\mu\nu},L_{\mu\lambda}]=ig_{\mu\mu}L_{\nu\lambda},$$

$$g_{\mu\nu}=\begin{bmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&1&&\\&&&&-1&\\&&&&&-1\end{bmatrix}. \tag{2.2}$$

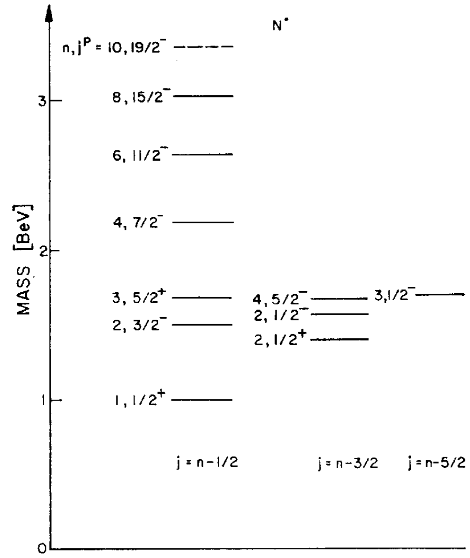

FIG. 1. The levels of the known isospin-$\frac{1}{2}$ baryons and a possible assignment of the $O(4,2)$ quantum numbers n, j^P are given.

Parity is defined by

$$a \xrightarrow{\Pi} b, \quad b \xrightarrow{\Pi} -a \tag{2.3}$$

under which L_k, L_{45}, L_{46}, and L_{56} are invariant, while M_i, L_{i5}, L_{i6} pick up a minus sign.

Using the maximally degenerate boson representation of the H atom, which was given by

$$|nlm\rangle=(-)^m(2l+1)^{1/2} \times\begin{pmatrix}\tfrac{1}{2}(n-1) & \tfrac{1}{2}(n-1) & l\\ \tfrac{1}{2}(m-n_1+n_2) & \tfrac{1}{2}(m+n_1-n_2) & -m\end{pmatrix}|n_1 n_2 m\rangle, \tag{2.4}$$

with the parabolic wave functions

$$|n_1n_2m\rangle=[n_1!(n_2+m)!n_2!(n_1+m)!]^{-1/2} \times a_1^{\dagger n_2+m}a_2^{\dagger n_1}b_1^{\dagger n_1+m}b_2^{\dagger n_2}|0\rangle \tag{2.5}$$

for $m\geq 0$ (for $m<0$ substitute $n_1\to n_1-m$ and $n_2\to n_2-m$ everywhere on the right side), we now construct the fermion states

$$|njm\pm\rangle=(-)^m(2j+1)^{1/2}\begin{pmatrix}\tfrac{1}{2} & j-\tfrac{1}{2} & j\\ r & m-r & -m\end{pmatrix}[n+j+\tfrac{1}{2}]^{-1/2} \times[a_r^\dagger\pm(-)^{j-1/2}ib_r^\dagger]|n\, j-\tfrac{1}{2}\, m-r\rangle. \tag{2.6}$$

Parity is again defined as in (2.3), but additionally, we must specify

$$\Pi|0\rangle=i|0\rangle; \tag{2.7}$$

hence the parities of $|njm\pm\rangle$ are $\pm$, respectively.

One can easily convince oneself that the space (2.6) is irreducible under $O(4,2)$ extended by parity, and that the levels are

$$J^P=\tfrac{1}{2}^\pm,\tfrac{3}{2}^\pm,\cdots,(n-\tfrac{1}{2})^\pm \tag{2.8}$$

for every $n=1, 2, 3, \cdots$.

Explicitly, the states for $n=1$, 2 are given by

$$|1\tfrac{1}{2}r,\pm\rangle=\frac{1}{\sqrt{2}}(a_r^\dagger+ib_r^\dagger)|0\rangle;\ r=1,\ 2,$$

$$|2\tfrac{3}{2}\tfrac{3}{2}\pm\rangle=\tfrac{1}{2}(a_1^\dagger\mp ib_1^\dagger)a_1^\dagger b_1^\dagger|0\rangle,$$

$$|2\tfrac{3}{2}\tfrac{1}{2}\pm\rangle=\frac{1}{2}\bigg[\frac{1}{\sqrt{3}}(a_2^\dagger\mp ib_2^\dagger)a_1^\dagger b_1^\dagger + \frac{1}{\sqrt{3}}(a_1^\dagger\mp ib_1^\dagger)(a_1^\dagger b_2^\dagger+a_2^\dagger b_1^\dagger)\bigg]|0\rangle, \quad (2.9)$$

$$|2\tfrac{1}{2}\tfrac{1}{2}\pm\rangle=\frac{1}{\sqrt{6}}(a_1^\dagger\pm ib_1^\dagger)(a_1^\dagger b_2^\dagger-a_2^\dagger b_1^\dagger)|0\rangle.$$

There exist other extensions of $O(4,2)$ by parity. As we shall show, however, in Sec. IV, the one given here is the only one which leads to a nontrivial theory with $O(4,2)$ as the dynamical group.

III. THE MOST GENERAL MODEL

Using the electromagnetic theory of the H atom as a guide, we construct the most general possible theory of the same type on the fermion representation space. Putting the initial particle at rest and boosting the final one in the z direction with rapidity ζ [$\equiv\tanh^{-1}(v/c)$], we obtain a current with the structure[8]

$$\mathfrak{F}^\mu(\zeta)=\langle 2|e^{-iS_2}\Gamma^\mu e^{iM_3\zeta}e^{iS_1}|1\rangle, \quad (3.1)$$

where the M_i ($i=1$, 2, 3) are Lorentz generators in the Lie algebra under which Γ^μ transforms like a four-vector, while S_1, S_2 are arbitrary rotational scalars. According to the dynamical group philosophy, Γ^μ and S_1, S_2 must also be elements of the Lie algebra.

Since M_i is a vector under the rotation subgroup of $O(4,2)$, it can at most be a linear combination of L_{i4}, L_{i5}, L_{i6}; and the commutation rules

$$[M_i,M_j]=-iL_{ij} \quad (3.2)$$

fix this combination to be of the form

$$M_i=\cosh\epsilon(\cos\tau\, L_{i5}+\sin\tau\, L_{i6})+\sinh\epsilon\, L_{i4}. \quad (3.3)$$

Observe now that this M_i can be rotated by operators of the form e^{iS} into L_{i5}, namely,

$$L_{i5}=e^{iL_{45}\epsilon}e^{-iL_{56}\tau}M_ie^{iL_{56}\tau}e^{-iL_{45}\epsilon}. \quad (3.4)$$

We therefore can assume, without loss of generality, that

$$M_i=L_{i5}; \quad (3.5)$$

otherwise we could bring M_i to this form by changing S and Γ^μ appropriately.

Next, we can assume that S contains only L_{45} and L_{46}:

$$S_{1,2}=\Theta_{1,2}L_{45}+\Delta_{1,2}L_{46}. \quad (3.6)$$

The only other possible term L_{56} could always be taken out of the matrix element (3.1), giving only an over-all phase change, since L_{56} is diagonal on the Hilbert space. Θ and Δ are *tilting angles* which we allow, as in the case of the H atom, to depend on n.

In this work we shall assume, for simplicity, that the current operator Γ^μ is a purely *algebraic* one. This means that we exclude for the time being expressions which use the external momenta of the interaction vertex to construct vector operators, like *convective currents* P^μ, $P^\mu L_{46}$, or like q^μ, $q^\mu L_{46}$, etc. Such expressions will, in general, be needed in a complete theory, since they also occur in the current of the H atom (see the second paper of Ref. 6). The effect of such terms will be studied in a forthcoming paper.

Under this assumption, the current operator can at most be a linear combination of all four-vectors in the Lie algebra, which are

$$\Gamma_1^\mu\equiv(L_{56},L_{i6})$$

and

$$\Gamma_2^\mu\equiv(L_{45},\ -L_{i4}). \quad (3.7)$$

Say that

$$\Gamma^\mu=a\Gamma_1^\mu+b\Gamma_2^\mu. \quad (3.8)$$

If we define now the tilted operators O' as

$$O'=e^{-iS_2}Oe^{iS_2}, \quad (3.9)$$

then the current F^μ can be written as

$$F^\mu=\langle 2|\Gamma^{\mu\prime}e^{-iS_2}e^{iM_3\zeta}e^{iS_1}|1\rangle = \sum_n \langle 2|\Gamma^{\mu\prime}|n\rangle\langle n|G(\zeta)|1\rangle, \quad (3.10)$$

where we have inserted a complete set of states and defined

$$G(\zeta)\equiv e^{-iS_2}e^{iM_3\zeta}e^{iS_1}. \quad (3.11)$$

We first bring G to a form in which its matrix elements can easily be calculated. We write G explicitly, inserting (3.5) and (3.6) into (3.11):

$$G(\zeta)=e^{-i(\Delta_2L_{46}+\Theta_2L_{45})}e^{iL_{35}\zeta}e^{i(\Delta_1L_{46}+\Theta_1L_{45})}. \quad (3.12)$$

Observe that the operators N_1^i and N_2^i ($i=1$, 2, 3) defined by

$$N_1^1=\tfrac{1}{2}(L_{35}+L_{46}),\quad N_1^2=\tfrac{1}{2}(L_{45}-L_{36}),$$
$$N_1^3=\tfrac{1}{2}(L_{56}+L_{34}),\quad N_2^1=\tfrac{1}{2}(-L_{35}+L_{46}), \quad (3.13)$$
$$N_2^2=\tfrac{1}{2}(L_{45}+L_{36}),\quad N_2^3=\tfrac{1}{2}(L_{56}-L_{34})$$

form an $O(2,1)\times O(2,1)$ algebra with the commutation rules

$$[N_{1,2}^i,N_{1,2}^j]=ig_{kk}N_{1,2}^k,\quad g=\begin{bmatrix}1&&\\&1&\\&&-1\end{bmatrix}, \quad (3.14)$$
$$[N_1^i,N_2^j]=0.$$

Then G can be brought to Euler angle form of $O(2,1)$

$\times O(2,1)$

$$G(\zeta)=e^{-i(\alpha_1 N_1{}^3-\alpha_2 N_2{}^3)}e^{-i(\beta_1 N_1{}^2+\beta_2 N_2{}^2)}\times e^{-i(\gamma_1 N_1{}^3-\gamma_2 N_2{}^3)} \quad (3.15)$$

by a simple parameter transformation. The outer factors can be taken out of the matrix elements of G as phases, since L_{34} and L_{56} are diagonal in the representation space. All that remains is the product

$$\langle n|e^{-i(\beta_1 N_1{}^2+\beta_2 N_2{}^2)}|1\rangle =\sum_{n^1}\langle n|e^{-i\beta_1 N_1{}^2}|n'\rangle\langle n'|e^{-i\beta_2 N_2{}^2}|1\rangle, \quad (3.16)$$

in which every factor is just a global representation of $O(2,1)$ $v_{mn}{}^k(\sinh\frac{1}{2}\beta)$ given by Bargmann,[10–12] which has been extensively used in dynamical group calculations[1–3] (see Appendix).

The Euler angles in (3.15) are readily evaluated. To simplify $G(\zeta)$ we move the right exponential to the left in (3.12). Then we obtain

$$G(\zeta)=e^{-i(\Delta_2 L_{46}+\Theta_2 L_{45})}e^{i(\Delta_1 L_{46}+\Theta_1 L_{45})}\times e^{i\zeta(uL_{35}+vL_{36}+wL_{34})}, \quad (3.17)$$

with

$$u=1+(\Theta^2/\nu^2)(\cosh\nu-1),\quad v=(\cosh\nu-1)(\Delta\Theta/\nu^2),\quad w=(\Theta/\nu)\sinh\nu. \quad (3.18)$$

Inserting the $O(2,1)\times O(2,1)$ operators (3.13), we find

$$G(\zeta)=e^{-i(\Delta_2 L_{46}+\Theta_2 L_{45})}e^{i(\Delta_1 L_{46}+\Theta_1 L_{45})}\times e^{i\zeta[u(N_1{}^1-N_2{}^1)-v(N_1{}^2-N_2{}^2)+w(N_1{}^3-N_2{}^3)}, \quad (3.19)$$

where the second factor can be separated into the product of two commuting operators $G_1(\zeta),G_2(\zeta)$ in the form

$$G_1(\zeta)\cdot G_2(\zeta)\equiv e^{i\zeta[uN_1{}^1-vN_1{}^2+wN_1{}^3]}\times e^{i\zeta[-uN_2{}^1+vN_2{}^2-wN_2{}^3]}. \quad (3.20)$$

In this work, we shall confine our attention to the form factors of the $j^P=\frac{1}{2}^+$ ground state. Then the angles Θ_1,Θ_2 and Δ_1,Δ_2 coincide and $G(\zeta)$ becomes $G(\zeta)=G_1(\zeta)\cdot G_2(\zeta)$.

We now calculate the Euler angles $\alpha\equiv\alpha_1$, $\beta\equiv\beta_1$, $\gamma\equiv\gamma_1$ for the first factor by going to the 2×2 quaternion representation

$$N_1{}^1=i\sigma_1/2,\quad N_1{}^2=i\sigma_2/2,\quad N_1{}^3=\sigma_3/2. \quad (3.21)$$

The quaternion for G_1 is then

$$G_1(\zeta)=\cosh\tfrac{1}{2}\zeta-(u\sigma_1-v\sigma_2-iw\sigma_3)\sinh\tfrac{1}{2}\zeta, \quad (3.22)$$

which has to be compared with the Euler quaternion of

$$e^{-i\alpha N_1{}^3}e^{-i\beta N_1{}^2}e^{-i\gamma N_1{}^3}, \quad (3.23)$$

which is

$$G_1(\zeta)=\cos\tfrac{1}{2}(\alpha+\gamma)\cosh\tfrac{1}{2}\beta-\sin\tfrac{1}{2}(\alpha-\gamma)\sinh\tfrac{1}{2}\beta\sigma_1+\cos\tfrac{1}{2}(\alpha-\gamma)\sinh\tfrac{1}{2}\beta\,\sigma_2-i\sin\tfrac{1}{2}(\alpha+\gamma)\cosh\tfrac{1}{2}\beta\,\sigma_3. \quad (3.24)$$

This yields the four equations

$$\cos\tfrac{1}{2}(\alpha+\gamma)\cosh\tfrac{1}{2}\beta=\cosh\tfrac{1}{2}\zeta, \quad (3.25a)$$

$$\sin\tfrac{1}{2}(\alpha-\gamma)\sinh\tfrac{1}{2}\beta=u\sinh\tfrac{1}{2}\zeta, \quad (3.25b)$$

$$\cos\tfrac{1}{2}(\alpha-\gamma)\sinh\tfrac{1}{2}\beta=v\sinh\tfrac{1}{2}\zeta, \quad (3.25c)$$

$$\sin\tfrac{1}{2}(\alpha+\gamma)\cosh\tfrac{1}{2}\beta=-w\sinh\tfrac{1}{2}\zeta. \quad (3.25d)$$

One can easily convince oneself that the same equations hold also for α_2, β_2, and γ_2. From (3.25b) and (3.25c) we find

$$\sinh\tfrac{1}{2}\beta=(u^2+v^2)^{1/2}\sinh\tfrac{1}{2}\zeta,\quad \cosh\tfrac{1}{2}\beta=[1+\sinh^2(\tfrac{1}{2}\beta)]^{1/2} \quad (3.26)$$

such that

$$\sin\tfrac{1}{2}(\alpha-\gamma)=\frac{u}{(u^2+v^2)^{1/2}},\quad \cos\tfrac{1}{2}(\alpha-\gamma)=\frac{v}{(u^2+v^2)^{1/2}}, \quad (3.27)$$

i.e., $\frac{1}{2}(\alpha-\gamma)$ is a constant angle. From (3.25a) and (3.25b) we see that for $\zeta\to 0$, $\alpha\to -\gamma$; hence $\frac{1}{2}(\alpha-\gamma)$ is just the limit of $\alpha(-\gamma)$ for $\zeta\to 0$. We do not give the general solution of (3.25) since only the following special combination of Euler angles, apart from $\frac{1}{2}(\alpha-\gamma)$, $\frac{1}{2}(\alpha+\gamma)$, will be needed for the form factor of the ground state:

$$\cos\tfrac{1}{2}(3\alpha+\gamma)=\cos\tfrac{1}{2}(\alpha-\gamma)+2(\sin^2\tfrac{1}{2}(\alpha+\gamma)\cos\tfrac{1}{2}(\alpha-\gamma)+\sin\tfrac{1}{2}(\alpha+\gamma)\cos\tfrac{1}{2}(\alpha+\gamma)\sin\tfrac{1}{2}(\alpha-\gamma), \quad (3.28)$$

which becomes, due to (3.25),

$$\cos\tfrac{1}{2}(3\alpha+\gamma)=\frac{v}{(u^2+v^2)^{1/2}}-2\times\left[\frac{w^2v}{(u^2+v^2)^{3/2}}\tanh^2\frac{\beta}{2}+\frac{uw}{u^2+v^2}\tanh\tfrac{1}{2}\beta\frac{\cosh\frac{1}{2}\zeta}{\cosh\frac{1}{2}\beta}\right]. \quad (3.29)$$

Consider the current $\mathfrak{F}^\mu$ generated by the operator $\Gamma^\mu=a\Gamma_1{}^\mu+b\Gamma_2{}^\mu$ in the ground state. In our particular kinematical configuration in which ζ points in the z direction, the electric and magnetic form factors can be defined in terms of the components F^0 and F^1 as

$$F^0\equiv\cosh\tfrac{1}{2}\zeta\,G_E,\quad F^1\equiv\sinh\tfrac{1}{2}\zeta\,G_M. \quad (3.30)$$

We shall use Eq. (3.10) to evaluate F^0 and F^1. For this the corresponding components of the current operator

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[11] W. Albrecht *et al.*, Phys. Rev. Letters **17**, 1192 (1966); S. Drell, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley 1966* (University of California Press, Berkeley, California, 1967).

[12] See the Appendix.

Γ^μ have to be tilted to $\Gamma^{\mu\prime}$ according to (3.9). We find

$$\Gamma_1{}^{0\prime}=\cosh\nu L_{56}+\sinh\nu\left(\frac{\Theta}{\nu}L_{46}-\frac{\Delta}{\nu}L_{45}\right),$$

$$\Gamma_2{}^{0\prime}=L_{45}-\frac{\Delta}{\nu}\times\left[\sinh\nu L_{56}+(\cosh\nu-1)\left(\frac{\Theta}{\nu}L_{46}-\frac{\Delta}{\nu}L_{45}\right)\right], \quad (3.31)$$

$$\Gamma_1{}^{1\prime}=L_{16}+\frac{\Delta}{\nu}\times\left[\sinh\nu L_{14}+(\cosh\nu-1)\left(\frac{\Theta}{\nu}L_{15}+\frac{\Delta}{\nu}L_{16}\right)\right],$$

$$\Gamma_2{}^{1\prime}=-\cosh\nu L_{14}-\sinh\nu\left(\frac{\Theta}{\nu}L_{15}+\frac{\Delta}{\nu}L_{16}\right),$$

and therefore have to calculate the currents associated with the operators L_{ab}:

$$F_{ab}=\langle 1|L_{ab}|n'\rangle\langle n'|G(\zeta)|1\rangle. \quad (3.32)$$

To do this we write the L_{ab}'s from (2.1) in the more convenient form

$$\begin{aligned}
L_{56}&=\tfrac{1}{2}(a^\dagger a+b^\dagger b+2),\\
L_{46}&=\tfrac{1}{2}(N_1{}^++N_2{}^++N_1{}^-+N_2{}^-),\\
L_{45}&=(1/2i)(N_1{}^++N_2{}^+-N_1{}^--N_2{}^-),\\
L_{14}&=-\tfrac{1}{2}(a_1{}^\dagger a_2+a_2{}^\dagger a_1-b_1{}^\dagger b_2-b_2{}^\dagger b_1),\\
L_{15}&=\tfrac{1}{2}(a_1{}^\dagger b_1{}^\dagger-a_2{}^\dagger b_2{}^\dagger+a_1b_1-a_2b_2),\\
L_{16}&=-(1/2i)(a_1{}^\dagger b_1{}^\dagger-a_2{}^\dagger b_2{}^\dagger-a_1b_1+a_2b_2),
\end{aligned} \quad (3.33)$$

where we have used the raising and lowering operators $N_1{}^\pm,N_2{}^\pm$ of the $O(2,1)$ algebras (3.13). In terms of $a^\dagger$, $b^\dagger$, etc., they can be written as

$$\begin{aligned}
N_1{}^+&=N_1{}^1+iN_1{}^2=-a_2{}^\dagger b_1{}^\dagger,\\
N_1{}^-&=N_1{}^1-iN_1{}^2=-a_2b_1,\\
N_2{}^+&=N_2{}^1+iN_2{}^2=a_1{}^\dagger b_2{}^\dagger,\\
N_2{}^-&=N_2{}^1-iN_2{}^2=a_1b_2.
\end{aligned} \quad (3.34)$$

The contribution of the term $N_1{}^-$ to F_{45} and F_{46} can then be calculated in the following fashion, using the result in the Appendix:

$$\begin{aligned}
F_{N_1^-}&=\tfrac{1}{2}\sum_n\langle a_1-ib_1|N_1{}^-|n\rangle\langle n|G(\xi)|a_1{}^\dagger+ib_1{}^\dagger\rangle\\
&=\tfrac{1}{2}\{\langle a_1|N_1{}^-|a_1{}^\dagger a_2{}^\dagger b_1{}^\dagger\rangle\langle a_1a_2b_1|G(\xi)|a_1{}^\dagger\rangle+\langle b_1|N_1{}^-|\tfrac{1}{2}\sqrt{2}a_2{}^\dagger b_1{}^{\dagger 2}\rangle\langle\tfrac{1}{2}\sqrt{2}a_2b_1{}^2|G(\xi)|b_1{}^\dagger\rangle\}\\
&=\tfrac{1}{2}\{-e^{-i(\alpha-\gamma)/2}v_{3/2,1/2}{}^{1/2}(\sinh\tfrac{1}{2}\beta)v_{11}{}^1(-\sinh\tfrac{1}{2}\beta)\\
&\quad-\sqrt{2}e^{-i(3\alpha+\gamma)/2}v_{21}{}^1(\sinh\tfrac{1}{2}\beta)v_{1/2.1/2}{}^{1/2}(-\sinh\tfrac{1}{2}\beta)\}\\
&=-\tfrac{1}{2}\{e^{-i(\alpha-\gamma)/2}+2e^{-i(3\alpha+\gamma)/2}\}\frac{\sinh(\frac{1}{2}\beta)}{\cosh^4(\frac{1}{2}\beta)}.
\end{aligned} \quad (3.35)$$

To get $F_{N_2^-}$ we observe that parity changes $N_1{}^-\to N_2{}^-$, hence one need only use $G(\zeta)$ in the formula above, or, equivalently, keep β the same and change $\alpha\to-\alpha$, $\gamma\to-\gamma$. Therefore,

$$F_{N_2^-}=(F_{N_1^-})^*. \quad (3.36)$$

From (3.33) we find then

$$F_{46}=\mathrm{Re}F_{N_1^-}=-\tfrac{1}{2}[\cos\tfrac{1}{2}(\alpha-\gamma)+2\cos\tfrac{1}{2}(3\alpha+\gamma)]\sinh(\tfrac{1}{2}\beta)/\cosh^4(\tfrac{1}{2}\beta) \quad (3.37)$$

and

$$F_{45}=iF_{46}. \quad (3.38)$$

Similarly, we obtain

$$\begin{aligned}
&F_{56}=\tfrac{3}{2}\cos\tfrac{1}{2}(\alpha+\gamma)\cosh^{-3}(\tfrac{1}{2}\beta),\quad F_{16}=-\tfrac{1}{2}\sin\tfrac{1}{2}(\alpha-\gamma)\sinh\tfrac{1}{2}\beta\cosh^{-4}(\tfrac{1}{2}\beta),\\
&F_{15}=iF_{16},\quad F_{14}=-\tfrac{1}{2}i\sin\tfrac{1}{2}(\alpha+\gamma)\cosh^{-3}(\tfrac{1}{2}\beta).
\end{aligned} \quad (3.39)$$

We collect all these terms according to (3.31) and (3.8), insert the results for the Euler angles (3.26)–(3.29), and obtain for the total currents

$$F^0=\left(a\cosh\nu-b\frac{\Delta}{\nu}\sinh\nu\right)\tfrac{3}{2}\cosh\tfrac{1}{2}\zeta-\left[a\frac{\Theta-i\Delta}{\nu}\sinh\nu-b\frac{\Delta(\Theta-i\Delta)}{\nu^2}(\cosh\nu-1)+ib\right]$$
$$\times\left[\frac{3}{2}\frac{v}{(u^2+v^2)^{1/2}}-2\left(\frac{w^2v}{(u^2+v^2)^{3/2}}\tanh^2(\tfrac{1}{2}\beta)-\frac{uw}{u^2+v^2}\tanh\tfrac{1}{2}\beta\frac{\cosh\frac{1}{2}\zeta}{\cosh\frac{1}{2}\beta}\right)\right]\sinh\tfrac{1}{2}\beta\cosh^{-4}(\tfrac{1}{2}\beta) \quad (3.40)$$

and

$$F^1=\mu \sinh\tfrac{1}{2}\zeta \cosh^{-4}(\tfrac{1}{2}\beta), \tag{3.41}$$

with the magnetic moment being

$$\mu=-\tfrac{1}{2}iu\left[a\frac{\Delta(\Theta-i\Delta)}{\nu^2}(\cosh\nu-1)-b\frac{\Theta-i\Delta}{\nu}\sinh\nu-ia\right]+\tfrac{1}{2}iw\left[a\frac{\Delta}{\nu}\sinh\nu-b\cosh\nu\right], \tag{3.42}$$

if the charge, i.e., $F^0(\zeta=0)$, has been normalized to unity.

Comparing (3.41) with (3.30), we obtain the first important result .The magnetic form factor is

$$G_M=\mu/\cosh^4(\tfrac{1}{2}\beta), \tag{3.43}$$

which becomes with (3.26)

$$G_M=\mu/[1+(u^2+v^2)\sinh^2(\tfrac{1}{2}\zeta)]^2. \tag{3.44}$$

Introducing the invariant momentum transfer through

$$t=q^2=-4M^2\sinh^2(\tfrac{1}{2}\zeta), \tag{3.45}$$

we find that

$$G_M(t)=\mu\bigg/\left(1-\frac{u^2+v^2}{4M^2}t\right)^2. \tag{3.46}$$

Hence, the magnetic form factor has the shape of a double pole formula with a singularity at

$$t=4M^2/(u^2+v^2). \tag{3.47}$$

Since $u^2+v^2=1+w^2\geq 1$, the pole position corresponds in general to an anomalous threshold. This is the essential feature distinguishing this theory from other models,[1,13] which all have the singularity at the normal place, $4M^2$. It is due to this property that by proper choice of w, the shape of the magnetic form factor can be brought into coincidence with the function used by experimentalists for a best fit of their data[11]

$$G_M=\mu/(1-t/0.71)^2. \tag{3.48}$$

Looking back at the definition of w in (3.18), we see that $w=0$ if and only if $\Theta=0$. Therefore, the tilting operation with L_{45} as the generator is the source of the anomalous pole position.

What are the electric form factor and the magnetic moment in this model? Aside from the charge normalization, we impose two physical conditions upon the currents: (a) They must be conserved. (b) They must be real, as follows from time-reversal invariance. These conditions restrict the theory to having only two solutions. As we shall show below, they force F^0 to be even in ζ. But then, from (3.40), we see that v must vanish. This is fulfilled if either $\Theta=0$ or $\Delta=0$. In the first case, (3.18) gives

$$u=1,\ v=0,\ w=0,$$

and the form factors become

$$G_E=\left(1-\frac{t}{4M^2}\right)^{-2},\quad G_M=\mu\left(1-\frac{t}{4M^2}\right)^{-2},\quad \mu=-\tfrac{1}{3}. \tag{3.49}$$

In the second case, we obtain

$$G_E=\left(1+\tfrac{4}{3}\tanh^2\Theta\frac{\cosh^2\Theta t/4M^2}{(1-\cosh^2\Theta t/4M^2)}\right)\times(1\text{-}\cosh^2\Theta t/0.71)^{-2}, \tag{3.50}$$

$$G_M=\mu(1-\cosh^2\Theta t/4M^2)^{-2},$$

$$\mu=-\tfrac{1}{3}.$$

As we have stated in the introduction, we may tentatively apply our results to the isoscalar properties of the nucleons, i.e.,

$$\tfrac{1}{2}G_E=G_E{}^S,\quad G_M=G_M{}^S,\quad \tfrac{1}{2}\mu=\mu^S. \tag{3.51}$$

The first solution reproduces very well the observed symmetry

$$G_M/\mu=G_E, \tag{3.52}$$

but the magnitude of μ and the shape of G_E and G_M differ considerably from the observed curve (3.48), since $4M^2\approx 3.5$ and $\mu^S=0.44$. The second solution is far better. Choosing Θ such that

$$\cosh^2\Theta\approx 5, \tag{3.53}$$

we reproduce the correct singularity of G_M at $t=0.71$ and obtain explicitly

$$G_E\approx\left[1+\frac{t/0.71}{1-t/0.71}\right](1-t/0.71)^{-2},\quad \frac{G_M}{\mu}=(1-t/0.71)^{-2}. \tag{3.54}$$

We have plotted these functions and compared them with the experimental data in Fig. 2, assuming

$$G_M{}^p/\mu_p=G_M{}^n/\mu_n,\quad G_E{}^n=0. \tag{3.55}$$

The agreement is excellent for $G_M{}^S/\mu^S$ and moderate for $G_E{}^S$. The value of μ^S, however, is as bad as in the first case.

We must yet show that current conservation forces the current $F^0(\zeta)$ to be even in ζ. From (3.1), we have for the ground state

$$F^\mu(\zeta)=\langle 1|e^{-iS_1}\Gamma^\mu e^{iM_{3}\zeta}e^{iS_1}|1\rangle. \tag{3.56}$$

[13] G. Cocho, C. Frondsal, Harun Ar-Rashid, and R. White, Phys. Rev. Letters **17**, 275 (1966); H. Leutwyler, *ibid.* **17**, 156 (1966); W. Rühl, Nuovo Cimento **44**, 572 (1966).

Therefore, $q_\mu F^\mu = 0$ becomes

$$\langle 1 | e^{-iS_1}[(M-p_0)\Gamma^0 - p^3\Gamma^3] e^{iM_3\zeta} e^{iS_1} | 1 \rangle = 0\,,$$

where p is the momentum of the final nucleon corresponding to the rapidity ζ. Using the vector property of Γ^μ, we can also write this equation in the form

$$\begin{aligned} MF^0(\zeta) &= M\langle 1 | e^{-iS_1}\Gamma^0 e^{iM_3\zeta} e^{iS_1} | 1 \rangle \\ &= M\langle 1 | e^{-iS_1} e^{iM_3\zeta} \Gamma^0 e^{iS_1} | 1 \rangle \\ &= MF^{0*}(-\zeta)\,. \end{aligned}$$

Since F^0 must be real, it must also be even in ζ. Q.E.D.

IV. OTHER POSSIBLE EXTENSIONS BY PARITY

Starting from an irreducible representation of $O(4,2)$ built upon the states

$$|njm\pm\rangle = (-)^m (2j+1)^{1/2} \begin{pmatrix} \frac{1}{2} & j-\frac{1}{2} & j \\ i & m-i & m \end{pmatrix} \times a_i^\dagger | n\ j-\tfrac{1}{2} m \rangle\,, \quad (4.1)$$

we may ask how many other extensions by parity one could construct on this space giving possibly different theories. According to the philosophy of the dynamical group approach, $O(4,2)$ must contain the current operators Γ^μ, the Lorentz generators M_i, and the tilting operator S as Lie algebra elements. Under these conditions, the group extension chosen in (2.3) turns out to be unique.

In Table I, we have listed all the possible ways that the L_{ab}'s could transform under parity. We have restricted L_{ij} to be an axial vector because its physical meaning as angular momentum is fixed. The parities of L_{i4}, L_{i5}, L_{i6} can be chosen freely while those of L_{45}, L_{46}, and L_{56} are then determined.

In order to construct a form factor of the structure (3.1), we need a tilting operator S and a Γ_0, both scalars under parity and rotation. Therefore, only cases (1) and (8) in the table are possible. Since we need, moreover, a Lorentz generator M_i, which is odd under parity, case (8) can also be excluded. Thus, only (1) remains and parity can be represented with doubling of the Hilbert space by the prescription (2.3).

TABLE I. Possible reflection properties of L_{ab} under parity.

Case	L_i	L_{i4}	L_{i5}	L_{i6}	L_{45}	L_{46}	L_{56}	Parity operation
(1)	+	−	−	−	+	+	+	$a \to b,\ b \to -a$
(2)	+	−	−	+	+	−	−	
(3)	+	−	+	−	−	+	−	
(4)	+	−	+	+	−	−	+	$a \to b,\ b \to a$
(5)	+	+	−	−	−	−	+	$a \to ia,\ b \to ib$
(6)	+	+	−	+	−	+	−	
(7)	+	+	+	−	+	−	−	
(8)	+	+	+	+	+	+	+	$a \to ia,\ b \to -ib$

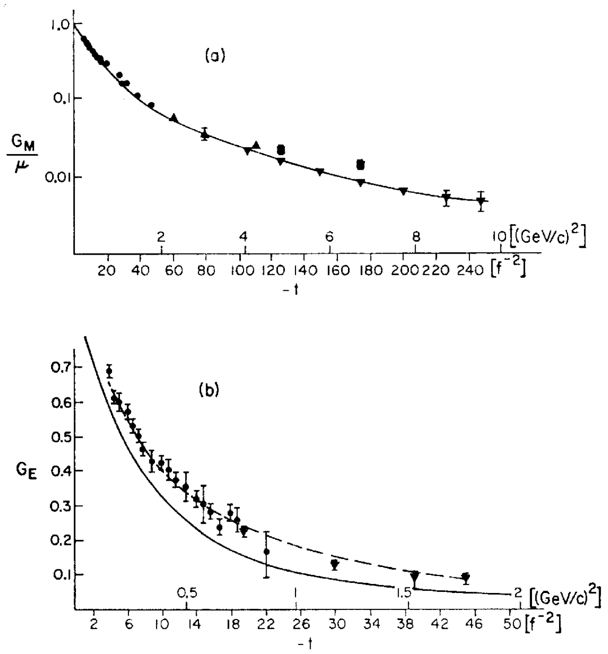

FIG. 2. The theoretical isoscalar form factors $G_M{}^S/\mu^S$ and $G_E{}^S$ are compared with the experimental form factors of the proton, assuming $G_M{}^p/\mu^p = G_M{}^n/\mu^n$ and $G_E{}^n = 0$, on (a) and (b), respectively. The dashed line in (b) shows the pure double-pole fit employed by Ref. 11. In (a) that fit coincides exactly with our curve.

That doubling is really needed to represent parity in case (1) can easily be seen in the following way: If there is no doubling, every state picks up at most a phase under parity

$$\Pi | njm \rangle = \eta (-1)^{f(n,j)} | njm \rangle\,; \quad (4.2)$$

$f(n,j)$ cannot depend upon m since L_i is an axial vector. From (4.1), we see that L_{56} is necessarily a scalar. We can also easily prove that L_{i4} has to be an axial vector.

Suppose L_{i4} were a vector. Then L_{i4} applied to $|njm\rangle$ must change the parity. On the other hand, L_{i4} conserves n and has matrix elements between equal as well as different j's. But this is impossible; there is no choice of $f(n,j)$ that can make L_{i4} a vector.

Consider now the case that L_{i4} is an axial vector. Then it is clear that $f(n,j)$ can depend only on n. Two cases can now be distinguished: (i) $L_{46}+iL_{45} = a^\dagger C b^\dagger$ is a scalar (L_{46} and L_{45} have the same parity since L_{56} has even parity); then $f(n)=0$, since $a^\dagger C b^\dagger$ changes n but not the parity. (ii) $L_{46}+iL_{45}$ is a pseudoscalar; then by the same kind of argument $f(n)=n$.

Hence only the cases (8) and (5) on the table can be verified without doubling of the representation space. They are explicitly given by

$$\Pi | njm \rangle = \eta | njm \rangle\,, \quad (4.3)$$

$$\Pi | njm \rangle = \eta (-1)^n | njm \rangle\,, \quad (4.4)$$

and can be defined operationally by

$$a \to ia, \quad b \to -ib, \quad |0\rangle \to i|0\rangle, \tag{4.5}$$

$$a \to ia, \quad b \to ib, \quad |0\rangle \to i|0\rangle, \tag{4.6}$$

respectively.

All other cases need doubling which can be achieved by using the direct product of L_{ab} with σ_0 or σ_3, according to whether they are scalars or pseudoscalars. The cases (1) and (4) also permit an operational definition of parity by

$$a \to b, \quad b \to -a, \quad |0\rangle \to i|0\rangle, \tag{4.7}$$

$$a \to b, \quad b \to a, \quad |0\rangle \to i|0\rangle, \tag{4.8}$$

respectively.

If we allow a Γ_0 from outside the Lie algebra, we need only one scalar as a tilting operator S, and the theory is much richer. $O(4,2)$ is then, however, no longer the dynamical group of the system. Such a model is discussed in Ref. 2.

V. CONCLUSION

We have discussed the most general theory of electromagnetic currents on the simplest fermion representation space of the dynamical group $O(4,2)$ using a current operator which is completely algebraic. Internal quantum numbers like isospin and hypercharge have been neglected in this approach. The results have tentatively been interpreted as applying to the isoscalar properties of the nucleons. The shape of the magnetic form factor is predicted to follow exactly the double-pole formula that has been used by the experimentalists for an empirical best fit of their data. The theoretical value of the magnetic moment, however, does not coincide with the experimental isoscalar magnetic moment. This defect will probably be corrected by including more general expressions in the current operator, such as convective currents.[14]

[14] *Note added in proof.* With such terms the complete electromagnetic form factors of proton and neutron can indeed be fitted in excellent agreement with experiment [A. O. Barut, D. Corrigan, and H. Kleinert (to be published)].

It will be interesting to see how a nontrivial inclusion of internal symmetries like $SU(2)$ or $SU(3)$ will modify our results for the isoscalar form factors.

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APPENDIX

We recall briefly the global representations of the discrete series D_k^+ of $O(2,1)$. D_k^+ is characterized by the spectrum of the (third) diagonal operator, which goes as $m=k, k+1, k+2, \cdots$. Let

$$\mathbf{a}=\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}, \quad |\mathbf{a}|=1 \tag{A1}$$

be the 2×2 representation of $O(2,1)$, then the corresponding D_k^+ matrix element is given by the v function

$$v_{mn}^k(a)=\Theta_{mn}\bar{\alpha}^{-m-n}\beta^{m-n} \times F(k-n, 1-n-k, 1+m-n, -\beta\bar{\beta}), \tag{A2}$$

with, for $m>n$,

$$\Theta_{mn}=\frac{1}{(m-n)!}\left[\frac{(m-k)!(m+k-1)!}{(n-k)!(n+k-1)!}\right]^{1/2}. \tag{A3}$$

For $m<n$, we use v^T and $\beta \to -\bar{\beta}$. In the text, we have used v_{mn}^k with the argument $\sinh\frac{1}{2}\beta$ which indicates the $O(2,1)$ transformation

$$\mathbf{a}=\begin{pmatrix}\cosh\frac{1}{2}\beta & \sinh\frac{1}{2}\beta \\ \sinh\frac{1}{2}\beta & \cosh\frac{1}{2}\beta\end{pmatrix}. \tag{A4}$$

The matrix element of $e^{-i\beta N_1^2}$, for example, is

$$\langle m|e^{-i\beta N_1^2}|n\rangle = v_{mn}^k(\sinh\tfrac{1}{2}\beta), \tag{A5}$$

where m,n are the eigenvalues of N_1^3 on the corresponding states and k is the lowest value they could take.