Path Integral on Spherical Surfaces
in $D$ Dimensions and on Group Spaces

H. Kleinert

Institut für Theorie der Elementarteilchen
Freie Universität Berlin
Arnimallee 14    D - 1000 Berlin 33

October 19, 1989

Abstract

I solve the path integral for a point particle moving on the surface of a sphere in $D$ dimensions and, exploiting the equivalence between a $D = 4$ surface and the group space $SU(2)$, for a spherical top. The $SU(2)$ case serves as a prototype for the path integration on arbitrary group spaces.
1) In Schrödinger quantum mechanics it is a simple problem to find the wave functions and energies of a point particle moving on the surface of a sphere in $D$ dimensions. The same thing holds for a spherical top. In both cases it is the quantization of the rotational Lie algebra not the canonical quantization rules which leads to the correct answer.

Surprisingly these two elementary problems have never been solved by path integration. The reason is that the standard Feynman path integral is intimately linked with the canonical quantization rule and these are known to fail for angular variables. In fact until very recently we did not possess any reliable quantum equivalence principle at the level of path integrals, i.e., a precise rule of how to generalize the well-known Feynman path integral formula in cartesian coordinates to curved space. The classical work of De Witt [1] and all subsequent modifications [2,3] are of little help since all of them produce different extra constants to the Schrödinger energy proportional to the scalar curvature $\tilde{R}$ which for a sphere of radius $R$ is $\tilde{R} = (D - 1)(D - 2)/R^2$ and for the top $3/2I$ where $I$ is the moment of inertia {for the asymmetric top with three moments of inertia it is $[(I_1 + I_2 + I_3)^2 - 2(I_1^2 + I_2^2 + I_3^2)]/2I_1I_2I_3}$. Apart from contradicting the most natural quantization via the rotational Lie algebra such constants if really present would change dramatically the gravitational properties of interstellar gases of rotating molecules and must therefore be rejected.

In a recent paper [4] we have finally succeeded in finding a unique correct set of rules for setting up the measure of path integration in spaces with curvature and torsion. The first major success of this quantum equivalence principle was the solution of an outstanding problem the time-sliced path integral of the $D = 3$ Coulomb system [4]. A solution had previously been given only structurally [5]; the proper treatment of the time sliced expression had unfortunately been limited to the unphysical case of $D = 2$ dimensions [6] the reason being that the combined coordinate and time transformations which make the system harmonic and integrable are holonomic in $D = 2$ and do not produce curvature nor torsion. In $D = 3$ where this happens the
correct quantum equivalence principle had to be found before the problem could be solved [7].

The purpose of the present note is to show that the new measure of path integration proposed in [4] finally solves the long-standing problems of the path integral on spheres and group spaces with results which are in agreement with the corresponding Lie algebras as they should.

2) A path integral that has been solved in the literature [8] is describes a point particle only near the surface of a sphere in $D$ dimensions. Its imaginary-time-sliced form reads

$$
(u_{\hat{n}} \tau_{\hat{n}} | u_n \tau_n) \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M R^2}} \prod_{n=1}^{N} \left[ \int \frac{d^{D-1} u_n}{\sqrt{2\pi\hbar\epsilon/M R^2}} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\}, \tag{1}
$$

with the sliced action

$$
A^N_0 = \frac{M}{2\epsilon} R^2 \sum_{n=1}^{N+1} (u_n - u_{n-1})^2 = \frac{M}{\epsilon} R^2 \sum_{n=1}^{N+1} (1 - \cos \Delta \vartheta_n), \tag{2}
$$

where $\Delta \vartheta$ is the small angle between $u_n$ and $u_{n-1}$ (the $\approx$ sign becomes an equality for $N \to \infty$). There are two reasons for using the term near rather than on the sphere.

i) The sliced action involves the shortest distances between the points in the embedding euclidean space rather than the intrinsic geodesic distances on the sphere. This fact has been observed before [9-10] and is easy to correct.

ii) There is an additional action associated with the measure of path integration to be taken from [4].

The exact solution of the path integral (1) goes as follows: For each time interval $\epsilon\ell$ the exponential $\exp \left\{ -MR^2 (1 - \cos \Delta \vartheta_n)/\hbar\epsilon \right\}$ is expanded into spherical harmonics according to formula

$$
\exp \left\{ -\frac{MR^2}{2\hbar\epsilon} (u_n - u_{n-1})^2 \right\} = \sum_{l=0}^{\infty} a_l(\hbar) \frac{l + D/2 - 1}{D/2 - 1} \frac{1}{S_D(l)} C_{l}^{(D/2-1)}(\cos \Delta \vartheta_n)
$$
\[ = \sum_{l=0}^{\infty} a_l(h) \sum_m Y_{lm}(u_n) Y_{lm}^*(u_{n-1}) \quad (3) \]

where \( a_l(h) \equiv (2\pi/h)^{(D-1)/2} \tilde{I}_{l+D/2-1}(h) \Gamma \tilde{I}_l(z) \equiv \sqrt{2\pi} z e^{-z} I_{l}(z) \) with \( I_{l}(z) = \) Bessel functions \( \Gamma \) and \( h \equiv MR^2/\hbar \varepsilon \). The functions \( C^l_{(z)}(z) \) are the Gegenbauer polynomials and \( Y_{lm}(u) \) the hyperspherical harmonics in \( D \) dimensions [11]. For each adjacent pair of such factors \( (n+1, n), (n, n-1) \) \( \Gamma \) the integration over the intermediate \( u_n \) variable can be done using the well-known orthogonormality relation for the hyperspherical harmonics. The combined two-step amplitude has the same expansion as \( (3) \) with \( a_l(h) \) replaced by \( (h/2\pi) a_l(h)^2 \). By successive integration in \( (1) \) we obtain the total time sliced amplitude

\[ (u_b \tau_b | u_a \tau_a) \approx \left( \frac{h}{2\pi} \right)^{(N+1)(D-1)/2} \sum_{l=0}^{\infty} a_l(h)^{N+1} \sum_m Y_{lm}(u_b) Y_{lm}^*(u_a). \quad (4) \]

We now go to the limit \( N \to \infty, \varepsilon = (\tau_b - \tau_a)/(N + 1) \to 0, \) where

\[ \left( \frac{h}{2\pi} \right)^{(N+1)(D-1)/2} a_l(h)^{N+1} = \left[ \tilde{I}_{l+D/2-1} \left( \frac{MR^2}{\hbar \varepsilon} \right) \right]^{N+1} \to \exp \left\{ -(\tau_b - \tau_a) \frac{\hbar}{2MR^2} \frac{(l + D/2 - 1)^2 - 1/4}{\hbar^2} \right\}, \quad (5) \]

and obtain the time displacement amplitude for the motion near the sphere as the spectral expansion

\[ (u_b \tau_b | u_a \tau_a) = \sum_{l=0}^{\infty} \exp \left\{ -\frac{\hbar L_2}{2MR^2} (\tau_b - \tau_a) \right\} \sum_m Y_{lm}(u_b) Y_{lm}^*(u_a), \quad (6) \]

with

\[ L_2 \equiv (l + D/2 - 1)^2 - 1/4, \quad (7) \]

and the energy eigenvalues \( E_i^{near} = \hbar^2 L_2/2MR^2 \). For \( D = 4\Gamma \) the most convenient expansion is in terms of the representation functions \( D_{nm}^l(\varphi, \theta, \gamma) \)
of the rotation group involving the Euler angle parametrization of the vectors on the unit sphere
\[
\begin{align*}
\dot{x}^1 &= \cos(\theta/2) \cos[(\varphi + \gamma)/2] \\
\dot{x}^2 &= -\sin(\theta/2) \sin[(\varphi + \gamma)/2] \\
\dot{x}^3 &= \sin(\theta/2) \cos[(\varphi - \gamma)/2] \\
\dot{x}^4 &= \sin(\theta/2) \sin[(\varphi - \gamma)/2].
\end{align*}
\]
(8)

In terms of these \(\Gamma(6)\) reads
\[
(u_b \tau_k | u_a \tau_a) = \sum_{l=0}^{\infty} \exp \left\{ -\frac{\hbar L_2}{2MR^2} (\tau_b - \tau_a) \right\} \times \sum_{m_1, m_2 = -l}^{l} \frac{l + 1}{2\pi^2} D_{m_1 m_2}^{l/2}(\varphi_b, \theta_b, \gamma_b) D_{m_1 \pm m_2}^{l/2}(\varphi_a, \theta_a, \gamma_a).
\]
(9)

These amplitudes display already the correct wave functions for the movement on the surface of the sphere as we know from Schrödinger theory. They do not however carry the correct energy eigenvalues which should be \(E_l = \hbar^2 \hat{L}_2^2 / 2MR^2\) with the eigenvalue of the squared angular momentum operator \(\hat{L}_2^2 = l(l + D - 2)\) rather than \(E_l^{n\text{car}}\) with \(L_2 = (l + D/2 - 1)^2 - 1/4\).

To have the correct energies the path integral needs the two changes announced above. First the time-sliced action must measure the proper geodesic distance rather than the euclidean distance in the embedding space and should thus read
\[
\mathcal{A}^N = \frac{M}{\epsilon} R^2 \sum_{n=1}^{N+1} \frac{(\Delta \vartheta_n)^2}{2},
\]
(10)
rather than (2). Since the time-sliced path integral is solved exactly with the action (2) it is convenient to expand the true action around the soluble one as \([12]\)
\[
\mathcal{A}^N = \mathcal{A}_0^N + \Delta_t \mathcal{A}^N = \frac{M}{\epsilon} R^2 \sum_{n=1}^{N+1} \left[ (1 - \cos \Delta \vartheta_n) + \frac{1}{24} \Delta \vartheta_n^4 + \ldots \right],
\]
(11)
and treat the correction perturbatively to lowest order. There is no need to go higher than quartic order since only the quartic term contributes to the relevant order $\epsilon$ in the limit $N \to \infty$. In $D = 2$ dimensions the quartic correction is sufficient to bring the path integral from near to on the sphere (here a circle). Indeed, with the measure of the path integration being

$$
\frac{1}{\sqrt{2\pi\hbar\epsilon/MR^2}} \prod_{n=1}^{N+1} \int_{-\pi/2}^{\pi/2} \frac{d\varphi_n}{\sqrt{2\pi\hbar\epsilon/MR^2}}
$$

and the leading action (10) the quartic term $\Delta\theta_n^4 = (\varphi_n - \varphi_{n-1})^4$ can be replaced by its expectation

$$
\langle \Delta\theta_n^4 \rangle_0 = \frac{3\epsilon\hbar}{MR^2},
$$

so that the correction term of the action is given by

$$
\langle \Delta_4\mathcal{A}^N \rangle_0 = (\tau_b - \tau_a) \frac{\hbar^2/4}{2MR^2}.
$$

where we have replaced $(N+1)\epsilon$ by $\tau_b - \tau_a$. For $D = 2\Gamma$ this supplies precisely the missing energy to raise $E^\text{near}_j$ up to $E_j$.

In higher dimensions we must change also the measure of path integration according to [4]. What we have to explain in any $D$ is the difference

$$
\Delta L_2 = \hat{L}^2 - L_2 = 1/4 - (D/2 - 1)^2.
$$

This vanishes at $D = 3$ where it changes sign. Note that the expectation of the quartic correction term $\Delta_4\mathcal{A}^N$ in (11) being always positive cannot account for the discrepancy by itself. Let us calculate its contribution in $D$ dimensions. For very small $\epsilon\Gamma$ the fluctuations near the sphere will lie close to the $D - 1$ dimensional tangent space. Let $\Delta x_n$ be the coordinates in this space. Then we can write

$$
\Delta_4\mathcal{A}^N \approx \frac{M}{\epsilon} \frac{R}{\hbar} \sum_{n=1}^{N+1} \frac{1}{24} \left( \frac{\Delta x_n}{R} \right)^4.
$$
The \( \Delta x_n \)'s have the lowest order correlation \( \langle \Delta x_i \Delta x_j \rangle_0 = (\hbar \epsilon/M) \delta_{ij} \). This shows that \( \Delta_4 A^N \) has the expectation

\[
\langle \Delta A^N \rangle_0 = (\tau - \tau_s) \frac{\hbar^2}{2MR^2} \Delta_4 L_2.
\]

(17)

where \( \Delta_4 L_2 \) is the contribution of the quartic term to the value \( L_2 \Gamma \)

\[
\Delta_4 L_2 = \frac{D^2 - 1}{12}.
\]

(18)

This result is obtained using the Wick contraction rules for the tensor \( \langle \Delta x_i \Delta x_j \Delta x_k \Delta x_l \rangle_0 = (\epsilon \hbar/M)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \). Thus we remain with a final discrepancy in \( D \) dimensions\( ^1 \)

\[
\Delta_{\text{meas}} L_2 = \Delta L_2 - \Delta_4 L_2 = -\frac{1}{3}(D-1)(D-2),
\]

(19)

to be explained now.

3) The final correction \( \Delta_{\text{meas}} L_2 \) is obtained from the proper treatment of the measure of the path integral à la [4]. Near the sphere in (1) we have used the measure

\[
\prod_{n=1}^{N} \left[ \int \frac{d^{D-1} \mathbf{u}_n}{\sqrt{2\pi \hbar \epsilon/M R^2}} \right].
\]

(20)

Although this measure seems quite natural for the surface of a sphere it requires a correction for path integration\( ^1 \) as was shown in [4]. To see this consider the amplitude in cartesian coordinates

\[
(\mathbf{x}_t | \mathbf{x'}_{t'}) = \frac{1}{\sqrt{2\pi \hbar \epsilon/M}} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^{\infty} d\Delta x_n \right] \prod_{n=1}^{N+1} K_0'(\Delta x_n),
\]

(21)

with the short-time amplitudes\( ^1 \)

\[
K_0'(\Delta x_n) = \langle x_{n} | \exp \left\{ -\frac{1}{\hbar} \hat{H} \right\} | x_{n-1} \rangle \equiv \frac{1}{\sqrt{2\pi \hbar \epsilon/M}} \left[ \exp \left\{ -\frac{1}{\hbar} \frac{M (\Delta x_n)^2}{2 \epsilon} \right\} \right]
\]

(22)
where $\Delta x_n \equiv x_n - x_{n-1}, x \equiv x_{N+1}, x' \equiv x_0$ (we may omit a possible extra potential which would enter trivially). We now transform $(\Delta x_n)^2$ by a non-holonomic mapping to a space with curvature and torsion [13] parametrized with coordinates $q^\mu$. For infinitesimal $\Delta x_n \approx dx_n$ the transformation would simply yield $(dx)^2 = g_{\mu\nu} dq^\mu dq^\nu$. For finite $\Delta x_n \Gamma$ we must expand $(\Delta x_n)^2$ up to forth order in $\Delta q_n^\mu = q_n^\mu - q_{n-1}^\mu \Gamma$ since we must find all terms that will eventually contribute to order $\epsilon$ [14,15]. We expand around the final point $q_0^\mu \Gamma$

$$
x^i(q_{n-1}) \equiv x^i(q_n - \Delta q_n) = x^i(q_n) - e^i_{\mu} \Delta q^\mu + \frac{1}{2} e^i_{\mu,\nu} \Delta q^\mu \Delta q^\nu - \frac{1}{3!} e^i_{\mu,\nu,\lambda} \Delta q^\mu \Delta q^\nu \Delta q^\lambda + \ldots .
$$

For brevity we have omitted the argument $q_n$ in the $e^i_{\mu}$'s as well a the subscripts $n$ of $\Delta q^\mu$. Squaring $\Delta x_n$ and expressing the everything in terms of the affine connection leads to the short-time sliced action expressed entirely in terms of intrinsic quantities (omitting again all sub $n$'s) $\Gamma$

$$
A_\gamma(q, q - \Delta q) = \frac{M}{2\epsilon} \{ g_{\mu\nu} \Delta q^\mu \Delta q^\nu - \Gamma_{\mu\nu,\lambda} \Delta q^\mu \Delta q^\nu \Delta q^\lambda \}
$$

$$
+ \left[ \frac{1}{3} g_{\mu\nu} (\partial^\kappa \Gamma_{\lambda\nu,\tau} + \Gamma_{\lambda\nu}^\rho \Gamma_{\rho\mu,\tau} \right) + \frac{1}{4} \Gamma_{\mu,\nu,\lambda} \Gamma_{\mu,\nu,\lambda} \Delta q^\mu \Delta q^\nu \Delta q^\lambda \Delta q^\kappa + \ldots \}
$$

with $g_{\mu\nu} \equiv e^j_{\mu} e^j_{\nu}$ and $\Gamma_{\mu,\nu,\lambda}$ evaluated at the final point $q$. The measure of path integration in (21) is transformed to $q^\mu$-space with a Jacobian following from (23) $\Gamma$

$$
J = \frac{\partial(\Delta x)}{\partial(\Delta q)} = \det(e^i_{\mu}) \det(\delta^\kappa_{\mu} - e^i_{\mu} e^i_{\nu,\mu} \Delta q^\nu + \frac{1}{2} e^i_{\mu} e^i_{\nu,\mu} \Delta q^\nu \Delta q^\lambda + \ldots ),
$$

where the curly brackets around the indices denote their symmetrization. Expanding the second factor in powers of $\Delta q^\mu \Gamma$ writing $\det(e^i_{\mu}) \equiv e(q) = \sqrt{\det g_{\mu\nu}(q)} \equiv \sqrt{g(q) \Gamma}$ and expressing the series in terms of a “Jacobian effective action” $A_J$ with the definition $J = \sqrt{g(q)} \exp \{ i A_J / \hbar \}$ with

$$
\frac{i}{\hbar} A_J = - \Gamma_{\nu,\mu} \Delta q^\nu
$$

(26)
we arrive at the time-sliced path integral in $q$-space

$$
\langle q | \exp \left\{ -\frac{1}{\hbar} (t - t') \hat{H} \right\} | q' \rangle \approx \frac{1}{\sqrt{2\pi \hbar \epsilon / M}} \prod_{n=2}^{N+1} \left[ \int d^D q_n \frac{\sqrt{g(q_n)}}{\sqrt{2\pi \hbar \epsilon / M}} \right] 
\times \exp \left\{ -\frac{1}{\hbar} \sum_{n=1}^{N+1} [A^c_j(q_n, q_n - \Delta q_n) + A_J] \right\}
$$

(27)

The integrals over $\Delta q_n$ are to be performed successively from $n = N$ down to $n = 1$. It is useful to reexpress the measure in terms of the naively expected group invariant measure which we write as follows

$$
\langle q | \exp \left\{ -\frac{1}{\hbar} (t - t') \hat{H} \right\} | q' \rangle \approx \frac{1}{\sqrt{2\pi \hbar \epsilon / M}} \prod_{n=2}^{N+1} \left[ \int d^D q_{n-1} \frac{\sqrt{g(q_{n-1})}}{\sqrt{2\pi \hbar \epsilon / M}} \right] 
\times \exp \left\{ -\frac{1}{\hbar} \sum_{n=1}^{N+1} [A^c_j(q_n, q_n - \Delta q_n) + A_{\text{meas}} A_J] \right\}
$$

(28)

The correction term $A_{\text{meas}} A_J$ due to the measure is obtained from the difference between $A_J$ and $A_{\hat{h}}$; the latter arising from the shift in the subscripts $n$ in the $\sqrt{g(q)}$ factors of (27); by one unit. From the ratio $\sqrt{g(q_n)} / \sqrt{g(q_{n-1})} = e(q_n) / e(q_{n-1}) \equiv \exp \{ i A_{\hat{h}} / \hbar \}$. Expanding the determinant $e(q_n) = e(q_{n-1} - \Delta q_n)$ in powers of $\Delta q_n$ we see that $A_{\hat{h}}$ is the same as $A_J$ in (26) except that symmetrization symbols are absent. Either (27) or (28) may be used as the correct path integral formulas in spaces with curvature and torsion.

4) The problem of interest here involves no torsion. Then a simple algebra shows that $A_{\text{meas}} A_J$ reduces to

$$
A_{\text{meas}} A_J = -\frac{\hbar}{6} \bar{R}_{\mu \nu} \Delta q^\mu \Delta q^\nu,
$$

where $\bar{R}_{\mu \nu}$ is the Ricci tensor which for a sphere of radius $R$ is $(D - 2) g_{\mu \nu} / R^2$. The perturbative treatment of (29) gives the only relevant
contribution to the energy \( \Gamma \)

\[
\langle \Delta_{mcst}, A \rangle_0 = -\frac{\hbar^2}{6M} \frac{(D - 1)(D - 2)}{R^2},
\]

thus producing precisely the missing energy required by (19).

5) The sphere in four dimensions is equivalent to the covering group of rotations in three dimensions \( SU(2) \). Knowing now how to solve the time-sliced path integral near and on the surface of the sphere \( \Gamma \) we can obtain the same quantities near and on the group space of \( SU(2) \) \[16\]. This puts us in a position to solve the time sliced path integral of a spinning spherical top by reduction to the \( SU(2) \) problem. We only have to go from \( SU(2) \) \( \Gamma \) which is the covering group of the rotation group \( \Gamma \) down to the rotation group itself \[17\]. The angular positions with Euler angles \( \gamma \) and \( \gamma + 2\pi \) are physically indistinguishable. The physical states must be a representation of this operation and the time-displacement amplitude must reflect this. The simplest possibility is the trivial even representation where one adds the amplitudes to go from the initial configuration \( \varphi_a, \theta_a, \gamma_a \) to the identical final ones \( \varphi_b, \theta_b, \gamma_b \) and \( \varphi_b, \theta_b, \gamma_b + 2\pi \) and forms the amplitude

\[
(\varphi_b, \theta_b, \gamma_b, \tau_b | \varphi_b, \theta_b, \gamma_b, \tau_a)_{top} = (\varphi_b, \theta_b, \gamma_b, \tau_b | \varphi_b, \theta_b, \gamma_b, \tau_a) + (\varphi_b, \theta_b, \gamma_b + 2\pi, \tau_b | \varphi_b, \theta_b, \gamma_b, \tau_a).
\]

The sum eliminates all half-integer representation functions \( d_{mm' \ell}(\theta) \) in the expansion (1) of the amplitude.

Instead of the sum we could also have formed another representation of the operation \( \gamma \rightarrow \gamma + 2\pi \Gamma \) the antisymmetric combination

\[
(\varphi_b, \theta_b, \gamma_b, \tau_b | \varphi_b, \theta_b, \gamma_b, \tau_a)_{fermions} =
(\varphi_b, \theta_b, \gamma_b, \tau_b | \varphi_b, \theta_b, \gamma_b, \tau_a) - (\varphi_b, \theta_b, \gamma_b + 2\pi, \tau_b | \varphi_b, \theta_b, \gamma_b, \tau_a).
\]

Here the expansion (9) retains only the half-integer angular momenta \( l/2 \). In nature such spins are associated with fermions such as electrons, protons, muons or neutrinos which carry only one specific value of \( l/2 \).
In principle there is no problem in treating also a non-spherical top. While the spherical top has “near the group space” a time-sliced action

\[ \frac{1}{\epsilon^2} \int \left[ 1 - \frac{1}{2} \text{tr}(g_n g_{n-1}^{-1}) \right], \]  

the asymmetric top with three moments of inertia \( I_{123} \) requires separating the three components of the angular velocities

\[ \omega_a = i \text{tr}(g \sigma_a g^{-1}), \quad a = 1, 2, 3. \]  

\((\sigma_a = \text{Pauli matrices})\) on the time lattice so that the action reads

\[
\frac{1}{\epsilon^2} \left\{ I_1 \left[ 1 - \frac{1}{2} \text{tr}(g_n \sigma_1 g_{n-1}^{-1}) \right] + I_2 \left[ 1 - \frac{1}{2} \text{tr}(g_n \sigma_2 g_{n-1}^{-1}) \right] + I_3 \left[ 1 - \frac{1}{2} \text{tr}(g_n \sigma_3 g_{n-1}^{-1}) \right] \right\},
\]

rather than (33). The amplitude “near the top” is then an appropriate generalization of (9). The calculation of the correction term \( \Delta E \) however is more complicated than before and is left to the reader following the rules explained above.
References


[3] For other discussions with results different from the above and ours see
   H. Kamo and T. Kawai Prog. Theor. Phys. 50 680 (1973)
   T. Kawai Found. Phys. 5 143 (1975)


[7] In the literature there is the claim to have also solved the $D = 3$ time-sliced path integral of the Coulomb system exactly by R. Ho and A. Inomata Phys. Rev. Lett. 48 231 (1982). This is not true however. After starting from Feynman’s path integral formula the authors do not proceed consistently. The correct (known) final result is found only thanks to a wrong transformation of the measure. See [6]. As we know now it is impossible to use the Feynman formula as a starting place due to path collapse. See H. Kleinert Phys. Lett. 82 1 313 (1989). This is why also the approach taken by F. Steiner Phys. Lett. 106 A 356 363 (1984) is incorrect.

   G. Junker and A. Inomata in Path Integrals from $meV$ to $MeV$ ed. by M. C. Gutswiller et al. World Scientific 1986
In these papers we must overlook the path collapse properties of many intermediate formulas and carefully select only the correct portions.


[12] This step was still done in [9]. Note however that the (known) correct final result stated in that paper is impossible to obtain from their calculation since the measure problem which is the main issue of the present paper was not solved at that time.


[17] This point was discussed by L. Schulman Phys. Rev. 174 1558 (1968). The author also gives a correct path integral but he does so a posteriori by reconstructing it from the known spectral representation of the Schrödinger result.