Path Integral on Spherical Surfaces in D Dimensions and on Group Spaces

H. Kleinert

Institut für Theorie der Elementarteilchen

Freie Universität Berlin

Arnimallee 14 D - 1000 Berlin 33

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Abstract

I solve the path integral for a point particle moving on the surface of a sphere in D dimensions and, exploiting the equivalence between a $D = 4$ surface and the group space $SU(2)$, for a spherical top. The $SU(2)$ case serves as a prototype for the path integration on arbitrary group spaces.

1) In Schrödinger quantum mechanics it is a simple problem to find the wave functions and energies of a point particle moving on the surface of a sphere in D dimensions. The same thing holds for a spherical top. In both cases it is the quantization of the rotational Lie algebra, not the canonical quantization rules, which leads to the correct answer.

Surprisingly, these two elementary problems have never been solved by path integration. The reason is that the standard Feynman path integral is intimitely linked with the canonical quantization rule, and these are known to fail for angular variables. In fact, until very recently we did not possess any reliable *quantum equivalence principle* at the level of path integrals, i.e. a precise rule of how to generalize the well-known Feynman path integral formula in cartesian coordinates to curved space. The classical work of De Witt [1] and all subsequent modifications [2, 3] are of little help since all of them produce, in the above systems, different extra constants to the Schrödinger energy, proportional to the scalar curvature $\bar{R}$, which for a sphere of radius R is $\bar{R} = (D-1)(D-2)/R^2$ and for the top $3/2I$ where I is the moment of inertia {for the asymmetric top with three moments of inertia it is $[(I_1+I_2+I_3)^2 - 2(I_1^2+I_2^2+I_3^2)]/2I_1I_2I_3$}. Apart from contradicting the most natural quantization via the rotational Lie algebra such constants, if really present, would change dramatically the gravitational properties of interstellar gases of rotating molecules and must therefore be rejected.

In a recent paper [4], we have finally succeded in finding a unique correct set of rules for setting up the measure of path integration in spaces with curvature and torsion. The first major success of this quantum equivalence principle. was the solution of an outstanding problem, the time-sliced path integral of the $D = 3$ Coulomb system [4]. A solution had previously been given only structurally [5]; the proper treatment of the time sliced expression had unfortunately been limited to the unphysical case of $D = 2$ dimensions [6], the reason being that the combined coordinate and time transformations which make the system harmonic and integrable, are holonomic in $D = 2$ and do not produce curvature nor torsion. In $D = 3$ where this happens the correct quantum equivalence principle had to be found before the problem could be solved [7].

The purpose of the present note is to show that the new measure of path integration proposed in [4] finally solves the long-standing problems of the path integral on spheres and group spaces with results which are in agreement with the corresponding Lie algebras, as they should.

2) A path integral that *has* been solved in the literature [8] is describes a point particle only *near* the surface of a sphere in D dimensions. Its imaginary-time-sliced form reads

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/MR^2}^{D-1}}\prod_{n=1}^{N}\left[\int\frac{d^{D-1}\mathbf{u}_n}{\sqrt{2\pi\hbar\epsilon/MR^2}^{D-1}}\right]\exp\left\{-\frac{1}{\hbar}\mathcal{A}^N\right\}, \tag{1}$$

with the sliced action

$$\mathcal{A}_0^N = \frac{M}{2\epsilon}R^2\sum_{n=1}^{N+1}(\mathbf{u}_n-\mathbf{u}_{n-1})^2 = \frac{M}{\epsilon}R^2\sum_{n=1}^{N+1}(1-\cos\Delta\vartheta_n), \tag{2}$$

where $\Delta\vartheta$ is the small angle between $\mathbf{u}_n$ and $\mathbf{u}_{n-1}$ (the $\approx$ sign becomes an equality for $N\to\infty$) There are two reasons for using the term *near* rather than *on* the sphere.

i) The sliced action involves the shortest distances between the points in the *embedding* euclidean space rather than the intrinsic geodesic distances on the sphere. This fact has been observed before [9, 10] and is easy to correct.

ii) There is an additional action associated with the measure of path integration to be taken from [4].

The exact solution of the path integral (1) goes as follows: For each time interval ϵ, the exponential $\exp\{-MR^2(1-\cos\Delta\vartheta_n)/\hbar\epsilon\}$ is expanded into spherical harmonics according to formula

$$\exp\left\{-\frac{MR^2}{2\hbar\epsilon}(\mathbf{u}_n-\mathbf{u}_{n-1})^2\right\}$$
$$= \sum_{l=0}^{\infty}a_l(h)\frac{l+D/2-1}{D/2-1}\frac{1}{S_D}C_l^{(D/2-1)}(\cos\Delta\vartheta_n)$$

$$= \sum_{l=0}^{\infty} a_l(h) \sum_{\mathbf{m}} Y_{l\mathbf{m}}(\mathbf{u}_n) Y^*_{l\mathbf{m}}(\mathbf{u}_{n-1}) \tag{3}$$

where $a_l(h) \equiv (2\pi/h)^{(D-1)/2}\, \tilde{I}_{l+D/2-1}(h)$, $\tilde{I}_\nu(z) \equiv \sqrt{2\pi z} e^{-z} I_\nu(z)$ with $I_\nu(z) =$ Bessel functions, and $h \equiv MR^2/\hbar\epsilon$. The functions $C_l^{(\nu)}(z)$ are the Gegenbauer polynomials and $Y_{l\mathbf{m}}(\mathbf{u})$ the hyperspherical harmonics in D dimensions [11]. For each adjacent pair of such factors $(n+1, n), (n, n-1)$, the integration over the intermediate $\mathbf{u}_n$ variable can be done using the well-known orthogonormality relation for the hyperspherical harmonics. The combined two-step amplitude has the same expansion as (3) with $a_l(h)$ replaced by $(h/2\pi) a_l(h)^2$. By successive integration in (1) we obtain the total time sliced amplitude

$$(\mathbf{u}_b \tau_b | \mathbf{u}_a \tau_a) \approx \left(\frac{h}{2\pi}\right)^{(N+1)(D-1)/2} \sum_{l=0}^{\infty} a_l(h)^{N+1} \sum_{\mathbf{m}} Y_{l\mathbf{m}}(\mathbf{u}_b) Y^*_{l\mathbf{m}}(\mathbf{u}_a). \tag{4}$$

We now go to the limit $N \to \infty, \epsilon = (\tau_b - \tau_a)/(N+1) \to 0$, where

$$\left(\frac{h}{2\pi}\right)^{(N+1)(D-1)/2} a_l(h)^{N+1} = \left[\tilde{I}_{l+D/2-1}\left(\frac{MR^2}{\hbar\epsilon}\right)\right]^{N+1}$$
$$\longrightarrow \exp\left\{-(\tau_b - \tau_a)\hbar \frac{(l+D/2-1)^2 - 1/4}{2MR^2}\right\}, \tag{5}$$

and obtain the time displacement amplitude for the motion *near* the sphere as the spectral expansion

$$(\mathbf{u}_b \tau_b | \mathbf{u}_a \tau_a) = \sum_{l=0}^{\infty} \exp\left\{-\frac{\hbar L_2}{2MR^2}(\tau_b - \tau_a)\right\} \sum_{\mathbf{m}} Y_{l\mathbf{m}}(\mathbf{u}_b) Y^*_{l\mathbf{m}}(\mathbf{u}_a), \tag{6}$$

with

$$L_2 \equiv (l + D/2 - 1)^2 - 1/4, \tag{7}$$

and the energy eigenvalues $E_l^{near} = \hbar^2 L_2/2MR^2$. For $D = 4$, the most convenient expansion is in terms of the representation functions $\mathcal{D}^l_{mm'}(\varphi, \theta, \gamma)$

of the rotation group, involving the Euler angle parametrization of the vectors on the unit sphere

$$\begin{aligned}
\hat{x}^1 &= \cos(\theta/2)\cos[(\varphi+\gamma)/2] \\
\hat{x}^2 &= -\sin(\theta/2)\sin[(\varphi+\gamma)/2] \\
\hat{x}^3 &= \sin(\theta/2)\cos[(\varphi-\gamma)/2] \\
\hat{x}^4 &= \sin(\theta/2)\sin[(\varphi-\gamma)/2].
\end{aligned} \tag{8}$$

In terms of these, (6) reads

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = \sum_{l=0}^{\infty}\exp\left\{-\frac{\hbar L_2}{2MR^2}(\tau_b-\tau_a)\right\} \tag{9}$$

$$\times \sum_{m_1,m_2=-l}^{l}\frac{l+1}{2\pi^2}\mathcal{D}^{l/2}_{m_1m_2}(\varphi_b,\theta_b,\gamma_b)\mathcal{D}^{l/2\,*}_{m_1m_2}(\varphi_a,\theta_a,\gamma_a).$$

These amplitudes display already the correct wave functions for the movement *on* the surface of the sphere, as we know from Schrödinger theory. They do not, however, carry the correct energy eigwnvalues which should be $E_l = \hbar^2\hat{L}^2/2MR^2$ with the eigenvalue of the squared angular momentum operator $\hat{L}^2 = l(l+D-2)$ rather than E_l^{near} with $L_2 = (l+D/2-1)^2 - 1/4$.

To have the correct energies, the path integral needs the two changes announced above. First, the time-sliced action must measure the proper geodesic distance rather than the euclidean distance in the embedding space and should thus read

$$\mathcal{A}^N = \frac{M}{\epsilon}R^2\sum_{n=1}^{N+1}\frac{(\Delta\vartheta_n)^2}{2}, \tag{10}$$

rather than (2). Since the time-sliced path integral is solved exactly with the action (2) it is convenient to expand the true action around the soluble one as [12]

$$\mathcal{A}^N = \mathcal{A}^N_0 + \Delta_4\mathcal{A}^N = \frac{M}{\epsilon}R^2\sum_{n=1}^{N+1}\left[(1-\cos\Delta\vartheta_n) + \frac{1}{24}\Delta\vartheta_n^4 + \dots\right], \tag{11}$$

and treat the correction perturbatively to lowest order. There is no need to go higher than quartic order since only the quartic term contributes to the relevant order ϵ in the limit $N \to \infty$. In $D = 2$ dimensions, the quartic correction is sufficient to bring the path integral from *near* to *on* the sphere (here a circle). Indeed, with the measure of the path integration being

$$\frac{1}{\sqrt{2\pi\hbar\epsilon/MR^2}} \prod_{n=1}^{N+1} \int_{-\pi/2}^{\pi/2} \frac{d\varphi_n}{\sqrt{2\pi\hbar\epsilon/MR^2}} \tag{12}$$

and the leading action (10), the quartic term $\Delta\vartheta_n^4 = (\varphi_n - \varphi_{n-1})^4$ can be replaced by its expectation

$$\langle \Delta\vartheta_n^4 \rangle_0 = 3\frac{\epsilon\hbar}{MR^2}, \tag{13}$$

so that the correction term of the action is given by

$$\langle \Delta_4 \mathcal{A}^N \rangle_0 = (\tau_b - \tau_a)\frac{\hbar^2/4}{2MR^2}. \tag{14}$$

where we have replaced $(N+1)\epsilon$ by $\tau_b - \tau_a$. For $D = 2$, this supplies precisely the missing energy to raise E_l^{near} up to E_l.

In higher dimensions, we must change also the measure of path integration is necessary according to [4]. What we have to explain in any D is the difference

$$\Delta L_2 = \hat{L}^2 - L_2 = 1/4 - (D/2 - 1)^2. \tag{15}$$

This vanishes at $D = 3$ where it changes sign. Note that the expectation of the quartic correction term $\Delta_4 \mathcal{A}^N$ in (11) being always positive cannot account for the discrepancy by itself. Let us calculate its contribution in D dimensions. For very small ϵ, the fluctuations near the sphere will lie close to the $D - 1$ dimensional tangent space. Let $\Delta\mathbf{x}_n$ be the coordinates in this space. Then we can write

$$\Delta_4 \mathcal{A}^N \approx \frac{M}{\epsilon} R^2 \sum_{n=1}^{N+1} \frac{1}{24} (\frac{\Delta\mathbf{x}_n}{R})^4. \tag{16}$$

The $\Delta\mathbf{x}_n$'s have the lowest order correlation $\langle\Delta x_i\Delta x_j\rangle_0 = (\hbar\epsilon/M)\delta_{ij}$. This shows that $\Delta_4\mathcal{A}^N$ has the expectation

$$\langle\Delta\mathcal{A}^N\rangle_0 = (\tau_b - \tau_a)\frac{\hbar^2}{2MR^2}\Delta_4 L_2. \tag{17}$$

where $\Delta_4 L_2$ is the contribution of the quartic term to the value L_2,

$$\Delta_4 L_2 = \frac{D^2-1}{12}. \tag{18}$$

This result is obtained using the Wick contraction rules for the tensor $\langle\Delta x_i\Delta x_j\Delta x_k\Delta x_l\rangle_0 = (\epsilon\hbar/M)(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$. Thus we remain with a final discrepancy in D dimensions,

$$\Delta_{meas}L_2 = \Delta L_2 - \Delta_4 L_2 = -\frac{1}{3}(D-1)(D-2), \tag{19}$$

to be explained now.
3) The final correction $\Delta_{meas}L_2$ is obtained from the proper treatment of the measure of the path integral à la [4]. *Near* the sphere in (1) we have used the measure

$$\prod_{n=1}^{N}\left[\int\frac{d^{D-1}\mathbf{u}_n}{\sqrt{2\pi\hbar\epsilon/MR^2}^{D-1}}\right]. \tag{20}$$

Although this measure seems quite natural for the surface of a sphere, it requires a correction for path integration, as was shown in [4]. To see this consider the amplitude in cartesian coordinates

$$(\mathbf{x}t|\mathbf{x}'t') = \frac{1}{\sqrt{2\pi\epsilon\hbar/M}^D}\prod_{n=2}^{N+1}\left[\int_{-\infty}^{\infty}d\Delta x_n\right]\prod_{n=1}^{N+1}K_0^\epsilon(\Delta\mathbf{x}_n), \tag{21}$$

with the short-time amplitudes,

$$K_0^\epsilon(\Delta\mathbf{x}_n) = \langle\mathbf{x}_n|\exp\left\{-\frac{1}{\hbar}\epsilon\hat{H}\right\}|\mathbf{x}_{n-1}\rangle \equiv \frac{1}{\sqrt{2\pi\epsilon\hbar/M}^D}\exp\left\{-\frac{1}{\hbar}\frac{M}{2}\frac{(\Delta\mathbf{x}_n)^2}{\epsilon}\right\} \tag{22}$$

where $\Delta \mathbf{x}_n \equiv \mathbf{x}_n - \mathbf{x}_{n-1}, \mathbf{x} \equiv \mathbf{x}_{N+1}, \mathbf{x}' \equiv \mathbf{x}_0$ (we may omit a possible extra potential which would enter trivially). We now transform $(\Delta \mathbf{x}_n)^2$ by a *non-holonomic* mapping to a space with curvature and torsion [13], parametrized with coordinates q^μ. For infinitesimal $\Delta \mathbf{x}_n \approx d\mathbf{x}_n$, the transformation would simply yield $(d\mathbf{x})^2 = g_{\mu\nu} dq^\mu dq^\nu$. For finite $\Delta \mathbf{x}_n$, however, we must expand $(\Delta \mathbf{x}_n)^2$ up to forth order in $\Delta {q_n}^\mu = {q_n}^\mu - {q_{n-1}}^\mu$, since we must find all terms that will eventually contribute to order ϵ [14, 15]. We expand around the final point q_n^μ,

$$
\begin{aligned}
& x^i(q_{n-1}) \equiv x^i(q_n - \Delta q_n) = \\
& \qquad x^i(q_n) - {e^i}_\mu \Delta q^\mu + \frac{1}{2} {e^i}_{\mu,\nu} \Delta q^\mu \Delta q^\nu - \frac{1}{3!} {e^i}_{\mu,\nu\lambda} \Delta q^\mu \Delta q^\nu \Delta q^\lambda + \ldots \; .
\end{aligned} \tag{23}
$$

For brevity, we have omitted the argument q_n in the ${e^i}_\mu$'s as well a the subscripts n of Δq^μ. Squaring $\Delta \mathbf{x}_n$ and expressing the everything in terms of the affine connection leads to the short-time sliced action expressed entirely in terms of intrinsic quantities (omitting again all sub n's),

$$
\begin{aligned}
\mathcal{A}^\epsilon_>(q, q - \Delta q) = & \frac{M}{2\epsilon} \{ g_{\mu\nu} \Delta q^\mu \Delta q^\nu - ,_{\mu\nu\lambda} \Delta q^\mu \Delta q^\nu \Delta q^\lambda \\
& + \left[\frac{1}{3} g_{\mu\tau} (\partial_\kappa ,_{\lambda\nu}{}^\tau + ,_{\lambda\nu}{}^\delta ,_{\kappa\delta}{}^\tau) + \frac{1}{4} ,_{\kappa\lambda}{}^\sigma ,_{\nu\kappa\sigma} \right] \Delta q^\mu \Delta q^\nu \Delta q^\lambda \Delta q^\kappa + \ldots \}
\end{aligned} \tag{24}
$$

with $g_{\mu\nu} \equiv {e^i}_\mu {e^i}_\nu$ and $,_{\mu\nu\lambda}$ evaluated at the final point q. The measure of path integration in (21) is transformed to q^μ-space with a Jacobian following from (23),

$$
J = \frac{\partial(\Delta x)}{\partial(\Delta q)} = \det({e^i}_\kappa) \det({\delta^\kappa}_\mu - {e_i}^\kappa e^i_{\{\mu,\nu\}} \Delta q^\nu + \frac{1}{2} e_i^\kappa e^i_{\{\mu,\nu\lambda\}} \Delta q^\nu \Delta q^\lambda + \ldots), \tag{25}
$$

where the curly brackets around the indices denote their symmetrization. Expanding the second factor in powers of Δq^μ, writing $\det({e^i}_\mu) \equiv e(q) = \sqrt{\det g_{\mu\nu}(q)} \equiv \sqrt{g(q)}$, and expressing the series in terms of a "Jacobian effective action" $\mathcal{A}_J$ with the definition $J = \sqrt{g(q)} \exp\{i\mathcal{A}_J/\hbar\}$, with

$$
\frac{i}{\hbar} \mathcal{A}_J \;\; = \;\; -,_{\{\nu\mu\}}{}^\mu \Delta q^\nu \tag{26}
$$

$$+ \quad \frac{1}{2}[\partial_{\{\mu,\ \nu,\kappa\}}{}^{\kappa} + {}_{,\ \{\nu,\kappa}{}^{\sigma}{}_{,\ \mu\},\sigma}{}^{\kappa} - {}_{,\ \{\nu\kappa\}}{}^{\sigma}{}_{,\ \{\mu,\sigma\}}{}^{\kappa}]\Delta q^{\nu}\Delta q^{\mu} + \dots,$$

we arrive at the time-sliced path integral in q-space

$$\langle q|\exp\{-\frac{1}{\hbar}(t-t')\hat{H}\}|q'\rangle \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M}^{D}}\prod_{n=2}^{N+1}\left[\int d^{D}\Delta q_n \frac{\sqrt{g(q_n)}}{\sqrt{2\pi\epsilon\hbar/M}^{D}}\right]$$
$$\times \exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N+1}[\mathcal{A}_{>}^{\epsilon}(q_n, q_n - \Delta q_n) + \mathcal{A}_J]\right\} \tag{27}$$

The integrals over Δq_n are to be performed successively from $n = N$ down to $n = 1$. It is useful to reexpress the measure in terms of the naively expected group invariant measure which we write as follows

$$\langle q|\exp\{-\frac{1}{\hbar}(t-t')\hat{H}\}|q'\rangle \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M}^{D}}\prod_{n=2}^{N+1}\left[\int d^{D}q_{n-1} \frac{\sqrt{g(q_{n-1})}}{\sqrt{2\pi\epsilon\hbar/M}^{D}}\right]$$
$$\times \exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N+1}[\mathcal{A}_{>}^{\epsilon}(q_n, q_n - \Delta q_n) + \Delta_{meas}\mathcal{A}_J]\right\}. \tag{28}$$

The correction term $\Delta_{meas}\mathcal{A}_J$ due to the measure is obtained from the difference between $\mathcal{A}_J$ and $\mathcal{A}_{J_0}$, the latter arising from the shift in the subscripts n in the $\sqrt{g(q)}$ factors of (27), (28) by one unit, i.e. from the ratio $\sqrt{g(q_n)}/\sqrt{g(q_{n-1})} = e(q_n)/e(q_{n-1}) \equiv \exp\{i\mathcal{A}_{J_0}/\hbar\}$. Expanding the determinant $e(q_n) = e(q_{n-1} - \Delta q_n)$ in powers of Δq_n we see that $\mathcal{A}_{J_0}$ is the same as $\mathcal{A}_J$ in (26) except that symmetrization symbols are absent. Either (27) or (28) may be used as the correct path integral formulas in spaces with curvature and torsion.

4) The problem of interest here involves no torsion. Then a simple algebra shows that $\Delta_{meas}\mathcal{A}_J$ reduces to

$$\Delta_{meas}\mathcal{A}_J = -\frac{\hbar}{6}\bar{R}_{\mu\nu}\Delta q^{\mu}\Delta q^{\nu}, \tag{29}$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor, which for a sphere of radius R is $(D-2)g_{\mu\nu}/R^2$. The perturbative treatment of (29) gives the only relevant contribution to the energy,

$$\langle \Delta_{meas} \mathcal{A}_J \rangle_0 = -\epsilon \frac{\hbar^2}{6M} \frac{(D-1)(D-2)}{R^2}, \tag{30}$$

thus producing precisely the missing energy required by (19).

5) The sphere in four dimensions is equivalent to the covering group of rotations in three dimensions, the group $SU(2)$. Knowing now how to solve the time-sliced path integral near and on the surface of the sphere, we can obtain the same quantities *near* and *on* the group space of $SU(2)$ [16]. This puts us in a position to solve the time sliced path integral of a spinning spherical top by reduction to the $SU(2)$ problem. We only have to go from $SU(2)$, which is the covering group of the rotation group, down to the rotation group itself [17]. The angular positions with Euler angles γ and $\gamma + 2\pi$ are physically indistinguishable. The physical states must be a representation of this operation and the time-displacement amplitude must reflect this. The simplest possibility is the trivial even representation where one adds the amplitudes to go from the initial configuration $\varphi_a, \theta_a, \gamma_a$ to the identical final ones $\varphi_b, \theta_b, \gamma_b$ and $\varphi_b, \theta_b, \gamma_b + 2\pi$ and forms the amplitude

$$\begin{aligned} &(\varphi_b, \theta_b, \gamma_b\ \tau_b | \varphi_b, \theta_b, \gamma_b\ \tau_a)_{top} = \\ &\qquad (\varphi_b, \theta_b, \gamma_b\ \tau_b | \varphi_b, \theta_b, \gamma_b\ \tau_a) + (\varphi_b, \theta_b, \gamma_b + 2\pi\ \tau_b | \varphi_b, \theta_b, \gamma_b\ \tau_a). \end{aligned} \tag{31}$$

The sum eliminates all half-integer representation functions $d^{l/2}_{mm'}(\theta)$ in the expansion (1) of the amplitude.

Instead of the sum we could also have formed another representation of the operation $\gamma \to \gamma + 2\pi$, the antisymmetric combination

$$\begin{aligned} &(\varphi_b, \theta_b, \gamma_b\ \tau_b | \varphi_b, \theta_b, \gamma_b\ \tau_a)_{fermions} = \\ &\qquad (\varphi_b, \theta_b, \gamma_b\ \tau_b | \varphi_b, \theta_b, \gamma_b\ \tau_a) - (\varphi_b, \theta_b, \gamma_b + 2\pi\ \tau_b | \varphi_b, \theta_b, \gamma_b\ \tau_a). \end{aligned} \tag{32}$$

Here the expansion (9) retains only the half-integer angular momenta $l/2$. In nature such spins are associated with fermions such as electrons, protons, muons, or neutrinos, which carry only one specific value of $l/2$.

In principle, there is no problem in treating also a non-spherical top. While the spherical top has "near the group space" a time-sliced action

$$\frac{1}{\epsilon^2} I \left[1 - \frac{1}{2} \mathrm{tr}(g_n g_{n-1}^{-1}) \right], \tag{33}$$

the asymmetric top with three moments of inertia I_{123} requires separating the three components of the angular velocities

$$\omega_a = i \mathrm{tr}(g \sigma_a \dot{g}^{-1}), \quad a = 1, 2, 3. \tag{34}$$

(σ_a= Pauli matrices) on the time lattice so that the action reads

$$\begin{aligned} \frac{1}{\epsilon^2} \Big\{ I_1 [1 - \frac{1}{2} \mathrm{tr}(g_n \sigma_1 g_{n-1}^{-1})] + I_2 [1 - \frac{1}{2} \mathrm{tr}(g_n \sigma_2 g_{n-1}^{-1})] \\ + I_3 [1 - \frac{1}{2} \mathrm{tr}(g_n \sigma_3 g_{n-1}^{-1})] \Big\} . \end{aligned} \tag{35}$$

rather than (33). The amplitude "near the top" is then an appropriate generalization of (9). The calculation of the correction term ΔE, however, is more complicated than before and is left to the reader, following the rules explained above.

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