The Extra Gauge Symmetry of String Deformations in Electromagnetism with Charges and Dirac Monopoles *

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Abstract

We point out that electromagnetism with Dirac magnetic monopoles harbors an *extra* local gauge invariance to be called *monopole gauge invariance*. The gauge transformations act on a *gauge field of monopoles* $F^P_{\mu\nu}$ and is *independent* of the ordinary *electromagnetic gauge invariance*. The extra invariance expresses the physical irrelevance of the shape of the Dirac strings attached to the monopoles. The independent nature of the new gauge symmetry is illustrated by comparison with two other systems, superfluids and solids, which are *not* gauge-invariant from the outset but which nevertheless possess a precise analog of the monopole gauge invariance in their vortex and defect structure, respectively. The extra monopole gauge invariance is shown to be responsible for the Dirac charge quantization condition $2eg/\hbar c =$ integer which can now be proved for any *fixed* particle orbits, i.e., without invoking fluctuating orbits which would corresond to the standard derivation using Schrödinger wave functions. The only place where quantum physics enters in our theory is by admitting the action to jump by $2\pi\hbar\times$ integer without physical consequences when moving the string at fixed orbit.

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1 Introduction

The purpose of this note is to point out that if Dirac magnetic monopoles [1] are added to electromagnetism, the theory possesses one more independent gauge symmetry which makes the Dirac string irrelevant and enforces the Dirac charge quantization for any *fixed* particle orbit. No use is made of the single-valuedness of particle wave functions as in other works [1, 2] which in our path integral discussion would correspond to summing over all fluctuating particle orbits. The only quantum feature entering is that the action is allowed to jump by $2\pi\hbar\times$ integer when moving around the string of a fixed particle orbit without physical consequences. With this admission, the action is invariant under two *mutually independent* types of gauge transformations, the usual *electromagnetic gauge transformations* of the vector potential A_μ which keep the electromagnetic field strengths $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ invariant, plus the *extra* gauge transformations on the *monopole gauge field* $F^P_{\mu\nu}$ which describes the world lines of the magnetic monopoles [the reason for using a superscript P will become clear below when discussing the non-gauge invariant sytems after Eq. (44)]. The various gauges of the latter correspond to various shapes of the Dirac strings emerging from the monopoles.

As usual, there exists a dual description of the system with a dual vector potential $\tilde{A}_\mu$. Correspondingly, there will be a *pair* of gauge symmetries, to be called *magnetoelectric* and *charge* gauge invariance, the latter being new. Both are completely independent of the initial electromagnetic gauge invariance.

The fact that all these extra gauge invariances are completely independent of the initial electromagnetic one will be illustrated by means of two completely different physical systems, elasic solids and superfluids. These are *not* gauge-invariant from the outset by having no analog of the *electromagnetic* gauge invariance. They do, however, exhibit a perfect analog of the extra *monopole* gauge invariance. They also contain analogs of the magnetoelectric and the charge gauge invariance which will demonstrate that also these are independent of the initial electromagnetic gauge invariance.

Although the physical properties of Dirac magnetic monopoles are theoretically quite well understood we feel that there is still a lack of an elegant description of the singular Dirac string which imports the monopole flux from spatial infinity. In particular, the existence of the additional monopole gauge invariance (and its dual partner) has apparently been overlooked. We see this as a reason why the literature displays frustrations when setting up and interpreting various field actions for studying the field and particle fluctuations. The most recent formulation [2] was restricted to a single electron-monopole pair and made use of two electromagnetic gauge fields depending on the relative position of the particles. In our formulation it would correspond to two specific choices of gauges of our monopole gauge field $F^P_{\mu\nu}$ depending on the position of the charge with respect to the monopole. Using these the authors of [2] have, as they call it, "exorcized" the Dirac string, but at the cost of having a term in the action which keeps requiring case distinctions. We find that by keeping all string degrees of freedom in the action of the system as in Dirac's original work [1], but by making full use of the the extra gauge invariance, we can ensure the physical irrelevance of the strings in a most natural and universal way.

2 Monopole Gauge Invariance

The classical free-field electromagnetic action without monopoles reads, in a euclidean formulation (using natural units with $c = 1$),

$$\mathcal{A}_0 = \frac{1}{16\pi} \int d^4x F^2_{\mu\nu}, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength expressed in terms of the usual electromagnetic vector potential A_μ. For brevity we have written $F_{\mu\nu}^2$ for the contracted $F_{\mu\nu}F_{\mu\nu}$. The field A_μ is single-valued and integrable, i.e., it satisfies the Schwartz condition

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) A_\lambda = 0. \tag{2}$$

Electromagnetic gauge transformations change

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \tag{3}$$

without changing $F_{\mu\nu}$. Thus Λ must be integrable as well,

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \Lambda = 0. \tag{4}$$

Magnetic monopoles are line-like objects in 4-space (worldlines) along which the observable field strength $F_{\mu\nu}^{\rm obs}$ satisfies

$$\frac{1}{2} \epsilon_{\mu\nu\lambda\kappa} \partial_\mu F_{\lambda\kappa}^{\rm obs} = -4\pi \tilde{\jmath}_\nu, \tag{5}$$

where $\tilde{\jmath}_\mu$ is the magnetic current which for a monopole of magnetic charge g is proportional to a δ-function singularity on its world line $\bar{x}_\mu(\sigma)$,

$$\delta_\mu(x; L) \equiv \int d\sigma \frac{d\bar{x}_\mu(\sigma)}{ds} \delta^{(4)}(x - \bar{x}(\sigma)), \tag{6}$$

namely,

$$\tilde{\jmath}_\mu = g \delta_\mu(x; L). \tag{7}$$

To make the Dirac string invisible in Bohm-Aharonov scattering with charged particles we shall later [below Eq. (25)] find the requirement that the magnetic charge g and the fundamental electric charge e are related by the Dirac quantization condition

$$\frac{2eg}{\hbar c} = \text{integer}, \tag{8}$$

with $c = 1$ in the natural units at hand.

Eq. (5) shows that $F^{\rm obs}_{\mu\nu}$ cannot be represented as a curl of a single-valued vector potential A_μ [the left-hand side would be $\epsilon_{\mu\nu\lambda\kappa}(\partial_\mu\partial_\lambda - \partial_\lambda\partial_\mu)A_\kappa)/2$]. The easiest way of circumventing this problem is by incorporating the monopole worldline into the electromagnetic field theory is via an extra monopole gauge field,

$$F^P_{\mu\nu} \equiv 4\pi g \tilde{\delta}_{\mu\nu}(x;S). \tag{9}$$

Here $\tilde{\delta}_{\mu\nu}(x;S)$ is the dual,

$$\tilde{\delta}_{\mu\nu}(x;S) \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}\delta_{\lambda\kappa}(x;S), \tag{10}$$

of the δ-function $\delta_{\lambda\kappa}$ which is singular on the world surface S,

$$\delta_{\mu\nu}(x;S) \equiv \int d\sigma d\tau \left[\frac{d\bar{x}_\mu(\sigma)}{d\sigma}\frac{d\bar{x}_\nu(\tau)}{d\tau} - (\mu \leftrightarrow \nu)\right] \delta^{(4)}(x - \bar{x}(\sigma,\tau)), \tag{11}$$

and S is any surface whose boundary coincides with the world line of the monopole, i.e.,

$$\frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}\partial_\nu\tilde{\delta}_{\lambda\kappa}(x;S) = \delta_\mu(x;L) \tag{12}$$

(this being Stokes' theorem in a local formulation). The precise location of the surface is irrelevant, only the boundary line L has a physical meaning. The surface S is the worldsurface of the Dirac string. Thus, for any line L there are many possible surfaces S. We can go over from one S to another, say S', at fixed boundary L as follows

$$\tilde{\delta}_{\lambda\kappa}(x;S) \to \tilde{\delta}_{\lambda\kappa}(x;S') = \tilde{\delta}_{\lambda\kappa}(x;S) + \partial_\mu\delta_\nu(x;V) - \partial_\nu\delta_\mu(x;V), \tag{13}$$

where

$$\delta_\nu(x;V) \equiv \epsilon_{\nu\mu\kappa\delta}\int d\sigma d\tau d\lambda \frac{d\bar{x}_\mu}{d\sigma}\frac{d\bar{x}_\kappa}{d\tau}\frac{d\bar{x}_\delta}{d\lambda}\delta^{(4)}\left(x - \bar{x}(\sigma,\tau,\lambda)\right) \tag{14}$$

is the δ-function that is singular on a three-dimensional volume V in 4-space swept out when the surface S moves through 4-space.

Many monopoles are, of course, represented by an appropriate additive superposition of various gauge fields of the form (9) with different surfaces S. Note that these gauge fields are of an entirely new type: If we approximate the spacetime continuum by a simple hypercubic lattice of spacing ϵ, the superpositions of (9) can be written as $F^P_{\mu\nu} = 4\pi g N_{\mu\nu}(x)/\epsilon^2$ where $N_{\mu\nu}(x)$ is an arbitrary *integer-valued* antisymmetric tensor field. It may be imagined as living on the plaquettes of the lattice.

We are now ready to set up the electromagnetic action in the presence of a monopole world line. It depends only on the difference between the total field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ of the integrable vector potential A_μ and the monopole gauge field $F^P_{\mu\nu}$ of (9), i.e., it is given by [3]

$$\mathcal{A}_0 + \mathcal{A}_{\rm mg} \equiv \mathcal{A}_{0,\rm mg} = \int d^4x \frac{1}{16\pi} \left(F_{\mu\nu} - F^P_{\mu\nu} \right)^2 . \tag{15}$$

The subtraction of $F^P_{\mu\nu}$ is essential in avoiding an infinite energy density that would otherwise be carried by the flux tube in $F_{\mu\nu}$ inside the Dirac string, the differences

$$F^{\rm obs}_{\mu\nu} \equiv F_{\mu\nu} - F^P_{\mu\nu}$$

being the regular observable field strengths. Since only fields with finite action are physical, the action contains no contributions from squares of δ-functions as it might initially appear.

The action (15) exhibits two types of gauge invariances. First, the original electromagnetic one under (3), under which $F^P_{\mu\nu}$ is trivially invariant. Second, there is gauge invariance under *monopole gauge transformations*

$$F^P_{\mu\nu} \to F^P_{\mu\nu} + \partial_\mu \Lambda^P_\nu - \partial_\nu \Lambda^P_\mu , \tag{16}$$

with integrable vector functions $\Lambda^P_\mu(x)$, which by (13) have the general form

$$\Lambda^P_\mu(x) = 4\pi g \delta_\mu(x; V), \tag{17}$$

with arbitrary choices of 3-volumes V. Certainly, $\Lambda^P_\mu(x)$ can also be a superposition of such functions with various V's, $\Lambda^P_\mu(x) = 4\pi g \sum_V \delta_\mu(x;V)$. In this way one obtains all functions of x whose values are integer multiples of $4\pi g$. To have invariance of (15), the transformation (16) must be accompanied by a shift in the electromagnetic gauge field [4]

$$A_\mu \to A_\mu + \Lambda^P_\mu . \tag{18}$$

From Eqs. (9), (12), and (13) we see that the physical significance of the part (16) of the monopole gauge transformation is to change the Dirac world surface withouth changing its boundary, the monopole world line. An exception is only the submanifold of the form $\Lambda^P_\mu = \partial_\mu \Lambda^P$ with $\Lambda^P = 4\pi g \sum_{V_4} \delta(x;V_4)$, where $\delta(x;V_4)$ is the δ-function on the four-volume V_4,

$$\delta(x;V_4) \equiv \epsilon_{\nu\mu\kappa\delta} \int d\nu d\sigma d\tau d\lambda \frac{d\bar{x}_\nu}{d\sigma} \frac{d\bar{x}_\mu}{d\sigma} \frac{d\bar{x}_\kappa}{d\tau} \frac{d\bar{x}_\delta}{d\lambda} \delta^{(4)} \left(x - \bar{x}(\nu,\sigma,\tau,\lambda) \right). \tag{19}$$

This does not give any change in $F^P_{\mu\nu}$ since it is a submanifold of the original gauge transformations (3). In general, the field strengths $F_{\mu\nu}$ *are* changed by moving the Dirac string through space, but the observable field strengths $F^{\text{obs}}_{\mu\nu}$ are invariant.

The part (18) of the monopole gauge transformations expresses the fact that in the presence of monopoles the gauge field A_μ is necessarily a *cyclic variable* for which A_μ and $A_\mu + 2n\pi$ are identical for any integer n.

It must be emphasized that this transformation has no relation with the original gauge transformation (3). This will become most obvious in the counterexample to be given in Eq. (43) of a theory which exhibits the analog of the monopole gauge invariance although it has no counterpart of the original electromagnetic gauge invariance. Hopefully, the counterexamle will eventually do away with an often-found misconception [4] that, since the movement of the Dirac string can be achieved by a transformation,

$$A_\mu \to A_\mu + g\partial_\mu \Omega, \tag{20}$$

where Ω is the spherical angle over which the string has swept, the invisibility of the string may be related to the electromagnetic gauge invariance (together with the single-valuedness of wave functions). After all, (20) looks precisely like (3), with $\Lambda = g\Omega$. This argument, however, is invalid since the spherical angle is a multivalued function which fails to satisfy the integrability condition (4). This is why (20) is *not* a gauge transformation in spite of its suggestive appearance. It cannot possibly be since it changes the magnetic field along the Dirac string. Sometimes, (20) is referred to as a "singular gauge transformation" or "general gauge transformation". Obviously, this terminology is strongly misleading and must be rejected. After all, if we were to allow for such "singular" (i.e. non-integrable) transformations Λ in (3) we could reach an arbitrary field $F_{\mu\nu}$ starting from $F_{\mu\nu} \equiv 0$ and the physics would certainly not be invariant under this [5].

3 Monopoles and Charge Quantization

The fundamental difference between the original and the monopole gauge invariance becomes relevant if one wants to describe also electric charges via the current interaction

$$\mathcal{A}_{\rm el} = i \int d^4x j_\mu(x) A_\mu(x) \tag{21}$$

where $j_\mu(x)$ is the electric current of the world line of a charged particle

$$j_\mu = e\delta_\mu(x; L). \tag{22}$$

Due to ordinary current conservation

$$\partial_\mu j_\mu = 0, \tag{23}$$

the action (21) is trivially invariant under electromagnetic gauge transformations (3). In contrast, it can remain invariant under monopole gauge transformations (16), (18) only if the monopole charge satisfies the quantization condition (8). Indeed, the change of the electric interaction, and thus also of the total action

$$\mathcal{A}_{\rm tot} \equiv \mathcal{A}_0 + \mathcal{A}_{\rm mg} + \mathcal{A}_{\rm el} \tag{24}$$

is

$$\mathcal{A}_{\rm el} \to \mathcal{A}_{\rm el} + i4\pi e g I \tag{25}$$

where

$$I \equiv \int d^4x \delta_\mu(L)\delta_\mu(V). \tag{26}$$

This is an integer number if L passes through V and zero if it misses V (i.e., if the string in the operation (13) sweeps across L or not). But physics is invariant under jumps of the action by $2\pi\times$integer since this does not contribute to any path integral since this involves only the exponential $e^{-\mathcal{A}/\hbar}$. In this way we derive the charge quantization (8). It must be emphasized that this is done here with much less quantum mechanical input than in earlier works [1]. Our derivation requires no wave functions as, for instance, the Refs. [1, 4]. In path integral language at hand, this would imply a sum over fluctuating particle orbits to be present. In our proof, however, the orbits are allowed to remain fixed, only the strings are moved around by the monopole gauge transformation.

4 Dual Gauge Fields

We now come to the other pair of gauge fields which are dual to the first ones, to be called *magnetoelectric* and the *charge* gauge fields. The first of these is introduced by going over from the action (15) to a first-order formalism, using an independent fluctuating field $f_{\mu\nu}$ and replacing the action (15) by the equivalent one

$$\tilde{\mathcal{A}}_{0,\mathrm{mg}} = \frac{1}{4\pi}\int d^4x \left[\frac{1}{4}f_{\mu\nu}^2 + \frac{i}{2}f_{\mu\nu}\left(F_{\mu\nu} - F_{\mu\nu}^P\right)\right], \tag{27}$$

with the two independent fields A_μ and $f_{\mu\nu}$. Since $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\mu A_\mu$ we can integrate out the field A_μ in the associated path integral (which is equivalent here to extremizing $\tilde{\mathcal{A}}_{0,\mathrm{mg}}$ in A_μ). This gives the constraint

$$\partial_\nu f_{\mu\nu} = 0. \tag{28}$$

It can be satisfied by introducing a dual magnetoelectric vector potential $\tilde{A}_\mu$ and writing

$$f_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda\kappa}\partial_\lambda \tilde{A}_\kappa. \tag{29}$$

If we also introduce a dual field tensor

$$\tilde{F}_{\mu\nu} \equiv \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu, \tag{30}$$

the action (27) takes the dual form

$$\tilde{\mathcal{A}}_{0,\mathrm{mg}} \equiv \tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_{\mathrm{mg}} = \int d^4x \left(\frac{1}{16\pi}\tilde{F}_{\mu\nu}^2 + i\tilde{A}_\mu \tilde{j}_\mu\right), \tag{31}$$

with the magnetoelectric source

$$\tilde{j}_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}\partial_\nu F_{\lambda\kappa}^P. \tag{32}$$

This is the current density of the magnetic monopole, as we see from (7), (9), and (12). This current density satisfies the conservation law

$$\partial_\mu \tilde{j}_\mu = 0, \tag{33}$$

expressing the fact that monopole world lines are closed so that $\partial_\mu \delta_\mu(x; L)=0$. As a consequence, the action (31) allows for an additional set of gauge transformations which are the magnetoelectric ones

$$\tilde{A}_\mu \to \tilde{A}_\mu + \partial_\mu \tilde{\Lambda} \tag{34}$$

with arbitrary integrable functions $\tilde{\Lambda}$,

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\tilde{\Lambda} = 0. \tag{35}$$

If we include the electric current (22) into the dual form of the action (31) it becomes

$$\tilde{\mathcal{A}}_{\rm tot} = \int d^4x \left\{ \frac{1}{4\pi} \left[\frac{1}{4} f_{\mu\nu}^2 + \frac{i}{2} f_{\mu\nu} \left(F_{\mu\nu} - F_{\mu\nu}^P \right) \right] + i j_\mu A_\mu \right\}. \tag{36}$$

Integrating out the field A_μ gives now

$$\partial_\mu f_{\mu\nu} = 4\pi i j_\nu, \tag{37}$$

rather than (28). The solution of this requires the introduction of a gauge field analog to (9), the *charge* gauge field

$$\tilde{F}_{\mu\nu}^P = 4\pi e \tilde{\delta}_{\mu\nu}(x; S). \tag{38}$$

Then (37) is solved by

$$f_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\lambda\kappa} (\tilde{F}_{\mu\nu} - \tilde{F}_{\mu\nu}^P). \tag{39}$$

The identity (12) ensures (37).

Note that when inserting (39) into (36) there appears at first also a term

$$\Delta\mathcal{A} = \frac{i}{8\pi} \int d^4x \epsilon_{\mu\nu\lambda\kappa} \tilde{F}_{\mu\nu}^P F_{\lambda\kappa}^P \tag{40}$$

When remembering the explicit forms (9) and (38) this is seen to be equal to

$$\Delta\mathcal{A} = 4\pi e g \frac{i}{2} \int d^4x \epsilon_{\mu\nu\lambda\kappa} \delta_{\mu\nu}(x; S) \delta_{\lambda\kappa}(x; S'). \tag{41}$$

Now, the integral is an integer as follows most easily by going to a lattice and writing $\delta_{\mu\nu}(x; S)$ in terms of the above introduced integer-valued tensor field $N_{\mu\nu}(x)$ defined on the plaquettes of the lattice as $\delta_{\mu\nu}(x; S) \equiv N_{\mu\nu}(x)/\epsilon^2$.

Invoking Dirac's quantization condition (8) [which was required after Eq. (26)] the resulting $e^{-\Delta\mathcal{A}/\hbar}$ is obviously equal to unity and can be ignored.

The dual version of the total action (24) of monopoles and charges is therefore

$$\tilde{\mathcal{A}}_{\rm tot} \equiv \tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_{\rm el} + \tilde{\mathcal{A}}_{\rm mg} = \int d^4x \left[\frac{1}{16\pi}(\tilde{F}_{\mu\nu} - \tilde{F}^P_{\mu\nu})^2 + i\tilde{A}_\mu \tilde{j}_\mu\right]. \tag{42}$$

It is the same as Eq. (29) extended by the charge gauge field $F^P_{\mu\nu}$.

With the (apparently absolute) predominance of electric charges in nature, however, this dual action is only of academic interest.

5 Independent Nature of New Gauge Invariance

We now turn to demonstrating that the new monopole gauge invariance is completely independent of the original electromagnetic gauge invariance. The same thing will be seen to be true for the dual magnetoelectric and charge gauge invariances. The demonstration is given by presenting two counterexamples of systems which do not possess an original gauge invariance at all while possesing the new gauge invariances.

Crytalline Solids First there is the three-dimensional crytalline solid with defects. Its elastic plus plastic energy is completely analogous to (15) and reads [3, 6]

$$E_{0,pl} \equiv E_0 + E_{pl} = \int d^3x \mu \left(u_{ij} - u^P_{ij}\right)^2, \tag{43}$$

where μ is the modulus of shear (we have omitted the second elastic modulus, for simplicity), u_{ij} is the *elastic strain*

$$u_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{44}$$

of the displacement fields $u_i(x)$, and u_{ij}^P is the *plastic strain* (this name explains the superscript P here and in (9)] which is the *defect gauge field* of the crystal. The energy (43) is a symmetric version of the electromagnetic action (15). It does *not* possess any initial gauge invariance. It is, however, invariant under *plastic* or *defect gauge transformations*, which are analogous to the monopole gauge transformations (16), (18),

$$\begin{aligned} u_{ij}^P &\rightarrow u_{ij}^P + \partial_i \Lambda_j^P + \partial_j \Lambda_i^P \\ u_i &\rightarrow u_i + \Lambda_i^P . \end{aligned} \tag{45}$$

The physical meaning of the first line is to move around the irrelevant Volterra sheets in the solid whose boundaries are the defect lines [6]. The second line expresses the fact that the displacement field is a cyclic variable. On a simple cubic lattice with a spacing normalized to 2π the displacements u_i can not be distinguished from $u_i + 2n\pi$ for any integer n since the identity of the lattice constituents makes it impossible to tell where the displaced atom has come from. In fact, after a long time self-diffusion will carry each particle through the entire crystal so that all such memories are lost.

Also the dual gauge invariance, corresponding to the magnetoelectric one in (34), can be exhibited in this system. One merely has to go to the first-order formalism via a symmetric stress field σ_{ij} [analogous to $f_{\mu\nu}$ in (27)]

$$E_{0,pl}^{eq} = \int d^3x \left[\frac{1}{4\mu} \sigma_{ij}^2 + i\sigma_{ij}(u_{ij} - u_{ij}^P) \right] \tag{46}$$

and finds by integrating the path integral for the partition function over the displacement fields u_i, the *stress conservation law* [3] [compare Eq. (28)]

$$\partial_i \sigma_{ij} = 0. \tag{47}$$

This is true in the absence of external body forces. The stress conservation calls for the introduction of a *stress gauge field* χ_{ij} [compare (29)]

$$\sigma_{ij} \equiv \epsilon_{ikl} \epsilon_{jmn} \partial_k \partial_m \chi_{ln}, \tag{48}$$

which is coupled locally to the *defect density* [analogous to (32)]

$$\eta_{ij} = \epsilon_{ikl}\epsilon_{jmn}\partial_n\partial_m u^P_{ln}, \tag{49}$$

with an energy [compare (31)],

$$\tilde{E}_{0,pl} \equiv \tilde{E}_0 + \tilde{E}_{pl} = \int d^3x \left[\frac{1}{4\mu}\sigma^2_{ij} + i\chi_{ij}\eta_{ij}\right]. \tag{50}$$

Due to *defect conservation* [analogous to (33)]

$$\partial_i\eta_{ij} = 0, \tag{51}$$

there is *stress gauge invariance*

$$\chi_{ij} \to \chi_{ij} + \partial_i\tilde{\Lambda}_j + \partial_j\tilde{\Lambda}_i, \tag{52}$$

[as in (34)].

Superfluids The second example is superfluid ^{4}He. Here the fundamental field is the phase of the complex order parameter $\Psi(\mathbf{x}) = e^{iu\mathbf{x})}$, the scalar field $u(\mathbf{x})$ with u and $u + 2n\pi$ being identical for any integer n. The energy of the superfluid part of the liquid is completely analogous to (15) and (43) [3, 7]:

$$E_{0,pl} \equiv E_0 + E_{pl} = \int d^3x\,\mu\left(u_i - u^P_i\right)^2, \tag{53}$$

where u^P_i corresponds to the *plastic strain*. It is the *gauge field of vortex lines*. The energy (43) is a symmetric version of the electromagnetic action (15). As (43) it does *not* possess any initial gauge invariance, but it is invariant under *vortex gauge transformations*, which are the analogs of the monopole gauge transformations (16), (18), (45):

$$\begin{aligned} u^P_i &\to u^P_i + \partial_i\Lambda^P \\ u &\to u + \Lambda^P. \end{aligned} \tag{54}$$

The physical meaning of the first line is to move around surfaces in space across which the phase u jumps by integer multiples of 2π and whose boundaries are vortex lines [7]. The second line expresses the cyclic nature of the phase field u.

The dual gauge invariance, corresponding to the magnetoelectric one in (34) and (52) is found as before by going to the first-order formalism via a vector field b_i [analogous to $f_{\mu\nu}$ in (27) and to σ_{ij} in (46)]

$$E^{eq}_{0,pl} = \int d^3x \left[\frac{1}{4\mu} b_i^2 + i b_i (u_i - u_i^P) \right]. \tag{55}$$

The field b_i is, in fact, the supercurrent. By integrating the path integral for the partition function over the displacement fields u_i one finds the *conservation law* of the supercurrent [3] [compare Eq. (28) and (47)]

$$\partial_i b_i = 0. \tag{56}$$

This calls for the introduction of a *gauge field of superflow* a_i [compare (29) and (48)] so that

$$b_i \equiv \epsilon_{ijk} \partial_j a_k, \tag{57}$$

which is coupled locally to the *vortex density* [analogous to (32) and (49)]

$$l_i = \epsilon_{ijk} \partial_j u_k^P, \tag{58}$$

with an energy [compare (31), (50)],

$$\tilde{E}_{0,pl} \equiv \tilde{E}_0 + \tilde{E}_{pl} = \int d^3x \left[\frac{1}{4\mu} b_i^2 + i a_i l_i \right]. \tag{59}$$

since vortex lines are closed there is *vortex conservation* [analogous to (33), (51)]

$$\partial_i l_i = 0, \tag{60}$$

which implies the *stress gauge invariance*

$$a_i \to a_i + \partial_i \tilde{\Lambda}, \tag{61}$$

[as in (34), (52)].

Thus crystalline solids and superfluids possess the analogs of monopole and magnetoelectric gauge invariances, the plastic and stress gauge invariances, without any initial gauge symmetry which happened to be present in the electromagnetic case. Note that the stress gauge transformations in ther latter case have precisely the same form as the original gauge transformations in three-dimensional electromagnetism.

6 Outlook

It is amusing that, from the observations in this paper, the initial gauge symmetry is of a more "accidental" feature of electromagnetism than the other two, the monopole and magnetoelectic gauge structures (i.e., the defect (vortex) and the stress gauge structures (gauge field of superflow). The latter are of a much more universal nature appearing in many systems containing massless and defect-like excitations. Further examples are solids with higher gradient elasticity [6] and gravity [6, 8].

An important application of the new type of gauge invariance will be the derivation of a simple quantum field theory of charges and Dirac monopoles.

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[1] P.A.M. Dirac, Proc. Roy. Soc. *A133*, 60 (1931); Phys. Rev. *74*, 817 (1948); M.N. Saha, Ind. J. Phys. **10**, 145 (1936). See also J. Schwinger, *Particles, Sources and Fields*, Vols. 1 and 2, Addison Wesley, Reading, Mass., 1970 and 1973, G. Wentzel, Progr. Theor. Phys. Suppl. *37-38*, 163 (1966), the review by E. Amaldi, in *Old and New Problems in Elementary Particles*, edited by G. Puppi, Academic Press, New York (1968), D. Villaroel, Phys. Rev. D *14*, 3350 (1972), Yu.D. Usachev, Sov. J. Particles Nuclei *4*, 92 (1973), A.O. Barut, J. Phys. A *11*, 2037 (1978), and the textbook J.D. Jackson, *Classical Electrodynamics*, John Wiley and Sons, New York, 1975, Sects. 6.12-6.13.

[2] T.T. Wu and C.N. Yang, Phys. Rev. D *14*, 437 (1976) "exorcise" the string via sections. While these may be useful in the mathematics of a single static monopole, they are awkward to handle in the dynamics of an arbitrary number of them. Note that, contrary to our approach, Wu and Yang are cannot derive the Dirac quantization condition at the level of classical electron and monopole orbits (see their remark 6 on page 443)], Phys. Rev. D *12*, 3845, (1075), D *16*, 1018 (1977); Nuclear Physics B *107*, 365 (1976), C.N. Yang, Lectures presented at the 1976 Erice summer school, in *Gauge Interactions*, Plenum Press, New York 1978, edited by A. Zichichi and at the 1982 Erice summer school, in *Gauge Interactions*, Plenum Press, New York 1984, edited by A. Zichichi.

[3] H. Kleinert, Lectures presented at the 1982 Erice summer school, ibid.; and Lectures presented at the 1983 Cargèse summer school, publ. in *Progress in Gauge Field Theory*, Plenum Press, 1984, edited by G. t'Hooft et al. .
H. Kleinert, Phys. Lett. *A130*, 59 (1988)

[4] See, most notably, the textbook by Jackson in Ref. [1], p. 258, where it is stated that "a choice of different string positions is equivalent to different choices of (electromagnetic) gauge"; also his Eq. (6.162) and the lines below it. In the unnumbered equation on p. 258 Jackson observes that the physical monopole field is $F_{\mu\nu}^{monop} = F_{\mu\nu} - F_{\mu\nu}^{P}$ but the independent gauge properties of $F_{\mu\nu}^{P}$ and the need to use the action (15) rather than (1) are not noticed.

[5] P. Goddard and D. Olive, Progress in Physics *41*, 1357 (1978) in their review article work with such "general gauge transformations" [see their Eq. (2.46)] and they do this with appropriate care. They point out that the field tensor $F_{\mu\nu}^{\rm obs} = F_{\mu\nu} + (1/a^3 a)\mathbf{\Phi} \cdot (\partial_\mu \mathbf{\Phi} \times \partial_\nu \mathbf{\Phi})$ introduced by 't Hooft into his $SU(2)$ gauge theory with Higgs fields $\mathbf{\Phi}$ to describe magnetic monopoles can be brought to the form $F_{\mu\nu}^{\rm obs} \equiv F_{\mu\nu} - F_{\mu\nu}^{P}$ by a gauge transformation within the SU(2) gauge group which moves magnetic fields from $F_{\mu\nu}$ to $F_{\mu\nu}^{P}$ without changing $F_{\mu\nu}^{\rm obs}$ [see their Eq. (4.30) and the last two equations in their Section 4.5]. Note that these are *not* permissible gauge transformation of the electromagnetic type while being similar to our *monopole gauge transformations* of the form (16)-(18), although with more general transformation functions than (17) being allowed in the SU(2) gauge theory.

[6] H. Kleinert, *Gauge Fields in Condensed Matter*, World Scientific, Vol 2, Singapore 1989.

[7] ibid., Vol 1.

[8] H. Kleinert, Ann. d. Physik *44*, 117 (1987)