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Exact Semiclassical Resistance of Thin Superconducting Wire

H. Kleinert and T. Sauer

Institut für Theorie der Elementarteilchen, Freie Universität Berlin, Berlin, Germany

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We give the exact analytic expression for the semiclassical rate of phase slips in a thin superconducting wire, thereby correcting earlier published results.

One of the most beautiful applications of Langer's[1] path-integral approach to tunneling processes concerns the rate of phase slips in a thin superconductive wire. An extensive discussion was given many years ago by Langer, Ambegaokar,[2] McCumber and Halperin[3] (LAMH). The resulting electrical resistance explains the experimental data[4] extremely well over a wide range of parameters and deviates only very near the transition temperature (see Fig. 1). The only unesthetical feature of the original result was that a fluctuation determinant had to be evaluated numerically. This was overcome some years ago by Duru, Kleinert, and Ünal (DKÜ)[5] who were able to simplify the path integration and obtained an analytic result. They separated the local complex order parameter along the wire coordinate z, the *order field* $\Delta(z)$, into radial and azimuthal part, $\rho(z)\, e^{i\gamma(z)}$, and integrated out $\gamma(z)$ exactly; the remaining "radial" path integral over $\rho(z)$ allowed for a simple semiclassical treatment.

The purpose of this note is to point out that a factor was overlooked in the DKÜ treatment. Taking this into account, the approximate interpolation formula given by MH can now be compared with the analytic expression and the agreement turns out to be excellent. Since the subject is meanwhile textbook material[6] we think it is worthwhile to record the exact analytic result.

Our starting point is the partition function of a thin wire expressed as the path integral over the order field in the presence of an external source j

$$\mathscr{Z}[j] = \int \rho\mathscr{D}\rho\mathscr{D}\gamma \exp\left\{-\frac{\sigma}{T}\int_{-L/2}^{L/2} dz\left[(\rho_z)^2 - \rho^2 + \frac{1}{2}\rho^4 + \rho^2(\gamma_z)^2 - 2j\gamma_z\right]\right\} \quad (1)$$

where σ is the wire's cross section, L its length measured in units of the coherence length, and T the temperature measured in units of $2f_c/k_B$ (f_c

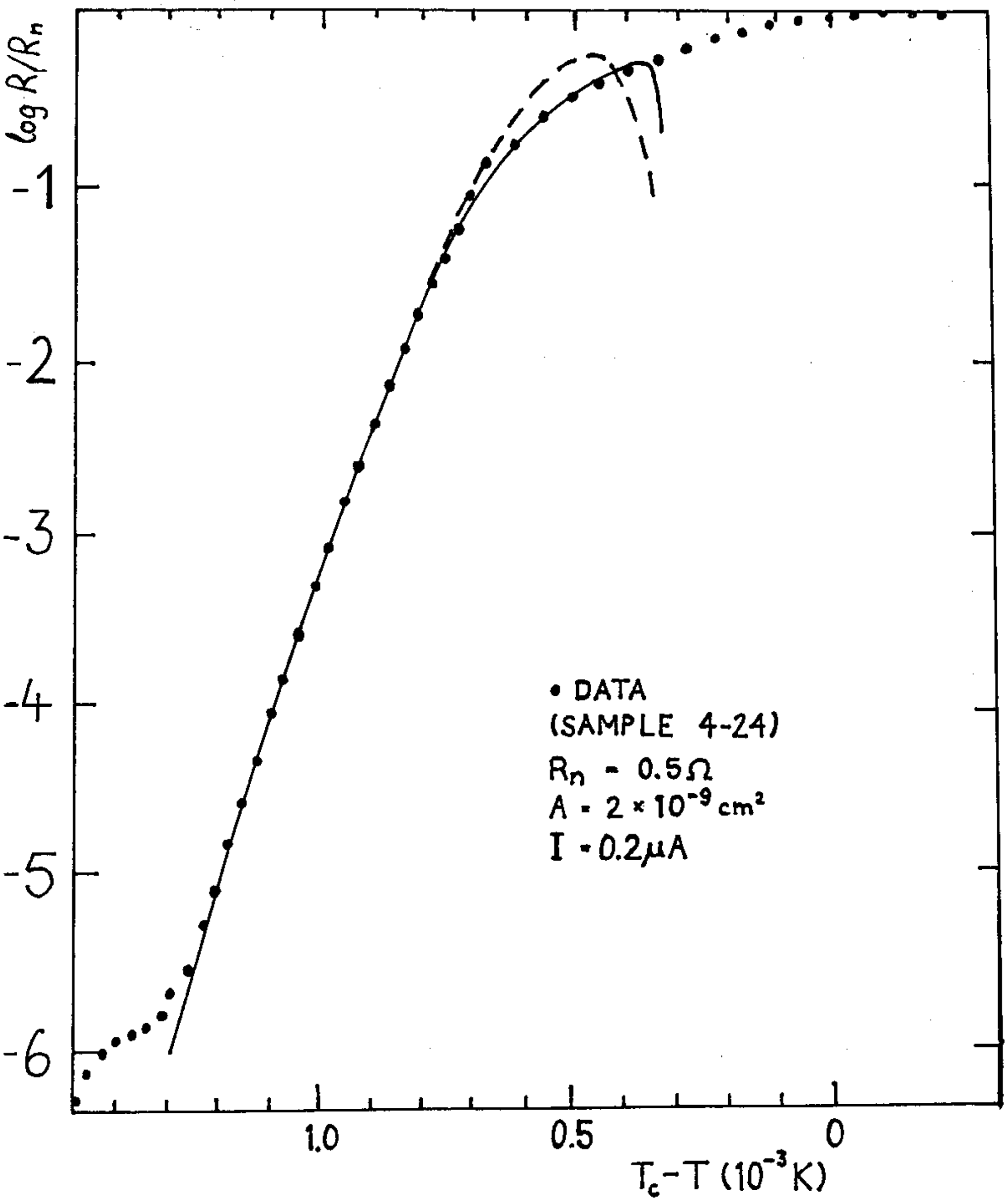

Fig. 1. Resistance of a thin superconducting wire near the transition temperature, experimental data in comparison with LAMH theory. Dashed line is resistance due to phase slips only; for the solid line, a normal background resistance has also been taken into account. The figure is taken from Ref. 4.

being the BCS condensation energy and k_B the Boltzmann constant). The source $2j\gamma_z$ enforces a stationary average current $\langle\rho^2(\partial_z\gamma)\rangle = j \equiv \kappa(1-\kappa^2)$ carried by the wire. We follow DKÜ and integrate out the angular variable γ, reducing the path integral (1) to a one-dimensional problem $\mathscr{Z}[j] = \int \mathscr{D}\rho \exp\{-(\sigma/T)\int dz[(\rho_z)^2 - \rho^2 + \rho^4/2 - j^2/\rho^2]\}$. Although it has recently been understood that this reduced path integral cannot be defined via Feynman's time slicing procedure,[7] there is no problem at the semiclassical level where fluctuations take place near the metastable minimum $\rho_0 \equiv (1-\kappa^2)^{1/2}$. The parameter κ is the field momentum of the order field, which at the metastable minimum reads

$$\Delta_0 \equiv \rho_0 \, e^{i\kappa z} \tag{2}$$

The nontrivial extremal fluctuations are [with $\omega^2 \equiv 2(1-3\kappa^2)$]

$$\Delta_b \equiv \left[1-\kappa^2-\frac{\omega^2/2}{\cosh^2(\omega z/2)}\right]^{1/2} \exp\left[i\left(\kappa z+\arctan\left[\frac{\omega}{2\kappa}\tanh\left(\frac{\omega z}{2}\right)\right]\right)\right]$$
$$\equiv \rho_b(z) e^{i\gamma_b(z)} \tag{3}$$

The subscript b stands for "critical bubble" in the sense of Langer.[1]

To fulfill periodic boundary conditions, we have for the minimum (2) the quantization

$$\kappa \equiv \frac{2\pi}{L} n_w \tag{4}$$

and for the critical bubble (3)

$$\kappa_b + \frac{2}{L}\arctan\left(\frac{\omega_b}{2\kappa}\right) \equiv \frac{2\pi}{L} n_w \tag{5}$$

where the integer n_w is the number of windings.

In the semiclassical evaluation of the reduced path integral, one has to compute the infinite ratio of eigenvalues

$$\prod_n \left(\frac{\lambda_n^0}{\lambda_n^b}\right)\Bigg|_{n_w \text{ fixed}} \tag{6}$$

where the $\lambda_n^{0,b}$ are eigenvalues of the differential operators

$$\mathcal{H}_{0,b} = -\frac{d^2}{dz^2}+\frac{1}{2}\frac{d^2}{d\rho^2}\left[-\rho^2+\frac{1}{2}\rho^4-j^2/\rho^2\right]\Bigg|_{\rho=\rho_{0,b}} \tag{7}$$

Due to (4) and (5), the eigenvalues λ_n^0 and λ_n^b contain, at fixed n_w, different values of κ. To simplify the calculation one may rewrite (6) as

$$\prod_n \left(\frac{\lambda_n^0(\kappa)}{\lambda_n^b(\kappa_b)}\right)\Bigg|_{n_w \text{ fixed}} = \prod_n \left(\frac{\lambda_n^0(\kappa)}{\lambda_n^0(\kappa_b)}\right)\Bigg|_{n_w \text{ fixed}} \prod_n \left(\frac{\lambda_n^0(\kappa_b)}{\lambda_n^b(\kappa_b)}\right) \tag{8}$$

The second product on the right-hand side was evaluated analytically by DKÜ.[5] The first product was overlooked. Evaluating this we find

$$\prod_n \left(\frac{\lambda_n^0(\kappa)}{\lambda_n^b(\kappa_b)}\right)\Bigg|_{n_w \text{ fixed}} = \exp\left[-\frac{12\kappa}{\omega}\arctan\left(\frac{\omega}{2\kappa}\right)\right] \tag{9}$$

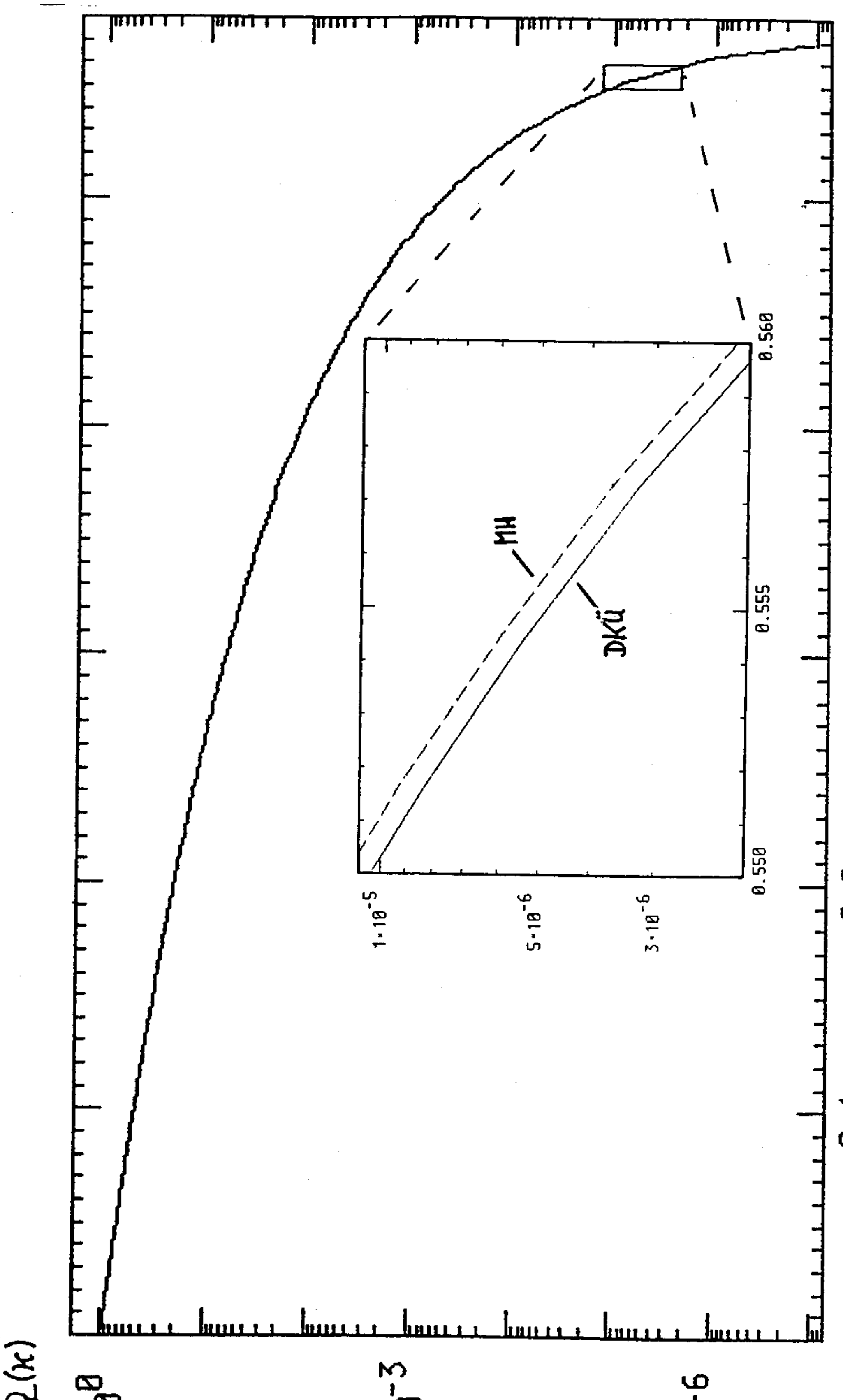

Fig. 2. Comparison of the κ-dependent part of the prefactor (11) and (12). Solid line is the full expression (11); dashed line is MH's approximate formula (12). Only the strongly magnified insert makes the difference visible.

With this additional factor the rate of (current reducing) phase slips from the state (2) reads (compare with Ref. 5, Eq. (59))

$$\text{Rate} = \frac{1}{2\pi\tau} \frac{L}{\sqrt{\pi T/\sigma}} 2^{5/4} \Omega(\kappa)\, e^{-F_b\sigma/T} \tag{10}$$

where

$$\Omega(\kappa) \equiv \frac{(1-3\kappa^2)^{7/4}}{(1-\kappa^2)^{1/2}} \exp\left\{-\frac{3\sqrt{2}\kappa}{(1-3\kappa^2)^{1/2}} \arctan\left[\frac{(1-3\kappa^2)^{1/2}}{\sqrt{2}\kappa}\right]\right\}$$
$$\times |(1+\kappa^2) - [(1+\kappa^2)^2 + 3(1-3\kappa^2)^2]^{1/2}| \tag{11}$$

Here τ is a time scale discussed in Ref. 3 and $F_b = 4\omega/3 - 4j \arctan(\omega/2\kappa)$ is the energy of the critical bubble. For the κ-dependent part MH gave an approximate interpolating formula of their numerical evaluation (see Ref. 3, Eq. (4.36))

$$\Omega_{\text{MH}}(\kappa) = (1-\sqrt{3}\,\kappa)^{15/4}(1-\kappa^2/4) \tag{12}$$

Comparing the two expressions we find that MH's interpolating formula agrees with our analytic expression in (10) to within a few percent, the deviations being largest for $\kappa \to \kappa_c = 1/\sqrt{3}$ (see Fig. 2). Indeed, the first three Taylor coefficients are very close to each other:

$$\Omega(\kappa) = 1 - \frac{3\sqrt{2}}{2}\pi\kappa + \left(\frac{9}{4}\pi^2 - \frac{15}{4}\right)\kappa^2 - \sqrt{2}\left(\frac{9}{8}\pi^3 - \frac{27}{8}\pi\right)\kappa^3 + \cdots$$
$$= 1 - 6.664\kappa + 18.46\kappa^2 - 34.34\kappa^3 + \cdots \tag{13}$$

$$\Omega_{\text{MH}}(\kappa) = 1 - \sqrt{3}\frac{15}{4}\kappa + \frac{503}{32}\kappa^2 - \sqrt{3}\frac{1275}{128}\kappa^3 + \cdots$$
$$= 1 - 6.495\kappa + 15.72\kappa^2 - 17.25\kappa^3 + \cdots \tag{14}$$

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