Summing the spectral representations of Pöschl–Teller and Rosen–Morse fixed-energy amplitudes

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The spectral representations of the fixed-energy amplitude of the symmetric and the general Pöschl–Teller potentials are summed via a Sommerfeld–Watson transformation which leads to a simple closed-form expression. The result is used to write down a similar expression for the symmetric and general Rosen–Morse potentials, exploiting the close correspondence that exists between the two systems within the Schrödinger theory and the path integral formalism via a Duru–Kleinert transformation. From the singularities of the latter amplitude the bound and continuum states of the general Rosen–Morse potential are extracted.

I. INTRODUCTION

In recent years, the introduction of new techniques in path integral theory has led to a renewed interest in the quantum mechanics of potentials of the Pöschl–Teller type,[1]

$$A/\sin^2\theta + B/\cos^2\theta,$$

considered between singularities in θ and for A, $B>0$, and of the closely related Rosen–Morse type:[2]

$$A'/\cosh^2(x/d) - B'\tanh(x/d), \quad x\in]-\infty,\infty[.$$

The path integral of the Pöschl–Teller potential can be solved if one observes that, at least for certain values of A and B, it is formally equivalent to the azimuthal projection of the path integral of a particle moving *near* the surface of a sphere embedded in a higher-dimensional space.[3] For the symmetric Pöschl–Teller potential (where $B'=0$) a three-dimensional embedding space must be used, while the full Pöschl–Teller path integral requires a four-dimensional embedding. The projection procedure circumvents the usual difficulties encountered when one tries to define a path integral with a singular potential energy.[4] The usual naive time slicing, inspired by Feynman's original path integral definition would lead, in these cases, to highly divergent integrals. However, a time slicing of the *complete* three- or four-dimensional Cartesian path integral in the Feynman manner and a subsequent transformation of the slices to polar coordinates allows to regularize the azimuthal path integral. In the time-sliced expression the singular potential is approximated by Bessel functions and the slice integrals over these are well defined. Such a calculation yields, of course, nothing more than the azimuthal Green functions of free particles moving in three or four dimensions, whose spectral representations are well known. In three dimensions, the Green's function is built from the Legendre polynomials $P_l^m(\cos\theta)$ and in four dimensions from the representation functions $d^l_{m_1,m_2}(\theta)$ of the rotation group.

Apart from being an interesting object of study in path integration serving as a testing ground for new evaluation methods, the Pöschl–Teller potential also arises in certain recent applications, e.g., in membrane physics, where the symmetric form with $B=0$ governs the thermodynamic field fluctuations in the functional saddle point approximation.[5,6]

In Schrödinger theory, the Pöschl–Teller system can quite easily be transformed into the Rosen–Morse system. In the path integral formalism, the correspondence can be established after a path-dependent time reparametrization, the Duru–Kleinert transformation which relates the fixed-energy amplitudes[7] of the two systems. Their relation has already been written down and explored before.[8] That earlier work, however, performs several incorrect manipulations so that its final Rosen–Morse amplitude misses out on the continuous states which will be properly obtained here. Another calculation of the Rosen–Morse path integral has been presented in Ref. 9. However, this path-integral analysis seems to us incomplete just as the algebraic one given in Ref. 10. Both articles relate the Rosen–Morse potential to the modified Pöschl–Teller potential,

$$\alpha/\cosh^2(x/d) - \beta/\sinh^2(x/d),$$

and try to exploit the inherent SU(1,1) symmetry of the latter. But neither distinguishes between the eigenvalues below and those above the upper bound B' of the potential. Consequently, both fail to take into account the twofold degeneracy that exists for states with energies larger than B'.

The purpose of this paper is to calculate the fixed-energy amplitude of the symmetric and of the general

Pöschl–Teller potentials in closed form, i.e., in a form that involves no summation. It is then fairly simple to write down the fixed-energy amplitude of the Rosen–Morse potential in closed form and to deduce from it *all* the eigenstates. We shall derive our results in a system of units where the constant d in the Rosen–Morse potential is equal to unity. The parameters A, B, A', B' will be considered as dimensionless energy values. Physical energies are obtained by multiplying these numbers with $\hbar^2/2md^2$.

II. SUMMING THE PÖSCHL–TELLER FIXED-ENERGY AMPLITUDE

A. Symmetric Pöschl–Teller potential

The fixed-energy amplitude of the Pöschl–Teller potential $A/\sin^2\theta$, $\theta\in]0,\pi[$ that is symmetric with respect to the point $\pi/2$ satisfies the following differential equation:

$$\left[-\frac{\hbar^2}{2M}\left(\frac{d^2}{d\theta^2}+\frac{A}{\sin^2\theta}\right)-E\right](\theta|\theta_a)_{A,E}$$

$$=-i\hbar\delta(\theta-\theta_a), \tag{1}$$

with boundary conditions

$$(\theta=0|\theta_a)_{A,E}=(\theta=\pi|\theta_a)_{A,E}=0.$$

This may be compared to the projection of the fixed-energy amplitude

$$(\cos\theta\;|\cos\theta_a)_{m,E}$$

for a free particle in three dimensions into a fixed azimuthal angular momentum m. It is a solution of the equation

$$\left[\frac{\hbar^2}{2M}\left(-\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+\frac{m^2}{\sin^2\theta}\right)-\tilde{E}\right]$$

$$\times(\cos\theta\;|\cos\theta_a)_{m,E}=(-i\hbar/\sin\theta)\delta(\theta-\theta_a). \tag{2}$$

If we renormalize the latter amplitude introducing

$$(\theta|\theta_a)_{m,E}\equiv(\cos\theta\;|\cos\theta_a)_{m,E}\sqrt{(\sin\theta\sin\theta_a)}, \tag{3}$$

we can bring the derivative term of the Schrödinger equation to the usual free-particle form $d^2/d\theta^2$ and obtain

$$\left[\frac{\hbar^2}{2M}\left(-\frac{d^2}{d\theta^2}+\frac{m^2-1/4}{\sin^2\theta}\right)-\left(\tilde{E}+\frac{1}{4}\right)\right]$$

$$\times(\theta|\theta_a)_{m,E}=-i\hbar\delta(\theta-\theta_a), \tag{4}$$

which is Eq. (1), if we identify the parameters

$$A=m^2-\tfrac{1}{4},\quad m \text{ integer},\quad E=\tilde{E}+\tfrac{1}{4}. \tag{5}$$

Due to this relation with the free-particle amplitude, the amplitude for the Pöschl–Teller potential has the obvious spectral representation in terms of associated Legendre polynomials P_l^m:

$$(\theta_b|\theta_a)_{m,E}$$

$$=\sqrt{\sin\theta_b\sin\theta_a}\sum_{l=m}^{\infty}\frac{i\hbar}{E-\hbar^2 l(l+1)/2M}$$

$$\times\left(l+\frac{1}{2}\right)\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta_b)P_l^m(\cos\theta_a). \tag{6}$$

So far, this spectral representation has been derived only for integer-valued m and l.

It is the first expression that we want to sum up explicitly. The standard tool to do this is to find an appropriate analytic continuation in the summation variable l and to perform a Sommerfeld–Watson transformation on the sum. As we shall see this leads to obtain the following closed-form expression valid for $\cos\theta_a > \cos\theta_b$:

$$(\theta_b|\theta_a)_{m,E}$$

$$=\sqrt{\sin\theta_b\sin\theta_a}\frac{iM}{\hbar}\frac{\pi}{\sin\pi(l_E-m)}$$

$$\times\frac{\Gamma(l_E+m+1)}{\Gamma(l_E-m+1)}P_{l_E}^{-m}(\cos\theta_a)P_{l_E}^{-m}(-\cos\theta_b), \tag{7}$$

where

$$l_E=-\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+(2M/\hbar^2)E},\quad m=\sqrt{A+\tfrac{1}{4}},\quad m \text{ integer}. \tag{8}$$

It is easy to remove the restriction of m to integer values by using the general associated Legendre functions, which are defined in terms of hypergeometric functions in the following manner:

$$P_\lambda^\mu(z)=\left(\frac{1+z}{1-z}\right)^{\mu/2}\frac{1}{\Gamma(1-\mu)}$$

$$\times F\left(-\lambda,\lambda+1;1-\mu;\frac{1-z}{2}\right). \tag{9}$$

The definition is valid for all complex values of μ and λ. For integer $\mu=m$, $\lambda=l$ there exists a limiting formula that leads back the familiar "polynomials" $P_l^m(\cos\theta)$ (see Ref. 11, formula 15.1.2). Using the well-known formula

$$\pi/\sin\pi\lambda=\Gamma(\lambda)\Gamma(1-\lambda),$$

we can rewrite the right-hand side of Eq. (7) as

$$\sqrt{\sin\theta_b\sin\theta_a}(-iM/\hbar)\Gamma(m-l_E)\Gamma(l_E+m+1)$$
$$\times P_{l_E}^{-m}(\cos\theta_a)P_{l_E}^{-m}(-\cos\theta_b), \qquad (10)$$

or in a more precise notation, which allows for the condition $\cos\theta_a > \cos\theta_b$:

$$\sqrt{\sin\theta_b\sin\theta_a}(-iM/\hbar)\Gamma(m-l_E)\Gamma(l_E+m+1)$$
$$\times\{\Theta(\theta_b-\theta_a)P_{l_E}^{-m}(\cos\theta_a)P_{l_E}^{-m}(-\cos\theta_b)$$
$$+\Theta(\theta_a-\theta_b)P_{l_E}^{-m}(\cos\theta_b)P_{l_E}^{-m}(-\cos\theta_a)\}. \qquad (11)$$

Here $\Theta(x)$ denotes the Heaviside function, which is zero for $x\leqslant 0$ and one for $x>0$.

Using Eq. (9) one easily convinces oneself that the amplitude (11) solves the differential equation (4) for arbitrary complex values of m. The associated boundary conditions are satisfied if the square root in (8) is chosen positive. This can be seen by inspecting Eq. (9), which shows that

$$P_\lambda^{-\mu}(1)=0, \quad \text{if } \operatorname{Re}\mu>0.$$

Having made this choice we can abandon the restriction to integer m.

Let us now derive Eq. (7) by summing the spectral representation (6). First we rewrite the sum as a contour integral in an appropriate manner. An obvious possibility would be to represent (6) as

$$\frac{1}{2\pi i}\int_{\mathscr{C}} d\lambda\,\frac{\pi}{\sin\pi\lambda}\,\frac{i\hbar}{E-\hbar^2\lambda(\lambda+1)/2M}$$
$$\times\left(\lambda+\frac{1}{2}\right)\frac{\Gamma(\lambda-m+1)}{\Gamma(\lambda+m+1)}P_\lambda^m(-z_b)P_\lambda^m(z_a), \qquad (12)$$

where the integration contour $\mathscr{C}$ encloses counterclockwise the real axis for $x>m-\epsilon$. However, the pole structure of the integrand becomes much more transparent if we use the formula (in Ref. 11 formula 8.2.5)

$$\tilde{P}_\lambda^{-\mu}(z)=\frac{\Gamma(\lambda-\mu+1)}{\Gamma(\lambda+\mu+1)}\left(\tilde{P}_\lambda^\mu(z)\right.$$
$$\left.-\frac{2}{\pi}e^{-i\mu\pi}\sin(\mu\pi)\tilde{Q}_\lambda^\mu(z)\right), \qquad (13)$$

which holds for $\operatorname{Im}z>0$ and expresses the Legendre function with negative index μ, $\tilde{P}_\lambda^{-\mu}$, as a linear combination of $\tilde{P}_\lambda^\mu$ and the Legendre function of the second kind $\tilde{Q}_\lambda^\mu$. The over tildes are there to distinguish our Legendre functions P_λ^μ, Q_λ^μ from those in Ref. 11, the difference being that the latter have a branch cut on the interval $[-1;1]$ and therefore are discontinuous along it, whereas we prefer the Legendre functions to be well-defined real functions on this interval. The relation between the two is given by either of the two limits

$$P_\lambda^\mu(x)=\lim_{\epsilon\to 0}e^{i\pi\mu/2}\tilde{P}_\lambda^\mu(x+i\epsilon) \qquad (14)$$

or

$$P_\lambda^\mu(x)=\lim_{\epsilon\to 0}e^{-i\pi\mu/2}\tilde{P}_\lambda^\mu(x-i\epsilon). \qquad (15)$$

Keeping this in mind we can now write down (13) for integer $\mu=m$, $\lambda=l$:

$$P_l^{-m}(x)=(-1)^m[(l-m)!/(l+m)!]P_l^m(x). \qquad (16)$$

When inserted into the sum (6), this relation gives rise to another analytic continuation in l that can be rewritten à la Sommerfeld–Watson,

$$\frac{1}{2\pi i}\int_{\mathscr{C}} d\lambda\,\frac{\pi}{\sin\pi(\lambda-m)}\,\frac{i\hbar}{E-\hbar^2\lambda(\lambda+1)/2M}$$
$$\times\left(\lambda+\frac{1}{2}\right)\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda-m+1)}P_\lambda^{-m}(-z_b)P_\lambda^{-m}(z_a), \qquad (17)$$

or, more conveniently, as

$$\frac{-1}{2\pi i}\int_{\mathscr{C}} d\lambda\,\frac{\pi}{\sin\pi(\lambda-m)}\,\frac{2iM}{\hbar}\,\frac{\lambda+\frac{1}{2}}{(\lambda-l_E)(\lambda+l_E+1)}$$
$$\times\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda-m+1)}P_\lambda^{-m}(-z_b)P_\lambda^{-m}(z_a). \qquad (18)$$

The pole function $\pi/\sin\pi(\lambda-m)$ has residues with alternating signs. This can be compensated for by using the negative-argument formula

$$\tilde{P}_\lambda^\mu(-z)=e^{-i\nu\pi}\tilde{P}_\lambda^\mu(z)-(2/\pi)\sin(\lambda+\mu)\tilde{Q}_\lambda^\mu(z), \qquad (19)$$

which is valid for $\operatorname{Im}z>0$ and which, for integer λ,μ with $\lambda>0$, reduces to

$$P_l^m(-x)=(-1)^{l-m}P_l^m(x). \qquad (20)$$

The poles of $\pi/\sin\pi l$ at $l=-m,\ldots,m-1$ are canceled by the zeros of $\Gamma(l+m+1)/\Gamma(l-m+1)$, so that the residues at these points are zero. With this integrand, the integration contour can be chosen to enclose the entire real axis. Given that the integrand is an antisymmetric function with respect to $l=-\frac{1}{2}$ [following from Eq. (9), which shows that $P_\lambda^\mu=P_{-\lambda-1}^\mu$], the residues at $-m-1$, $-m-2$, $-m-3,\ldots$, are equal to the residues at $m,m+1,m+2,\ldots$, respectively. The integral, therefore, represents twice the sum in (11). If it can be shown that the integrand vanishes rapidly enough at infinity, integrals over semicircles at infinity will not contribute and the integral will be equal to the sum of all residues off the real axis. This is indeed the case, as will be shown in Appendix A. According to formulas (A16) and (A22), the integrand behaves asymptotically as

$$\mathscr{F}|\lambda|^{-2}(e^{(|\theta_a|+|\pi-\theta_b|)\,\mathrm{Im}\,\lambda|}/e^{|\pi\,\mathrm{Im}\,\lambda|}),\tag{21}$$

where $\mathscr{F}$ is a constant depending on α, β, θ, $\arg\lambda$. For $\theta_b>\theta_a$, this expression goes faster to zero than $|\lambda|^{-1}$, which is enough to make semicircle integrals at $|\lambda|=\infty$ vanish.

B. General Pöschl–Teller potential

The differential equation for the fixed-energy amplitude $(\theta|\theta_a)_{A,B,E}$, of this potential reads

$$\frac{\hbar^2}{2M}\left[-\frac{d^2}{d\theta^2}+\frac{A}{\sin^2\theta}+\frac{B}{\cos^2\theta}-\frac{2M}{\hbar^2}E\right](\theta|\theta_a)_{A,B,E}$$
$$=-i\hbar\delta(\theta-\theta_a),\tag{22}$$

with $\theta\in]0,\pi/2[$. We shall assume that $A\neq B$ since, otherwise, the potential would be symmetric with respect to $\theta=\pi/4$ and the transformation $\theta\to\theta/2$ would bring us back to the case treated in the previous section. Let the motion of a free particle in four dimensions be parametrized by the radial distance r and the Euler angles θ, ϕ, and ψ. Then, its azimuthal fixed-energy amplitude with $r=1$ and at fixed quantum numbers m_1,m_2—the latter are associated with the angles ϕ, ψ—satisfies the Schrödinger equation:

$$\left\{\frac{\hbar^2}{2M}\left[-\frac{4}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+\frac{3}{4}\right.\right.$$
$$\left.\left.+\frac{4(m_1^2+m_2^2)-8m_1m_2\cos\theta}{\sin^2\theta}\right]-E\right\}$$
$$\times(\cos\theta\,|\cos\theta_a)_{m_1,m_2,E}=-i\hbar\delta\,(\cos\theta-\cos\theta_a),\tag{23}$$

and $(\cos\theta\,|\cos\theta_a)_{m_1,m_2,E}$ vanishes at the boundaries $\theta=0$ and $\theta=\pi/2$. A similar renormalization as for the symmetric case, namely,

$$(\theta|\theta_a)_{m_1,m_2,E}\equiv 2\,(\cos 2\theta\,|\cos 2\theta_a)_{m_1,m_2,E}$$
$$\times\sqrt{(\sin 2\theta\sin 2\theta_a)},\tag{24}$$

transforms this to the equation

$$\frac{\hbar^2}{2M}\left[-\frac{d^2}{d\theta^2}+\frac{(m_1-m_2)^2-\frac14}{\sin^2\theta}\right.$$
$$\left.+\frac{(m_1+m_2)^2-\frac14}{\cos^2\theta}-\frac{2ME}{\hbar^2}-\frac14\right]$$
$$\times(\theta|\theta_a)_{m_1,m_2,E}=-i\hbar\delta(\theta-\theta_a),\tag{25}$$

which is just Eq. (22) for the special cases where $A=(m_1-m_2)^2-\frac14$ and $B=(m_1+m_2)^2-\frac14$ with integer m_1,m_2. For these A, B, the solution for (22) can immediately be deduced from the solution of Eq. (23), which is well known to be

$$(\cos\theta_b\,|\cos\theta_a)_{m_1,m_2,E}$$
$$=\sum_{L=M}^{\infty}\frac{i\hbar}{E-\hbar^2[(2L+1)^2-\frac14]/2M}\frac{2L+1}{2}$$
$$\times d^L_{m_1,m_2}(\theta_b)d^L_{m_1,m_2}(\theta_a),\tag{26}$$

where the $d^L_{m_1,m_2}$ are the representation functions of the rotation group. The variables $M=\max\{m_1,m_2\}$ and m_1,m_2,L are either *all* integer or *all* half-integer. Owing to Eq. (IV.2.1) in Ref. 12,

$$d^L_{m_1,m_2}(\theta)=(-1)^{m_1-m_2}d^L_{m_2,m_1}(\theta),\tag{27}$$

the product $d^L_{m_1,m_2}(\theta_b)d^L_{m_1,m_2}(\theta_a)$ is symmetric with respect to interchange of m_1 and m_2. Thus, we can suppose without any loss of generality that $m_1\geqslant m_2$ and omit the absolute values in Eq. (IV.2.6) of Ref. 12, which then becomes

$$d^L_{m_1,m_2}(\theta)=\sqrt{\frac{\Gamma(L+m_1+1)\Gamma(L-m_2+1)}{\Gamma(L-m_1+1)\Gamma(L+m_2+1)}}\frac{1}{\Gamma(m_1-m_2+1)}\left(\frac{1-\cos\theta}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\cos\theta}{2}\right)^{(m_1+m_2)/2}$$
$$\times F\left(-L+m_1,L+m_1+1;m_2+1;\frac{1-\cos\theta}{2}\right).\tag{28}$$

This equation entails the relation

$$d_{m_1,m_2}^{-L-1}(\theta)=d_{m_1,m_2}^{L}(\theta), \tag{29}$$

which holds even when d is analytically continued to complex L, as long as m_1,m_2 are integers or half-integers. It follows that the expression summed over in (26) is antisymmetric with respect to $L = -\frac{1}{2}$. Furthermore, the d function satisfies the following symmetry property in the independent variable:[12]

$$d_{m_1,m_2}^{L}(\theta)=(-1)^{L-m_1}d_{m_1,-m_2}^{L}(\theta-\pi). \tag{30}$$

Using this we can represent the sum again as a contour integral,

$$\frac{1}{2\pi i}\int_{\mathscr{C}} dL\,\frac{\pi}{\sin\pi(L-m_1)}\,\frac{i\hbar}{E-\hbar^2[(2L+1)^2-\frac{1}{4}]/2M}$$

$$\times[(2L+1)/2]d_{m_1,-m_2}^{L}(\theta_b-\pi)d_{m_1,m_2}^{L}(\theta_a), \tag{31}$$

and rewrite this as

$$\frac{-1}{2\pi i}\int_{\mathscr{C}} dL\,\frac{\pi}{\sin\pi(L-m_1)}\,\frac{iM}{4\hbar}\,\frac{2L+1}{(L-L_E)(L+L_E+1)}$$

$$\times d_{m_1,-m_2}^{L}(\theta_b-\pi)d_{m_1,m_2}^{L}(\theta_a), \tag{32}$$

where $\mathscr{C}$ encircles the entire real axis in a counterclockwise direction and

$$L_E=-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{2M}{\hbar^2}E+\frac{1}{4}}. \tag{33}$$

Just as in the symmetric case, the real poles of the integrand are situated at $l=\ldots,\ -m_1-3,\ -m_1-2,\ -m_1-1$ and $l=m_1,\ m_1+1,\ m_1+2,\ldots,$ the poles of $\pi/\sin\pi l$ for $-m_1-1<l<m_1$ are again canceled by the Γ function in the denominator and the sum over residues in the left half-plane is equal to that in the right one. According to Appendix A, formulas (A16) and (A22), the integrand behaves asymptotically as

$$\mathscr{F}|L|^{-2}e^{(|\theta_a|+|\pi-\theta_b|)\,|\mathrm{Im}\,L|}/e^{|\pi\,\mathrm{Im}\,L|}, \tag{34}$$

i.e., vanishes faster than $|\lambda|^{-1}$ as long as $\cos\theta_b < \cos\theta_a$. Thus, as in the symmetric case, the result is equal to the sum of nonreal residues and, due to Eq. (28), can be written as

$$(\cos\theta_b\,|\cos\theta_a)_{m_1,m_2,E}=\frac{iM\pi/(4\hbar)}{\sin\pi(L_E-m_1)}\,d_{m_1,-m_2}^{L_E}(\theta_b-\pi)d_{m_1,m_2}^{L_E}(\theta_a)$$

$$=-\frac{iM}{4\hbar}\,\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\left(\frac{1-\cos\theta_a}{2}\right)^{(m_1-m_2)/2}$$

$$\times\left(\frac{1+\cos\theta_a}{2}\right)^{(m_1+m_2)/2}\left(\frac{1+\cos\theta_b}{2}\right)^{(m_1+m_2)/2}\left(\frac{1-\cos\theta_b}{2}\right)^{(m_1-m_2)/2}$$

$$\times F\left(-L_E+m_1,L_E+m_1+1;m_1-m_2+1;\frac{1-\cos\theta_a}{2}\right)$$

$$\times F\left(-L_E+m_1,L_E+m_1+1;m_1+m_2+1;\frac{1+\cos\theta_b}{2}\right),\quad \text{for } \theta_b>\theta_a, \tag{35}$$

where

$$m_1=\tfrac{1}{2}\sqrt{B+\tfrac{1}{4}}+\tfrac{1}{2}\sqrt{A+\tfrac{1}{4}},$$
$$m_2=\tfrac{1}{2}\sqrt{B+\tfrac{1}{4}}-\tfrac{1}{2}\sqrt{A+\tfrac{1}{4}}. \tag{36}$$

In analogy with the symmetric Pöschl–Teller potential, this function satisfies (23) and the associated boundary conditions for arbitrary A,B if the square roots in the equations for m_1 and m_2 are chosen positive. Therefore, the fixed-energy amplitude for the general Pöschl–Teller potential with general A,B is now known from the transformation (24) of (35).

III. FIXED-ENERGY AMPLITUDES OF SYMMETRIC AND GENERAL ROSEN–MORSE POTENTIAL

A. Relation between Pöschl–Teller and Rosen–Morse potentials

We shall first show how the two types of amplitudes can be related in Schrödinger theory. For completeness, we also sketch the corresponding Duru–Kleinert path integral transformations to achieve the same goal.

1. Schrödinger theory

(i) *Symmetric potentials.* The azimuthal fixed-energy amplitude in three dimensions satisfies the differential equation (2) for which we have found the solution

$$(\cos\theta_b|\cos\theta_a)_{m,E}=(-iM/\hbar)\Gamma(m-l_E)\Gamma(l_E+m+1)$$
$$\times[\Theta(\theta_b-\theta_a)P_{l_E}^{-m}(\cos\theta_a)P_{l_E}^{-m}$$
$$\times(-\cos\theta_b)+\{\theta_a\leftrightarrow\theta_b\}]. \tag{37}$$

The transformation $\theta\to x$ with $\cos\theta=\tanh x$ brings Eq. (2) with (37) inserted to the form:

$$-\frac{\hbar^2}{2M}\left[\frac{d^2}{dx^2}+\frac{l(l+1)}{\cosh^2 x}-m^2\right](\cos\theta(x_b)\ |\cos\theta(x_a))_{m,E}$$
$$=-\hbar i\delta(x-x_a). \tag{38}$$

This is to be compared with the equation for the fixed-energy amplitude of the symmetric Rosen–Morse potential, which satisfies

$$\left[\frac{\hbar^2}{2M}\left(-\frac{d^2}{dx^2}+\frac{A'}{\cosh^2 x}\right)-E\right]G(x,x_a,E)$$
$$=-\hbar i\delta(x-x_a). \tag{39}$$

Obviously, the solution is obtained from the previous equation by making the substitutions:

$$m=m(E)\equiv\sqrt{-2ME/\hbar^2},\ l=-\tfrac{1}{2}\pm\sqrt{\tfrac{1}{4}-A'}, \tag{40}$$

so that it reads

$$(-iM/\hbar)\Gamma(m(E)-l)\Gamma(l+m(E)+1)$$
$$\times[\Theta(x-x_a)P_l^{-m(E)}(\tanh x)$$
$$\times P_l^{-m(E)}(-\tanh x_a)+\{x\leftrightarrow x_a\}]. \tag{41}$$

Note that the energy dependence of the amplitude has been moved from the lower to the upper index of the Legendre functions. Similarly, as in the case of the symmetric Pöschl–Teller potential we have to choose $\mathrm{Re}\, m(E)>0$ to satisfy the boundary conditions that require the amplitude $G(x,x_a,E)$ to vanish at $x=\pm\infty$. In the second equation of (40) the sign of the square root can be chosen at will since $l_+=-l_--1$ and the fixed-energy amplitude is invariant with respect to the replacement $l\to-l-1$. In the following section we shall demonstrate how this affects the eigenstates of the system. Contrary to the azimuthal problem, the symmetric Rosen–Morse potential admits free states since the potential is bounded from above, and we must be able to recover them from the above expression.

(ii) *General potentials.* Undergoing a transformation of variables $\theta\to x$ with

$$\cos\theta=\tanh x,$$

Eq. (23) becomes

$$\frac{\hbar^2}{2M}\left\{-4\frac{d^2}{dx^2}+4(m_1^2+m_2^2)\right.$$
$$\left.-\frac{8m_1m_2\sinh x}{\cosh x}-\frac{\epsilon-\frac{3}{4}}{\cosh^2 X}\right\}(x|x_a)_{m_1,m_2,E}$$
$$=-i\hbar\delta(x-x_a), \tag{42}$$

with $\epsilon=2ME/\hbar^2$. This may be compared with the equation for the fixed-energy amplitude of the Rosen–Morse potential:

$$\left\{\frac{\hbar^2}{2M}\left[-\frac{d^2}{dx^2}-B'\tanh x+A'\operatorname{sech}^2 x\right]-\widetilde{E}\right\}$$
$$\times G(x,x',E)=-i\hbar\delta(x-x'). \tag{43}$$

We are thus led to the expression

$$G(x_a,x_b,E)=(iM\pi/\hbar)/\sin\pi(L_{A'}-m_1)$$
$$\times d^{L_{A'}}_{-m_1,m_2}(\arccos(\tanh x_b)-\pi)$$
$$\times d^{L_{A'}}_{-m_1,-m_2}(\arccos(\tanh x_a)), \tag{44}$$

with

$$m_1=\tfrac{1}{2}\sqrt{B'-2M\widetilde{E}/\hbar^2}+\tfrac{1}{2}\sqrt{-B'-2M\widetilde{E}/\hbar^2},$$
$$m_2=\tfrac{1}{2}\sqrt{B'-2M\widetilde{E}/\hbar^2}-\tfrac{1}{2}\sqrt{-B'-2M\widetilde{E}/\hbar^2},$$
$$L_{A'}=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-4A'}. \tag{45}$$

In terms of hypergeometric functions, the solution can also be written as

$$G(x_a,x_b,E)$$

$$= -\frac{iM}{\hbar}\frac{\Gamma(m_1-L_{A'})\Gamma(L_{A'}+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\left(\frac{1-\tanh x_a}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\tanh x_a}{2}\right)^{(m_1+m_2)/2}$$

$$\times\left(\frac{1+\tanh x_b}{2}\right)^{(m_1+m_2)/2}\left(\frac{1-\tanh x_b}{2}\right)^{(m_1-m_2)/2}F\left(-L_{A'}+m_1,L_{A'}+m_1+1;m_1-m_2+1;\frac{1-\tanh x_a}{2}\right)$$

$$\times F\left(-L_{A'}+m_1,L_{A'}+m_1+1;m_1+m_2+1;\frac{1+\tanh x_b}{2}\right). \tag{46}$$

Since Eq. (23) is valid for arbitrary m_1,m_2 the expression (44) is the desired Green's function if the signs of the two square roots containing E in (45) are chosen positive. Then it vanishes at infinity and thus respects the proper boundary conditions, as can be seen from the definition of the d functions in terms of hypergeometric series in (28).

2. The Duru–Kleinert transformation

For completeness, we outline here how the Duru–Kleinert path integral transformation arrives at the same result. It proceeds in two steps:[13]

a. The f transformation. Singularities in the potential are removed by extending the resolvent operator with an appropriately chosen regulating function $f(x)>0$ using the obvious identity:

$$\frac{1}{\hat{H}-E}=f(x)^{1-\lambda}\frac{1}{f(x)^{\lambda}(\hat{H}-E)f(x)^{1-\lambda}}f(x)^{\lambda}. \tag{47}$$

More general regulating functions that depend on the momentum p as well as on the spatial variable may be admitted, but need not be considered in our application. The operator $1/f(x)^{\lambda}(\hat{H}-E)f(x)^{1-\lambda}$ is now rewritten as

$$\frac{1}{f(x)^{\lambda}(\hat{H}-E)f(x)^{1-\lambda}}$$

$$=\int_0^\infty ds\exp[isf(x)^{\lambda}(\hat{H}-E)f(x)^{1-\lambda}]. \tag{48}$$

The amplitude

$$\langle x_a|\exp[isf(x)^{\lambda}(\hat{H}-E)f(x)^{1-\lambda}]|x_b\rangle$$

$$=\langle x_a|\exp[is\hat{H}_f]|x_b\rangle$$

can now, by time slicing the parameter s, be represented as a path integral in the same manner as the time displacement operator in the familiar form of the theory. This regularization procedure amounts to slicing the time axis in nonequidistant steps, the length of the intervals being proportional to $f(x)$.

b. The h transformation. When going over to $\hat{H}_f$, the kinetic energy term has no longer the standard form, but reads

$$f(x)^{\lambda}\frac{d^2}{dx^2}f(x)^{1-\lambda}.$$

To restore the standard form of a pure second derivative one performs a variable transformation to new coordinates $x=h(q)$ with some suitable function $h(q)$, which satisfies $h'(q)^2=f(x)$. Then

$$\frac{d}{dx}\to\frac{1}{h'(q)}\frac{d}{dq},\quad dx\to h'(q)dq,$$

$$f(x)^{\lambda}\frac{d^2}{dx^2}f(x)^{1-\lambda}\to\frac{d^2}{dq^2}+\cdots. \tag{49}$$

The dots denote new terms collected in an effective potential energy $V_{\rm eff}(q)$ arising from the variable transformation which has to be added to the classical action of the transformed system. It can be explicitly calculated as

$$V_{\rm eff}(q)=-\frac{\hbar^2}{M}\left[\frac{1}{4}\frac{h'''}{h'}-\frac{3}{8}\left(\frac{h''}{h'}\right)^2\right]. \tag{50}$$

Let us apply this method to our systems.

(i) *Symmetric potentials.* The fixed-energy amplitude of a particle moving in a symmetric Pöschl–Teller potential reads in configuration space:

$$\int_0^\infty dt\int\mathscr{D}\theta\exp\left(\frac{i}{\hbar}\mathscr{A}[\theta]\right), \tag{51}$$

with the classical action

$$\mathscr{A}[\theta(\tau)]=\int_0^t d\tau\left[\frac{M}{2}\left(\frac{d}{d\tau}\theta\right)^2+\frac{\hbar^2}{8M}\right.$$
$$\left.-\frac{\hbar^2}{2M}\frac{m^2-\frac{1}{4}}{\sin^2\theta}+E\right]. \qquad (52)$$

Using the regulating function

$$f(\theta)=\sin^2\theta,$$

with $\lambda=0$, we can rewrite this as the f-transformed action

$$\mathscr{A}_f[\theta]=\int_0^s d\sigma\left[\frac{M}{2\sin^2\theta}\left(\frac{d}{d\sigma}\theta\right)^2+\frac{\hbar^2}{8M}\sin^2\theta\right.$$
$$\left.-\frac{\hbar^2}{2M}\left(m^2-\frac{1}{4}\right)+E\sin^2\theta\right]. \qquad (53)$$

The kinetic energy is brought back to conventional form by the variable substitution $\theta\to x$ with

$$\sin\theta=1/\cosh x, \qquad (54)$$

mapping the interval $]0;\pi[$ into $]-\infty,\infty[$. The effective potential is then

$$V_{\text{eff}}(x)=(\hbar^2/8M)(1+1/\cosh^2 x), \qquad (55)$$

and the Duru–Kleinert transformed action becomes

$$\mathscr{A}_{\text{DK}}[x]=\int_0^s d\sigma\left[\frac{M}{2}\left(\frac{d}{d\sigma}x\right)^2-\frac{\hbar^2m^2}{8M}+E\frac{1}{\cosh^2 x}\right]. \qquad (56)$$

(ii) *General potentials.* The fixed-energy amplitude $(\cos\theta\,|\cos\theta_a)$ from Eq. (23) can be represented as the following path integral:

$$(\cos\theta\,|\cos\theta_a)$$
$$=\frac{1}{\sqrt{\sin\theta\sin\theta_a}}\int_0^\infty dt\int\mathscr{D}\theta\exp\left(\frac{i}{\hbar}\mathscr{A}[\theta]\right), \qquad (57)$$

where the action is now given by

$$\mathscr{A}[\theta(\tau)]=\int_0^t d\tau\left[\frac{\mu}{2}\left(\frac{d}{d\tau}\theta\right)^2+\frac{\hbar^2}{32\mu}\right.$$
$$\left.-\frac{\hbar^2}{2\mu}\frac{m_1^2+m_2^2-2m_1m_2\cos\theta}{\sin^2\theta}+E\right], \qquad (58)$$

with the auxiliary mass $\mu=M/4$. Exactly the same Duru–Kleinert transformation as in the symmetric case leads to the transformed action:

$$\mathscr{A}_{\text{DK}}[x]=\int_0^s d\sigma\left[\frac{\mu}{2}\left(\frac{d}{d\sigma}x\right)^2\right.$$
$$-\frac{\hbar^2}{8\mu}(m_1^2+2m_1m_2\tanh x)$$
$$\left.+\left(E-\frac{3\hbar^2}{32\mu}\right)\frac{1}{\cosh^2 x}\right]. \qquad (59)$$

IV. EXTRACTION OF EIGENSTATES

The method we use for extracting the eigenstates from a fixed-energy amplitude is well known. It is described in detail in the textbook Ref. 13, Chap. 9 and in Ref. 16. The bound-state eigenvalues are given by the poles of the fixed-energy amplitude $G(E)$, the residues at these points give, up to $i\hbar$, a (tensorial) product of the bound states themselves. The continuous part of the spectrum corresponds to a branch cut of $G(E)$, a tensor product of free states can be extracted by calculating the discontinuity of $G(E)$ across the branch cut, which must be chosen to coincide with an interval $[a;\infty]$ on the real axis. For simplicity, we shall refer to this free-state discontinuity also as "residue." When citing fixed-energy amplitudes in this section we shall omit all Θ functions and inequalities specifying the order of their arguments, since, in the residues, the order becomes irrelevant. The free states will be normalized such that

$$\langle\Psi_E^{(j)}|\Psi_{E'}^{(j')}\rangle=\delta_{j,j'}\delta(E-E'),$$

where j is some degeneracy index that may be necessary, as will be seen below.

A. Eigenstates of symmetric Rosen–Morse potential

1. Calculation of bound states

The bound states are calculated from the residues of the fixed-energy amplitude $G(E)$ in Eq. (41). Its poles E_n are all contained in the first Γ function of the normalization constant. They can be calculated from the equation

$$\sqrt{-(2M/\hbar^2)E_n}-l=-n, \quad n\in N\cup\{0\}. \qquad (60)$$

The fixed-energy amplitude has the correct boundary values only if we choose $\text{Re}\sqrt{-E}>0$. From this it follows that bound states only exist if $l>0$, i.e., $A'<0$ and that their quantum numbers are subject to the condition

$$0 \leqslant n \leqslant l = -\tfrac{1}{2} + \sqrt{1-4A'}/2.$$

For these n we have

$$E_n = -(\hbar^2/2M)(\sqrt{1-4A'}/2 - n - \tfrac{1}{2})^2. \tag{61}$$

This result agrees with the one found in Ref. 14. Using the formula

$$\lim_{E\to E_n} (E-E_n)\Gamma[f(E)] = [1/f'(E_n)][(-1)^n/n!],$$

which is valid for $f(E_n) = -n$ and can be proved with de l'Hôpital's rule and using (19), we obtain for the residues of the fixed-energy amplitude:

$$\lim_{E\to E_n} (E-E_n)G(\zeta,\zeta',E)$$
$$= i\hbar\sqrt{-E_n}(1/n!)\Gamma(l+\sqrt{-2ME_n/\hbar^2}+1)$$
$$\times P_l^{n-l}(\zeta)P_l^{n-l}(\zeta'). \tag{62}$$

The correctly normalized wave functions corresponding to the bound states of the system are thus given by

$$\psi_n(x) = \sqrt{l-n}\sqrt{\Gamma(2l-n+1)/n}P_l^{n-l}(\tanh x). \tag{63}$$

Comparing this with the formula (see Ref. 15, Sec. 7.122.2):

$$\int_0^1 dz \frac{[P_\nu^\mu(z)]^2}{1-z^2} = -\frac{\Gamma(1+\mu+\nu)}{2\mu\Gamma(1-\mu+\nu)},$$

which is valid for $\mathrm{Re}\,\mu<0$ and positive integer $\nu+\mu$ – and taking into account that, due to (19), the integrand is a symmetric function—confirms our result for the normalization constant.

2. Calculation of free states

Crossing the branch cut on the positive real axis changes μ into $-\mu$. The potential being symmetric with respect to $x=0$ we expect the eigenvalues in the continous spectrum to be doubly degenerate, the two linearly independent eigenstates corresponding to waves that are propagating to the left and to the right. We shall use the abbreviation

$$y := \tanh x.$$

If we succeed in writing the discontinuity across the cut in the form

$$\frac{1}{2\pi\hbar}[G(y_a,y_b,\mu) - G(y_a,y_b,-\mu)]$$
$$= \sum_{j=1}^{2} \Psi_E^{(j)}(y_a)\Psi_E^{(j)}(y_b), \tag{64}$$

where j distinguishes the degenerate states, we will be able to read off directly a set of orthonormalized free states for each $E>0$. The discontinuity of our problem is

$$2\pi\hbar \sum_j \Psi_E^{(j)}(y_a)\Psi_E^{(j)}(y_b)$$
$$= (-iM/\hbar)\Gamma(\mu-\lambda)\Gamma(\lambda+\mu+1)P_\lambda^{-\mu}(y_a)$$
$$\times P_\lambda^{-\mu}(-y_b) + (iM/\hbar)\Gamma(-\mu-\lambda)$$
$$\times \Gamma(\lambda-\mu+1)P_\lambda^{\mu}(y_a)P_\lambda^{\mu}(-y_b). \tag{65}$$

We now define the abbreviations

$$\mathscr{K} := (-iM/\hbar)\Gamma(\mu-\lambda)\Gamma(\lambda+\mu+1),$$
$$\psi_1 := P_\lambda^\mu(y_a), \quad \psi_2 := P_\lambda^\mu(-y_b). \tag{66}$$

Given that μ is purely imaginary when E belongs to the continous spectrum and all other parameters are real, we can conclude that

$$\begin{aligned} \psi_1^* &:= P_\lambda^{-\mu}(y_a), \\ \psi_2^* &:= P_\lambda^{-\mu}(-y_b). \end{aligned} \tag{67}$$

The residue can therefore be written as

$$[G(y_a,y_b,\mu) - G(y_a,y_b,-\mu)]$$
$$= 2\pi\hbar \sum_{j=1}^{2} \Psi_E^{(j)}(y_a)\Psi_E^{(j)}(y_b)$$
$$= \mathscr{K}\psi_1^*(y_a)\psi_2^*(y_b) + \mathscr{K}^*\psi_1(y_a)\psi_2(y_b). \tag{68}$$

Using formulas (13) and (19), together with (14) and (15), we see that

$$\begin{aligned} \psi_2^* &= \mathscr{A}\psi_2 + \mathscr{B}\psi_1, \\ \psi_1 &= \mathscr{A}^*\psi_1^* + \mathscr{B}^*\psi_2^*, \end{aligned} \tag{69}$$

with

$$\mathscr{A} = \frac{\Gamma(\lambda-\mu+1)}{\Gamma(\lambda+\mu+1)} \frac{\sin \pi\lambda}{\sin \pi(\mu+\lambda)},$$

$$\mathscr{B} = \frac{\Gamma(\lambda-\mu+1)}{\Gamma(\lambda+\mu+1)} \frac{\sin \pi\mu}{\sin \pi(\mu+\lambda)}. \tag{70}$$

The residue (68) then becomes

$$2\pi\hbar \sum_{j=1}^{2} \Psi_E^{(j)}(y_a)\Psi_E^{(j)}(y_b)$$
$$=(\mathscr{K}\mathscr{A}+\mathscr{K}^*\mathscr{A}^*)\psi_1^*(y_a)\psi_2(y_b)$$
$$+\mathscr{K}\mathscr{B}\psi_1^*(y_a)\psi_1(y_b)$$
$$+\mathscr{K}^*\mathscr{B}^*\psi_2^*(y_a)\psi_2(y_b). \quad (71)$$

Now the fact that μ is the only nonreal parameter and purely imaginary, at that, entails

$$\mathscr{K}\mathscr{A}=\frac{-iM}{\hbar}\Gamma(\mu-\lambda)\Gamma(\lambda-\mu+1)\frac{\sin\pi\lambda}{\sin\pi(\mu+\lambda)}$$
$$=\frac{-iM}{\hbar}\frac{\pi}{\sin\pi(\mu-\lambda)}\frac{\sin\pi\lambda}{\sin\pi(\mu+\lambda)}$$
$$=\frac{iM}{\hbar}\frac{\pi\sin\pi\lambda}{|\sin\pi(\mu+\lambda)|^2}, \quad (72)$$

so that $\mathscr{K}\mathscr{A}$ is purely imaginary, too, and the first term drops out whereas $\mathscr{K}\mathscr{B}$ is real and can be calculated to equal

$$\mathscr{K}\mathscr{B}=\frac{-iM}{\hbar}\Gamma(\mu-\lambda)\Gamma(\lambda-\mu+1)\frac{\sin\pi\mu}{\sin\pi(\mu+\lambda)}$$
$$=\frac{iM}{\hbar}\frac{\pi\sin(-i\pi|\mu|)}{|\sin\pi(\mu+\lambda)|^2}$$
$$=\frac{M\pi}{\hbar}\frac{\sinh\pi|\mu|}{|\sin\pi(\mu+\lambda)|^2}, \quad (73)$$

where $\mu=-i|\mu|$, because Eq. (64) is only valid if values of the Green's function *below* the branch cut are subtracted from values *above* it. Now, we had agreed to take the real part of $\mu=\sqrt{-E}$ to be larger than zero. From this we deduce, with $E=re^{i\varphi}$, for the imaginary part above the cut, i.e., for $\varphi\in]0,\pi[$:

$$(1/\sqrt{r})\,\mathrm{Im}\sqrt{-E}=\mathrm{Im}\, e^{i(-\pi/2+\varphi/2)}$$
$$=\mathrm{Im}[\sin(\varphi/2)-i\cos(\varphi/2)]$$
$$=-\cos(\varphi/2)<0. \quad (74)$$

A pair of orthonormalized free states with energy $E=\mu^2\hbar^2/2M$ is finally given by

$$\Psi_E^{(+)}(x):=\sqrt{\frac{M}{2\hbar^2}}\frac{\sqrt{\sinh\pi|\mu|}}{|\sin\pi(\mu+\lambda)|}P_\lambda^\mu(\tanh x),$$
$$\Psi_E^{(-)}(x):=\sqrt{\frac{M}{2\hbar^2}}\frac{\sqrt{\sinh\pi|\mu|}}{|\sin\pi(\mu+\lambda)|}P_\lambda^\mu(-\tanh x). \quad (75)$$

B. Eigenstates of general Rosen–Morse potential

1. Calculation of bound states

Using (46) and the abbreviations

$$z=(1+\tanh x)/2,\ r_1=\tfrac{1}{2}\sqrt{B'-2M\tilde{E}/\hbar^2},$$
$$r_2=\tfrac{1}{2}\sqrt{-B'-2M\tilde{E}/\hbar^2},\ L=L_{A'},\ E=2M\tilde{E}/\hbar^2, \quad (76)$$

where we shall assume that $\mathrm{Re}\, r_1$, $\mathrm{Re}\, r_2>0$. We can write $G(x_a,x_b,E)$ as

$$\frac{-iM}{\hbar}\frac{\Gamma(L+r_1+r_2+1)\Gamma(r_1+r_2-L)}{\Gamma(2r_1+1)\Gamma(2r_2+1)}$$
$$\times z_a^{r_2}(1-z_a)^{r_1}\times z_b^{r_2}(1-z_b)^{r_1}$$
$$\times F(-L+r_1+r_2,L+r_1+r_2+1;2r_2+1;z_a)$$
$$\times F(-L+r_1+r_2,L+r_1+r_2+1;2r_1+1;1-z_b). \quad (77)$$

The Γ functions in the denominator of this expression make sure its hypergeometric part is well defined for any values of r_1,r_2. Its poles in the energy-variable E are all caused by the Γ functions in the numerator. Therefore the energy eigenvalues of the problem can be calculated from the conditions

$$r_1(E)+r_2(E)-L=-n,\quad n\in N\cup\{0\} \quad (78)$$

or

$$r_1(E)+r_2(E)+1+L=-n,\quad n\in N\cup\{0\}. \quad (79)$$

Because of our convention $\mathrm{Re}\, r_1$, $\mathrm{Re}\, r_2>0$, only the first equation will yield physical energy levels that are, moreover, subject to the additional constraint

$$L=\tfrac{1}{2}(\sqrt{1-4A'}-1)>\sqrt{2|B'|}/2=\{r_1+r_2\}_{\min},$$

where "min" denotes the smallest real value the function $r_1(E)+r_2(E)$ can take. Incidentally, this implies, of course, that A' has to be smaller than zero. If this condition is satisfied, then it follows

$$\tilde{E}_n=-\frac{\hbar^2}{2M}\frac{B'^2+4(L-n)^4}{4(L-n)^2}$$
$$=-\frac{\hbar^2}{2M}\frac{4B'^2+(2n+1-\sqrt{1-4A'})^4}{4(2n+1-\sqrt{1-4A'})^2}, \quad (80)$$

defined for

$$0\leqslant 2n\leqslant\sqrt{1-4A'}-\sqrt{2|B'|}-1.$$

Our result agrees with the one found in Ref. 2. For a function f with $f(E_n)=-n$, we have

$$\lim_{E\to E_n}(E-E_n)\Gamma[f(E)]=\frac{1}{f'(E_n)}\frac{(-1)^n}{n}.$$

The residues of the fixed-energy amplitude can therefore be given as

$$\lim_{E\to E_n}(E-E_n)G(z,z',E)=i\hbar\frac{4r_1r_2}{r_1+r_2}\frac{(-1)^n}{n!}\Gamma(2L-n+1)(zz')^{r_1}[(1-z)(1-z')]^{r_2}$$

$$\times\frac{F(2L-n+1,-n;1+2r_1;z)}{\Gamma(1+2r_1)}\frac{F(2L-n+1,-n;1+2r_2;1-z')}{\Gamma(1+2r_2)}. \tag{81}$$

Using Eq. 5.2.49 in Ref. 16,

$$F(a,b;a+b-c+1;1-z)=\frac{\Gamma(a+b-c+1)\Gamma(1-c)}{\Gamma(b-c+1)\Gamma(a-c+1)}F(a,b;c;z)+\frac{\Gamma(a+b-c+1)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}$$

$$\times z^{1-c}F(b-c+1,a-c+1;2-c;z), \tag{82}$$

with $a=r_1(E_n)+r_2(E_n)-L, b=r_1(E_n)+r_2(E_n)+L+1, c=2r_1+1$, and observing that the second term that would be singular at $z=0$ vanishes for $E=E_n$, since

$$1/\Gamma(r_1(E_n)+r_2(E_n)-L)=0,$$

we finally obtain

$$\lim_{E\to E_n}(E-E_n)G(z,z',E)=\frac{i\hbar 4r_1r_2}{r_1+r_2}\frac{(-1)^n}{n!}(zz')^{r_1}[(1-z)(1-z')]^{r_2}\frac{\Gamma(r_2+r_1+L+1)\Gamma(-2r_1)}{\Gamma(r_2-r_1-L)\Gamma(2r_1+1)\Gamma(r_2-r_1+L+1)}$$

$$\times F(2L-n+1,-n;1+2r_1;z)F(2L-n+1,-n;1+2r_1;z'). \tag{83}$$

Owing to the relation

$$\Gamma(b+n)/\Gamma(b)=(-1)^n[\Gamma(1-b)/\Gamma(1-b-n)]$$

applied to

$$b=r_2-r_1-L,\quad n=-r_1-r_2+L,\quad b+n=-2r_1, \tag{84}$$

we can rewrite this as

$$i\hbar\frac{4r_1r_2}{r_1+r_2}\frac{\Gamma(r_2+r_1+L+1)\Gamma(r_1-r_2+L+1)}{n!\Gamma(1+2r_1)^2\Gamma(r_2-r_1+L+1)}(zz')^{r_1}[(1-z)(1-z')]^{r_2}F(2L-n+1,-n;1+2r_1;z)$$

$$\times F(2L-n+1,-n;1+2r_1;z'). \tag{85}$$

The normalized wave functions are thus

$$\Psi_n(x)=\sqrt{\frac{4r_1r_2}{r_1+r_2}\frac{\Gamma(r_2+r_1+L+1)\Gamma(r_1-r_2+L+1)}{n!\Gamma(1+2r_1)^2\Gamma(r_2-r_1+L+1)}}\left(\frac{1+\tanh x}{2}\right)^{r_1}\left(\frac{1-\tanh x}{2}\right)^{r_2}$$

$$\times F\left(2L-n+1,-n;1+2r_1;\frac{1+\tanh x}{2}\right). \tag{86}$$

They contain Jacobi polynomials

$$P_l^{(\alpha,\beta)}(z)=\frac{\Gamma(\alpha+l+1)}{\Gamma(\alpha+1)l}F[-l,1+\alpha+\beta+l;\alpha+1;(1-z)/2], \tag{87}$$

in the following form:

$$\Psi_n(x)=\sqrt{\frac{4r_1r_2}{r_1+r_2}\frac{\Gamma(r_2+r_1+L+1)n}{\Gamma(r_1-r_2+L+1)\Gamma(r_2-r_1+L+1)}}\left(\frac{1+\tanh x}{2}\right)^{r_1}\left(\frac{1-\tanh x}{2}\right)^{r_2}P_n^{(2r_1,2r_2)}(-\tanh x). \tag{88}$$

Our result for the normalization constant is in agreement with the one found in Ref. 17.

2. Calculation of free states

Case I: $-B'<E<B'$. For $-B'<E<B'$ crossing the branch cut on the real axis transforms r_2 into $-r_2$. A free state $\Psi_E(z)$ must then arise from

$$\Psi_E(z)\Psi_E(z')$$

$$=(1/2\pi\hbar)[G(z,z',r_1,r_2)-G(z,z',r_1,-r_2)]. \tag{89}$$

Therefore

$$\begin{aligned}\Psi_E(z)\Psi_E(z')=&(-iM/2\pi\hbar^2)\Gamma(r_1+r_2+L+1)\Gamma(r_1+r_2-L)(zz')^{r_1}[(1-z)(1-z')]^{r_2}\\&\times\frac{F(r_1+r_2+L+1,r_1+r_2-L;1+2r_1;z)}{\Gamma(2r_1+1)}+\frac{iM}{2\pi\hbar^2}\Gamma(r_1-r_2+L+1)\Gamma(r_1-r_2-L)\\&\times(zz')^{r_1}[(1-z)(1-z')]^{-r_2}\frac{F(r_1-r_2+L+1,r_1-r_2-L;1+2r_1;z)}{\Gamma(2r_1+1)}\\&\times\frac{F(r_1-r_2+L+1,r_1-r_2-L;1-2r_2;1-z')}{\Gamma(-2r_2+1)}.\end{aligned} \tag{90}$$

In Appendix B we have proved that

$$z^{r_1}(1-z)^{-r_2}F(r_1-r_2+L+1,r_1-r_2-L;1+2r_1;z)=z^{r_1}(1-z)^{r_2}F(r_1+r_2+L+1,r_1+r_2-L;1+2r_1;z). \tag{91}$$

Therefore we can factorize the first function in (90) to obtain

$$\begin{aligned}\Psi_E(z)\Psi_E(z')=&\frac{-iM}{2\pi\hbar^2}(zz')^{r_1}[(1-z)(1-z')]^{r_2}F(r_1+r_2+L+1,r_1+r_2-L;1+2r_1;z)\\&\times\frac{1}{2r_2}\left\{\frac{\Gamma(r_1+r_2+L+1)\Gamma(r_1+r_2-L)}{\Gamma(2r_1+1)\Gamma(2r_2)}F(r_1+r_2+L+1,r_1+r_2-L;1+2r_2;1-z')\right.\\&\left.+\frac{\Gamma(r_1-r_2+L+1)\Gamma(r_1-r_2-L)}{\Gamma(2r_1+1)\Gamma(-2r_2)}(1-z')^{-2r_2}F(r_1-r_2+L+1,r_1-r_2-L;1-2r_2;1-z')\right\}.\end{aligned} \tag{92}$$

If we make the substitutions $a+b-c+1\to c$ and $z\to 1-z$ in (82) and set

$$a=r_1+r_2+L+1,\quad b=r_1+r_2-L,\quad c=2r_1+1. \tag{93}$$

The expression in curly brackets can be simplified to

$$\frac{\Gamma(r_1-r_2-L)\Gamma(r_1+r_2-L)\Gamma(r_1-r_2+L+1)\Gamma(r_1+r_2+L+1)}{[\Gamma(2r_1+1)]^2\Gamma(-2r_2)\Gamma(2r_2)}F(r_1+r_2+1+L,r_1+r_2-L;1+2r_1;z). \tag{94}$$

Equation (91) shows that the function

$$z^{r_1}(1-z)^{r_2}F(r_1+r_2+L+1,r_1+r_2+L;1+2r_1;z)$$

is real valued as long as z is real, so the correctly normalized wave functions can now be read off:

$$\Psi_E(z)=\sqrt{\frac{M}{4\pi\hbar^2|r_2|}}\left|\frac{\Gamma(r_1+r_2-L)\Gamma(r_1+r_2+L+1)}{\Gamma(2r_1+1)\Gamma(2r_2)}\right|[(1+\tanh x)/2]^{r_1}[(1-\tanh x)/2]^{r_2}$$
$$\times F[r_1+r_2+L+1,r_1+r_2-L;1+2r_1;(1+\tanh x)/2] \tag{95}$$

for the square root in the coefficient we used $-\ i/r_2\ =\ 1/|r_2|$ [compare with the discussion following Eq. (73)].

Case II: $B'<E$. For $B'<E$, crossing the cut amounts to making the *two* substitutions $r_2\to -r_2$ and $r_1\to -r_1$. Besides, we must expect each of the eigenvalues above the upper bound of the potential to be doubly degenerate. The residue of the fixed-energy amplitude will therefore have to be represented by a sum over a degeneracy index λ:

$$\sum_\lambda \Psi_E^{(\lambda)}(z)^*\Psi_E^{(\lambda)}(z')=\frac{1}{2\pi\hbar}[G(z,z',r_1,r_2)-G(z,z',-r_1,-r_2)]. \tag{96}$$

Thus

$$\sum_\lambda \Psi_E^{(\lambda)}(z)^*\Psi_E^{(\lambda)}(z')=\frac{-iM}{2\pi\hbar^2}\Gamma(r_1+r_2+L+1)\Gamma(r_1+r_2-L)(zz')^{r_1}[(1-z)(1-z')]^{r_2}$$
$$\times\frac{F(r_1+r_2+L+1,r_1+r_2-L;1+2r_1;z)}{\Gamma(2r_1+1)}\frac{F(r_1+r_2+L+1,r_1+r_2-L;1+2r_2;1-z')}{\Gamma(2r_2+1)}$$
$$+\frac{iM}{2\pi\hbar^2}\Gamma(-r_1-r_2+L+1)\Gamma(-r_1-r_2-L)(zz')^{-r_1}[(1-z)(1-z')]^{-r_2}$$
$$\times\frac{F(-r_1-r_2+L+1,-r_1-r_2-L;1-2r_1;z)}{\Gamma(-2r_1+1)}$$
$$\times\frac{F(-r_1-r_2+L+1,-r_1-r_2-L;1-2r_2;1-z')}{\Gamma(-2r_2+1)}, \tag{97}$$

with the abbreviations

$$K(r_1,r_2)=\frac{-iM}{2\pi\hbar^2}\frac{\Gamma(r_1+r_2+L+1)\Gamma(r_1+r_2-L)}{\Gamma(2r_1+1)\Gamma(2r_2+1)},$$

$$F_1(r_1,r_2,z)=z^{r_1}(1-z)^{r_2}F(r_1+r_2+L+1,r_1+r_2-L;1+2r_1;z), \tag{98}$$

$$F_2(r_1,r_2,1-z)=z^{r_1}(1-z)^{r_2}F(r_1+r_2+L+1,r_1+r_2-L;1+2r_2;1-z),$$

we can write as well

$$\sum_\lambda \Psi_E^{(\lambda)}(z)^*\Psi_E^{(\lambda)}(z')=K(r_1,r_2)F_1(r_1,r_2,z)F_2(r_1,r_2,1-z')-K(-r_1,-r_2)F_1(-r_1,-r_2,z)F_2(-r_1,-r_2,1-z'). \tag{99}$$

In Appendix B we have shown $F_1(r_1,r_2,z)$ to be invariant with respect to a change of sign in r_2. An analogous proof can be carried out for $F_2(r_1,r_2,1-z')$ and r_1, so that we have

$$F_1(r_1,-r_2,z)=F_1(r_1,r_2,z),$$

$$F_2(-r_1,r_2,1-z)=F_2(r_1,r_2,1-z).$$

Using this and Eq. (82) we can deduce the following relations:

$$\begin{aligned}F_2(-r_1,-r_2,1-z)&=\frac{1}{2r_1K(r_1,-r_2)}F_1(-r_1,-r_2,z)\\&\quad+\frac{1}{-2r_1K(-r_1,-r_2)}F_1(r_1,r_2,z),\\F_1(r_1,r_2,z)&=\frac{1}{-2r_2K(r_1,-r_2)}F_1(r_1,r_2,1-z)\\&\quad+\frac{1}{2r_2K(r_1,r_2)}F_1(-r_1,-r_2,1-z).\end{aligned}\tag{100}$$

Solving for $F_1(r_1,r_2,z)$ and $F_1(-r_1,-r_2,1-z)$ and substituting the resulting expressions in (99) leads to

$$\begin{aligned}2\pi\hbar\sum_\lambda\Psi_E^{(\lambda)}(z)^*\Psi_E^{(\lambda)}(z')&=-2r_1K(r_1,r_2)K(-r_1,-r_2)F_2(-r_1,-r_2,\\&\quad 1-z)F_2(r_1,r_2,1-z')-2r_2K(r_1,r_2)K(-r_1,-r_2)\\&\quad\times F_1(-r_1,-r_2,z)F_1(r_1,r_2,z').\end{aligned}\tag{101}$$

Now we have

$$\begin{aligned}-2r_1K(r_1,r_2)K(-r_1,-r_2)&=\frac{iM/2\hbar^2\sin(\pi 2r_1)}{|\sin\pi(r_1+r_2+L)|^2\Gamma(1+2r_2)\Gamma(1-2r_2)}\\-2r_2K(r_1,r_2)K(-r_1,-r_2)&=\frac{iM/2\hbar^2\sin(\pi 2r_2)}{|\sin\pi(r_1+r_2+L)|^2\Gamma(1+2r_1)\Gamma(1-2r_1)}.\end{aligned}\tag{102}$$

As, furthermore,

$$F_1(-r_1,-r_2,z)=F_1(r_1,r_2,z)^*,$$

$$F_2(-r_1,-r_2,1-z)=F_2(r_1,r_2,1-z)^*,\tag{103}$$

$$\Gamma(1-2r_2)=\Gamma(1+2r_2)^*,$$

the normalized wave functions are

$$\begin{aligned}\Psi_E^1(z)&=\frac{1}{\Gamma(1+2r_1)}\frac{\sqrt{M\sinh(\pi|2r_2|)/2}}{\hbar\,|\sin\pi(r_1+r_2+L)|}\\&\quad\times\left(\frac{1+\tanh x}{2}\right)^{r_1}\left(\frac{1-\tanh x}{2}\right)^{r_2}\\&\quad\times F\left(r_1+r_2+L+1,r_1+r_2-L;1\right.\\&\quad\left.+2r_1;\frac{1+\tanh x}{2}\right),\\\Psi_E^2(z)&=\frac{1}{\Gamma(1+2r_2)}\frac{\sqrt{M\sinh(\pi|2r_1|)/2}}{\hbar\,|\sin\pi(r_1+r_2+L)|}\\&\quad\times\left(\frac{1+\tanh x}{2}\right)^{r_1}\left(\frac{1-\tanh x}{2}\right)^{r_2}\\&\quad\times F\left(r_1+r_2+L+1,r_1+r_2-L;1\right.\\&\quad\left.+2r_2;\frac{1-\tanh x}{2}\right),\end{aligned}\tag{104}$$

which properly reduces to (75) for $B'=0$, i.e., for $r_1=r_2=\mu/2$. This can be seen by using Eqs. (100) and (9). It is here that we see a difference between our results and those obtained in earlier works. In Refs. 9 and 10, the Rosen–Morse states are derived from the eigenstates of the modified Pöschl–Teller potential, which has boundary conditions that do not permit degeneracy of the free states. (Due to the singularity in the latter potential the wave functions have to vanish at 0 and ∞.) Our derivation yields two orthogonal states per energy value above the Rosen–Morse potential barrier. It would appear that physical intuition is on our side since in a one-dimensional potential without singularities one always expects to have two linearly independent scattering states corresponding to waves that are propagating to the left and to the right.

V. CONCLUSION

We have carried out the summation in the spectral representation of symmetric and general Pöschl–Teller fixed-energy amplitudes. The close relationship between the Pöschl–Teller potentials and azimuthal systems on the one hand and the symmetric and general Rosen–

Morse potentials on the other enabled us also to write down a closed-form expression for the fixed-energy amplitude for the latter systems. We have seen how *all* the eigenstates of the Rosen–Morse potential together with their normalization constants can quite easily be extracted from this amplitude. In the transition from the Pöschl–Teller (or azimuthal) problem to the Rosen–Morse potentials, free states emerge, which are not present in the initial potentials. Their appearance is due to the fact that when transforming one system into the other the energy dependence of the fixed-energy amplitude is shifted to the analytically continued azimuthal quantum number. In the case of the symmetric Rosen–Morse potential, it is shifted from the lower index λ to the upper index μ of the associated Legendre function. The Pöschl–Teller amplitude contains the function $P^{\mu}_{\lambda(E)}$, which has no branch cut in the variable E, even though $\lambda(E)$ contains a square root. Indeed, it can easily be seen that $P^{\mu}_{\lambda(E)}$ is continuous across the branch cut of this square root on the real axis. For real E,

$$P^{\mu}_{\lambda(E+i\epsilon)} \overset{\epsilon\to 0}{\to} P^{\mu}_{-1/2+\sqrt{1/4+2ME/\hbar^2}} \tag{105}$$

and

$$P^{\mu}_{\lambda(E-i\epsilon)} \overset{\epsilon\to 0}{\to} P^{\mu}_{-1/2-\sqrt{1/4+2ME/\hbar^2}}. \tag{106}$$

Now, Eq. (9) shows that

$$P^{\mu}_{\lambda}=P^{\mu}_{-\lambda-1},$$

so that the two functions above are identical for $\epsilon=0$. When the energy dependence is moved to the upper index μ, on the other hand, the discontinuity over the branch cut of $\mu(E)=\sqrt{-2ME/\hbar^2}$ is no longer zero, since from Eq. (13) we see that the two functions $P^{\mu(E)}_{\lambda}$ and $P^{-\mu(E)}_{\lambda}$ are, in general, linearly independent.

ACKNOWLEDGMENTS

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APPENDIX A: ASYMPTOTIC BEHAVIOR OF HYPERGEOMETRIC FUNCTION

1. Integral representation

We wish to find the asymptotic behavior of

$$F[-\lambda,\lambda+\alpha+\beta+1;1+\alpha;(1-z)/2],$$

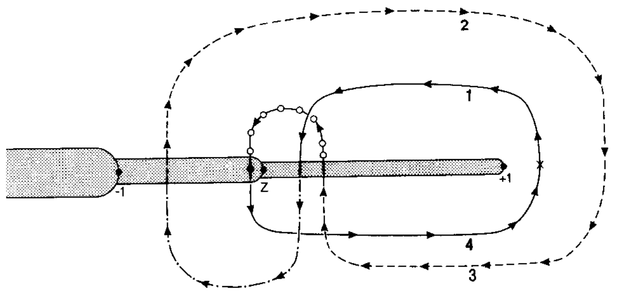

FIG. 1. Integration contour for hypergeometric function.

for $|\lambda|\to\infty$. We start with the following integral representation of this hypergeometric function [see Ref. 16, Eq. (5.3.19)]:

$$\frac{\Gamma(1+\alpha)}{\Gamma(\lambda+1+\alpha+\beta)\Gamma(-\lambda-\beta)}\left(\frac{1-z}{2}\right)^{-\alpha} \times\int_0^{(1-z)/2} dt[(1-z)/2-t]^{-\lambda-\beta-1}t^{\lambda+\alpha+\beta} \times(1-t)^{\lambda}. \tag{A1}$$

Under the transformation

$$t=(1-\tau)/2,$$
$$dt=-\tfrac{1}{2}d\tau, \tag{A2}$$

this becomes

$$\frac{\Gamma(1+\alpha)2^{-\lambda-\alpha}}{\Gamma(\lambda+1+\alpha+\beta)\Gamma(-\lambda-\beta)}\left(\frac{1-z}{2}\right)^{-\alpha} \times\int_z^1 d\tau(\tau-z)^{-\lambda-\beta-1}(1-\tau)^{\lambda+\alpha+\beta}(1+\tau)^{\lambda}. \tag{A3}$$

The integrand is supposed to be real on the interval $[z;1]$. This representation is valid for

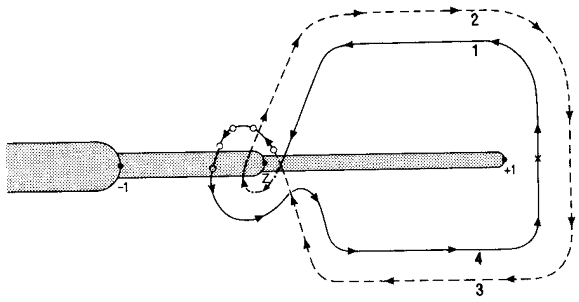

FIG. 2. Deformed integration contour for hypergeometric function.

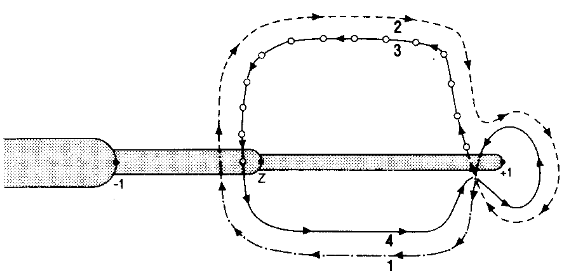

FIG. 3. Deformed integration contour for hypergeometric function.

$$\mathrm{Re}(1+\alpha) > \mathrm{Re}(\lambda+1+\alpha+\beta) > 0.$$

Another representation that is valid for arbitrary complex values of λ,α,β can be found by rewriting (A3) as a contour integral over a contour $\mathscr{C}$, which is depicted in Fig. 1.

The integrand has branch points as -1, z, $+1$, their branch cuts are chosen to run to $-\infty$ along the real axis. The broadening of the cut is to indicate that there is, in fact, superposition of branch cuts, but, for general α,β,λ, no cancellation. Since the cuts are crossed twice in opposite directions, the contour is closed on the Riemann surface of the integrand. Note that it can be deformed in the two manners shown in Figs. 2 and 3.

The latter contours will be useful for understanding certain more complicated deformations to be performed further below. The contour $\mathscr{C}$ being closed, the function

$$\mathscr{A}\left(\frac{1-z}{2}\right)^{-\alpha}\int_{\mathscr{C}} d\tau(\tau-z)^{-\lambda-\beta-1} \times(1-\tau)^{\lambda+\alpha+\beta}(1+\tau)^{\lambda}, \tag{A4}$$

according to Morse–Feshbach, must be a solution of the hypergeometric equation. What is more, this solution is defined for arbitrary λ,α,β. We shall now show that, choosing an appropriate $\mathscr{A}$, (A4) can be made into an analytic continuation of (A3). We first note that, for $\mathrm{Re}(1+\alpha)>\mathrm{Re}(\lambda+1+\alpha+\beta)>0$, integrals along circular integration contours around the branch points z, 1 will go to zero when the radius of these contours goes to zero. Going around the branch points will then provide the integrals over the contour sections 1,2,3,4 with the following phase factors:

$$\begin{aligned}
&\text{along 1:}\quad e^{i\pi(\lambda+\alpha+\beta)},\\
&\text{along 2:}\quad e^{i\pi(3\lambda+\alpha+3\beta)},\\
&\text{along 3:}\quad e^{i\pi(\lambda-\alpha+\beta)},\\
&\text{along 4:}\quad e^{i\pi(-\lambda-\alpha-\beta)}.
\end{aligned} \tag{A5}$$

This shows again that $\mathscr{C}$ is a closed contour. In what follows we will always write the integrand as

$$\text{phase}\times(\tau-z)^{-\lambda-\beta-1}(1-\tau)^{\lambda+\alpha+\beta}(1+\tau)^{\lambda}.$$

In this way, we will be able to suppose, that $(\tau-z)^{-\lambda-\beta-1}(1-\tau)^{\lambda+\alpha+\beta}(1+\tau)^{\lambda}$ is always real on the interval $[z;1]$. For $\mathrm{Re}(1+\alpha)>\mathrm{Re}(\lambda+1+\alpha+\beta)>0$, we can thus write

$$\begin{aligned}
&\int_{\mathscr{C}} d\tau(\tau-z)^{-\lambda-\beta-1}(1-\tau)^{\lambda+\alpha+\beta}(1+\tau)^{\lambda}\\
&=[e^{-i\pi(\lambda+\alpha+\beta)}-e^{i\pi(\lambda+\alpha+\beta)}-e^{i\pi(\lambda-\alpha+\beta)}+e^{i\pi(3\lambda+\alpha+3\beta)}]\int_z^1 d\tau(\tau-z)^{-\lambda-\beta-1}(1-\tau)^{\lambda+\alpha+\beta}(1+\tau)^{\lambda}\\
&=-4e^{i\pi(\lambda+\beta+1)}\sin\pi(\lambda+\alpha+\beta)\sin\pi(\lambda+\beta+1)\int_z^1 d\tau(\tau-z)^{-\lambda-\beta-1}(1-\tau)^{\lambda+\alpha+\beta}(1+\tau)^{\lambda}.
\end{aligned} \tag{A6}$$

Thus

$$\begin{aligned}
F\left(-\lambda,\lambda+\alpha+\beta+1;1+\alpha;\frac{1-z}{2}\right)&=\frac{e^{-i\pi(\lambda+\beta+1)}}{\pi^2 2^{\lambda+\alpha+2}}\Gamma(1+\alpha)\Gamma(\lambda+1+\beta)\Gamma(-\lambda-\alpha-\beta)\left(\frac{1-z}{2}\right)^{-\alpha}\\
&\quad\times\int_{\mathscr{C}} d\tau(\tau-z)^{-\lambda-\beta-1}(1-\tau)^{\lambda+\alpha+\beta}(1+\tau)^{\lambda}.
\end{aligned} \tag{A7}$$

2. Application of saddle-point approximation

This representation is amenable to saddle-point analysis, as will now be shown. The integrand can be written as

$$\exp[(-\lambda-\beta-1)\ln(\tau-z)+(\lambda+\alpha+\beta)\times\ln(1-\tau)+\lambda\ln(1+\tau)].$$

The saddle points can be found by setting the first derivative of the exponent equal to zero:

$$-\frac{\lambda+\beta+1}{\tau-z}-\frac{\lambda+\alpha+\beta}{1-\tau}+\frac{\lambda}{1+\tau}=0.$$

There are thus two saddle points situated at

$$\tau_\pm=\frac{(2\lambda+\alpha+\beta)z-\alpha-\beta}{2(\lambda+\alpha-1)}\pm\sqrt{\left[\frac{(2\lambda+\alpha+\beta)z-\alpha-\beta}{2(\lambda+\alpha-1)}\right]^2-\frac{\lambda+\beta+1-(\alpha+\beta)z}{\lambda+\alpha-1}}. \tag{A8}$$

For $|\lambda|\to\infty$ the positions of the saddle points converge toward

$$\tau_\pm=z\pm\sqrt{z^2-1}.$$

We shall assume that the usual saddle-point method with fixed saddle points can be extended without modifications to the present situation, where we have "moving" saddle points, and expand the exponent in the integrand to second order around the limiting values of the saddle points. The second-order coefficients are given by

$$\frac{\lambda+\beta+1}{(\tau_\pm-z)^2}-\frac{\lambda+\alpha+\beta}{(\tau_\pm-1)^2}-\frac{\lambda}{(\tau_\pm+1)^2}.$$

After a change of variables,

$$z\to\theta,\quad z=\cos\theta,$$

the limiting positions of the saddle points take on the form

$$\tau_\pm=e^{\pm i\theta}.$$

The value of the integrand at these two points is

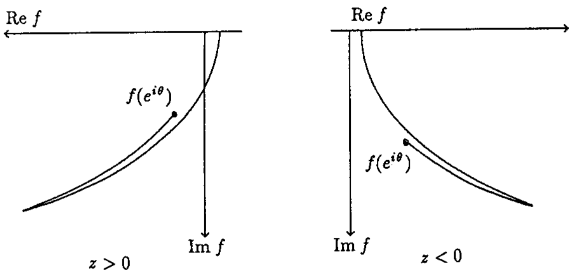

FIG. 4. Mapping of paths by the integrand factor $f(\tau):=(1-\tau^2)/(\tau-z)$.

$$[(1-e^{\pm i\theta})^\beta/(\pm i\sin\theta)^{\beta+1}](1-e^{\pm i\theta})^\alpha(-2)^\lambda e^{\pm i\lambda\theta}$$
$$=:e^{a_0^\pm}. \tag{A9}$$

The phase of $(-2)^\lambda$ is determined by the requirement that the function

$$(\tau-z)^{-\beta-1}(1-\tau)^{\alpha+\beta}[(1-\tau^2)/\tau-z]^\lambda, \tag{A10}$$

be real on the interval $[z;1]$. Its value for $\tau=e^{i\theta}$ can be determined by analytically continuing it from the real axis, i.e., along the path

$$z(\varphi)=[1-\epsilon(1-\varphi/\theta)]e^{i\varphi};\quad \varphi\in[0,\theta],$$

with ϵ a positive number smaller than unity. We can assume that all three factors are real on this interval. Now the function

$$(1-\tau^2)/(\tau-z)$$

maps the path $z(\varphi)$—qualitatively—into one of the paths in Fig. 4, depending on whether $z>0$ or $z<0$. The length of the path depends on θ. The phases along these two paths can be parametrized by

$$e^{-i\xi},\quad \xi\in[0,\Xi],\quad 0\leqslant\Xi\leqslant\pi,$$

where $e^{-i\Xi}=-e^{i\theta}$. The phase of the third factor must therefore have the form

$$e^{-i\lambda\Xi},$$

where $\Xi\in[0;\pi]$. For $\tau=e^{-i\theta}$ an analogous reasoning applies and the phase we are look ing for has finally to be chosen as

$$(-2)^\lambda e^{\pm i\lambda\theta} = 2^\lambda e^{\pm i\lambda(\theta-\pi)}. \tag{A11}$$

The second-order coefficient is equal to

$$\frac{(\lambda+\alpha+\beta)(1+e^{\pm i\theta})^2+\lambda(1-e^{\pm i\theta})^2-4(\lambda+\beta+1)e^{\pm 2i\theta}}{e^{\pm 2i\theta}4\sin^2\theta}.$$

For $|\lambda|\to\infty$, this behaves as

$$\mp\lambda(e^{\mp i\theta}i/\sin\theta) = e^{-i(\phi\pm\theta\pm\pi/2)}(|\lambda|/\sin\theta),$$

where $\lambda = |\lambda| e^{-i\phi}$. We shall write

$$a_2 := |\lambda|/\sin\theta. \tag{A12}$$

For the saddle-point method to work, the integration contour has to be deformed in such a manner that, on the saddle points, it is tangential to the two straights, defined by

$$e^{\pm i\theta}+\xi e^{i(\pm\theta/2+\phi/2\mp\pi/4)}, \xi\in]-\infty;\infty[.$$

If it can be shown that only the saddle points of the integrand contribute to the asymptotic behavior of the function, then, for sufficiently large $|\lambda|$, the integral in (A7) can be evaluated approximately as

$$\sum_j \Big\{\sigma_j e^{\eta_j} e^{i(gq/2+\phi/2-\pi/4)} \int_{-\infty}^{+\infty} d\tau \exp[a_0^+ - a_2(\tau-\tau_+)^2] + \sigma_j e^{\xi_j} e^{i(-\theta/2+\phi/2+\pi/4)} \int_{-\infty}^{+\infty} d\tau$$
$$\times \exp[a_0^- - a_2(\tau-\tau_-)^2]\Big\} = \sum_j \Big\{\sigma_j e^{\eta_j} e^{i(\theta/2+\phi/2-\pi/4)} \sqrt{\frac{\pi}{a_2}} e^{a_0^+} + \sigma_j e^{\xi_j} e^{i(-\theta/2+\phi/2+\pi/4)} \sqrt{\frac{\pi}{a_2}} e^{a_0^-}\Big\}, \tag{A13}$$

where the summation extends over the different sheets of the Riemann surface that take part in the integration, η_j,ξ_j are the corresponding phases and $\sigma_j = \pm 1$, depending on the orientation of the path. In applying the method we have to distinguish two cases.

A. Case 1: *Re*: $\lambda>0$

For sufficiently large $|\lambda|$, the integrand has zeros at $\tau=-1$ and $\tau=1$. The contour $\mathscr{C}$ can then be deformed as shown in Fig. 5 (cf. Fig. 3), since then the small circlelike contours around $+1$ in Fig. 3 can be contracted to a point. The "$+$" signs in the figure indicate the limiting positions of the saddle points. The paths 2 and 3 are running on top of each other across the upper saddle point and the paths 1 and 4 across the lower one. In the figure they were separated to be distinguishable. The integrand vanishes at the limits of the integrations along these four paths. According to Dingle,[18] this is a sufficient condition for the two saddle points to contribute additively to the asymptotic behavior of the integral. The asymptotic behavior of (A7) is therefore given by

$$2ie^{i(\theta/2+\phi/2-\pi/4)}e^{i\lambda\theta}\sqrt{\frac{\pi}{|\lambda|}}\sqrt{\sin\theta}\frac{(1-e^{i\theta})^\beta}{(i\sin\theta)^{\beta+1}}(1-e^{i\theta})^\alpha\frac{e^{i\pi\beta}}{\pi 2^{\alpha+2}}\frac{\Gamma(1+\alpha)\Gamma(\lambda+\beta+1)}{\Gamma(\lambda+\alpha+\beta+1)}\left(\frac{1-\cos\theta}{2}\right)^{-\alpha}$$
$$-2ie^{i(-\theta/2+\phi/2+\pi/4)}e^{-i\lambda\theta}\sqrt{\frac{\pi}{|\lambda|}}\sqrt{\sin\theta}\frac{(1-e^{-i\theta})^\beta}{(-i\sin\theta)^{\beta+1}}(1-e^{-i\theta})^\alpha$$
$$\times\frac{e^{-i\pi\beta}}{\pi 2^{\alpha+2}}\frac{\Gamma(1+\alpha)\Gamma(\lambda+\beta+1)}{\Gamma(\lambda+\alpha+\beta+1)}\left(\frac{1-\cos\theta}{2}\right)^{-\alpha}. \tag{A14}$$

The contribution of the Γ coefficients can be evaluated with the formula (see Ref. 16, p. 443)

$$\Gamma(z+1)\sim\sqrt{2\pi}z^{z+1/2}e^{-z}, \tag{A15}$$

for $|z|\to\infty$. It shows that

$$\Gamma(\lambda+\beta+1)/\Gamma(\lambda+\alpha+\beta+1)\sim\lambda^{-\alpha}. \tag{A16}$$

The result of (A14) thus becomes

$$\mathscr{F}_{\alpha,\beta}(\theta)\lambda^{-\alpha-1/2}e^{i\lambda\theta}+\mathscr{G}_{\alpha,\beta}(\theta)\lambda^{-\alpha-1/2}e^{-i\lambda\theta}, \tag{A17}$$

where

$$\mathcal{F}_{\alpha,\beta}(\theta)=2ie^{i(\theta/2-\pi/4)}\sqrt{\pi\sin\theta}\,\frac{(1-e^{i\theta})^{\beta}}{(i\sin\theta)^{\beta+1}}(1-e^{i\theta})^{\alpha}\frac{e^{i\pi\beta}}{\pi2^{\alpha+2}}\Gamma(1+\alpha)\left(\frac{1-\cos\theta}{2}\right)^{-\alpha},$$
$$\mathcal{G}_{\alpha,\beta}(\theta)=-2ie^{-i(\theta/2-\pi/4)}\sqrt{\pi\sin\theta}\,\frac{(1-e^{-i\theta})^{\beta}}{(-i\sin\theta)^{\beta+1}}(1-e^{-i\theta})^{\alpha}\frac{e^{-i\pi\beta}}{\pi2^{\alpha+2}}\Gamma(1+\alpha)\left(\frac{1-\cos\theta}{2}\right)^{-\alpha}. \tag{A18}$$

B. Case 2: *Re*: $\lambda<0$

For sufficiently large $|\lambda|$, the integrand has a zero at $\tau=z$. The modulus of the integrand behaves for $|\tau|\to\infty$ as const $\times|\tau|^{-|\mathrm{Re}\,\lambda|}$. Quarter-circle integrals at infinity therefore do not contribute and the contour $\mathcal{C}$ can be deformed as shown in Fig. 6 (cf. Fig. 2) because, for $\mathrm{Re}\,\lambda<0$, the integrand has zeros at z and ∞. The right-hand part of the contour in Fig. 2 can be stretched to infinity and rotated so as to run parallel to the imaginary axis. The small circlelike contours around z can be contracted to a point.

The asymptotic behavior of (A4) is therefore given by

$$2ie^{i(\theta/2+\phi/2-\pi/4)}e^{i\lambda\theta}\sqrt{\frac{\pi\sin\theta}{|\lambda|}}\frac{(1-e^{i\theta})^{\beta}}{(i\sin\theta)^{\beta+1}}(1-e^{i\theta})^{\alpha}\frac{e^{i\pi(\beta+\alpha)}}{\pi2^{\alpha+2}}\frac{\Gamma(1+\alpha)\Gamma(-\lambda-\alpha-\beta)}{\Gamma(-\lambda-\beta)}\left(\frac{1-\cos\theta}{2}\right)^{-\alpha}$$
$$+2ie^{i(-\theta/2+\phi/2+\pi/4)}e^{-i\lambda\theta}\sqrt{\frac{\pi\sin\theta}{|\lambda|}}\frac{(1-e^{-i\theta})^{\beta}}{(-i\sin\theta)^{\beta+1}}(1-e^{-i\theta})^{\alpha}$$
$$\times\frac{e^{-i\pi(\beta+\alpha)}}{\pi2^{\alpha+2}}\frac{\Gamma(1+\alpha)\Gamma(-\lambda-\alpha-\beta)}{\Gamma(-\lambda-\beta)}\left(\frac{1-\cos\theta}{2}\right)^{-\alpha}. \tag{A19}$$

After calculating the contribution of the Γ functions as in (A16),

$$\Gamma(-\lambda-\alpha-\beta)/\Gamma(-\lambda-\beta)\sim(-\lambda)^{-\alpha}, \tag{A20}$$

we can write the final result as:

$$e^{i\pi\alpha}e^{i\phi/2}\mathcal{F}_{\alpha,\beta}(\theta)(-\lambda)^{-\alpha}|\lambda|^{-1/2}e^{-i\lambda\theta}$$
$$-e^{-i\pi\alpha}e^{i\phi/2}\mathcal{G}_{\alpha,\beta}(\theta)(-\lambda)^{-\alpha}|\lambda|^{-1/2}e^{i\lambda\theta}. \tag{A21}$$

3. Summary of Appendix A

The results (A17) and (A21) can be summarized in the following formula:

$$\left|F\left(-\lambda,\lambda+\alpha+\beta+1;1+\alpha;\frac{1-\cos\theta}{2}\right)\right|$$
$$\xrightarrow{|\lambda|\to\infty} e^{\phi\,\mathrm{Im}\,\alpha}|\mathcal{H}_{\alpha,\beta}(\theta)|\,|\lambda|^{-\mathrm{Re}\,\alpha-1/2}e^{|\theta\,\mathrm{Im}\,\lambda|}, \tag{A22}$$

where $\lambda=e^{-i\phi}|\lambda|$ and $\mathcal{H}_{\alpha,\beta}(\theta)$ is independent of λ. The formula is valid for $\theta\in]0;\pi[$.

APPENDIX B: PROOF OF EQ. (91)

Both sides of the equation are solutions of the Papperitz equation:[16]

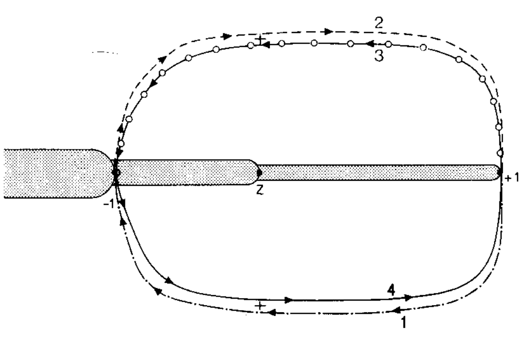

FIG. 5. Deformed integration contour for $\mathrm{Re}\,\lambda>0$.

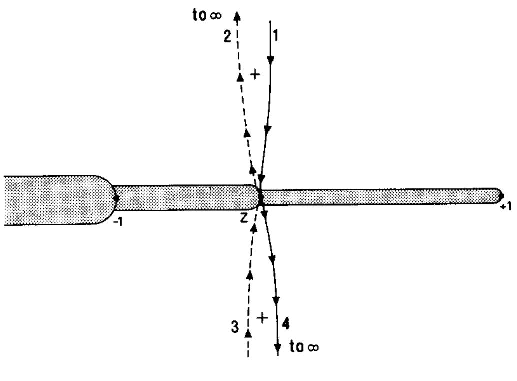

FIG. 6. Deformed integration contour for $\mathrm{Re}\,\lambda<0$.

$$\left[\frac{d^2}{dz^2}+\left(\frac{1-\lambda-\lambda'}{z}+\frac{1-\mu-\mu'}{z-1}\right)\frac{d}{dz}+\frac{\mu\mu'}{z(z-1)^2}-\frac{\lambda\lambda'}{z^2(z-1)}-\frac{\nu(\lambda+\lambda'+\mu+\mu'+\nu-1)}{z(z-1)}\right]G(z)=0, \qquad \text{(B1)}$$

with

$$\lambda=-\lambda'=r_1,\quad \mu=-\mu'=r_2,\quad \nu=L+1. \qquad \text{(B2)}$$

Obviously, after substituting $r_2 \to -r_2$, we still have a solution of the same equation. This means that the function resulting from this substitution must be a superposition of the function before the substitution took place and any other, linearly independent solution. Consequently, there must exist constants $\mathscr{A}$ and $\mathscr{B}$, such that

$$\begin{aligned} &z^{r_1}(1-z)^{-r_2}F(r_1-r_2+L+1,r_1-r_2-L;1+2r_1;z)\\ &\quad=\mathscr{A}z^{r_1}(1-z)^{r_2}F(r_1+r_2+L+1,r_1+r_2-L;2r_1+1;z)+\mathscr{B}z^{-r_1}(1-z)^{r_2}\\ &\qquad\times F(-r_1+r_2+L+1,r_1+r_2-L;-2r_1+1;z). \end{aligned} \qquad \text{(B3)}$$

But, the function on the left-hand side has a zero at $z=0$, whereas the second function on the right-hand side is singular for $z=0$. We therefore conclude that

$$\mathscr{B}=0.$$

On multiplying with z^{-r_1} and setting $z=0$ we obtain

$$\mathscr{A}=1,$$

so that the substitution $r_2 \to -r_2$ actually does not change the function.

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