H. KLEINERT, *et al.*
1° Ottobre 1969
Il Nuovo Cimento
Serie X, Vol. **63** A, pag. 657-674

Invariant Functions and Discrete Symmetries. - II

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(ricevuto il 21 Gennaio 1969)

Summary. — A Lorentz-covariant spinor basis for the scattering of spin configurations $(s, \frac{1}{2}) \to (s, \frac{1}{2})$ is constructed for massive particles producing scalar amplitudes which are free of kinematic singularities, real analytic, and eigenstates of P, T and $s \leftrightarrow u$ crossing.

1. – Introduction.

Due to the bad analyticity properties of the helicity decomposition of the scattering amplitude (1) Reggeization procedures have been complicated by unpleasant factors introduced to avoid kinematic singularities (2-5) and constraints taking account of threshold and pseudothreshold behaviour (3-5). Therefore it has been pointed out (6) that the natural way of parametrizing

(1) M. JACOB and G. C. WICK: *Ann. of Phys.*, **7**, 404 (1959).
(2) Y. HARA: *Phys. Rev.*, **136** B, 507 (1964); L.-L. CHAN WANG: *Phys. Rev.*, **142**, 1187 (1965); **153**, 1664 (1967); K. Y. LIN: *Phys. Rev.*, **155**, 1515 (1967); H. P. STAPP: *Phys. Rev.*, **160**, 1251 (1967).
(3) G. COHEN TANNOUDJI, A. MOREL and H. NAVELET: *Ann. of Phys.*, **46**, 239 (1968); J. D. JACKSON and G. E. HITE: *Phys. Rev.*, **169**, 1248 (1968); J. FRANKLIN: *Phys. Rev.*, **170**, 1606 (1968).
(4) G. C. FOX: *Phys. Rev.*, **157**, 1493 (1967).
(5) H. F. JONES: *Nuovo Cimento*, **50** A, 814 (1967); B. DIU and M. LE BELLAC: *Nuovo Cimento*, **53** A, 158 (1968).
(6) K. H. MÜTTER: *Nucl. Phys.*, B **8**, 311 (1968); J. GEICKE and K. H. MÜTTER: Berlin preprint (1968); K. H. MÜTTER and E. TRÄNKLE: Berlin preprint (1968); P. KROLL and E. TRÄNKLE: Berlin preprint.

high-energy scattering is to introduce Regge-type poles in the invariant amplitudes of an expansion of the S-matrix in terms of Hepp and Williams (hereafter referred to as HW) (7) standard covariants. This and related expansions provide at present the only general method of reducing the scattering of arbitrarily spinning particles to a scalar problem without introducing the difficulties mentioned above. As in the Toller description of the S-matrix (8), the automatic fulfilment of constraints at pseudothreshold, manifests itself in the occurrence of families of usual Regge poles. However, the families generated in this fashion are much smaller since they provide a minimal set of poles enforcing the constraints. In this way they introduce a natural mixing of Toller's $O_{3,1}$ representations. Conversely, one Toller representation introduces poles in several invariant amplitudes at a time together with all their family members. The success of fitting nucleon-nucleon-scattering and simultaneously its crossed process, photoproduction and certainly good old pion-nucleon scattering (6) by very few Regge-type poles in the invariant amplitudes has led us to study the HW decomposition in more detail.

The spinor covariants were constructed by HW considering only the orthochronous proper Lorentz group. For applications to strong interactions this basis is not very useful since the discrete symmetries of parity and time-reversal invariance produce linear relations among the invariant amplitudes. Even though HEPP later proved that a decomposition free of kinematical singularities exists also with covariants invariant under P and T (9), his proof does not give any hint as to how these covariants can be formed explicitly. Furthermore, if we want the invariant amplitudes to be real analytic in order to satisfy a Mandelstam representation (10) it is necessary to have the covariants Hermitian. Finally, since we plan to later Reggeize these amplitudes in the t-channel, they should have simple crossing properties with respect to the exchange of s and u-channel. This requirement will be seen to cause the covariants to factorize into a product of covariants containing kinematical variables of the two vertices separately. This property has led in the applications (6) to the factorization of the residues of the individual Regge poles generated by a pole in the invariant amplitude and can probably be used to prove a general factorization theorem.

Spinor covariants satisfying all these requirements with invariant amplitudes free of kinematic singularities are called parity eigenstandard covariants (PES).

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(7) K. HEPP: *Helv. Phys. Acta*, **37**, 55 (1964); D. N. WILLIAMS: UCLA Report No. UCRL-11113 (1963).

(8) M. TOLLER: *Nuovo Cimento*, **53** A, 671 (1968); D. Z. FREEDMAN and J. M. WANG: *Phys. Rev.*, **166**, 1560 (1967).

(9) K. HEPP: *Helv. Phys. Acta*, **37**, 55 (1964).

(10) J. BROS, H. EPSTEIN and V. GLASER: *Nuovo Cimento*, **31**, 1265 (1964); *Comm. Math. Phys.*, **1**, 240 (1965).

We will see in the later discussion that there is an intimate connection between the invariance under P and T and the crossing property with respect to the t-channel. In fact, all the amplitudes constructed by us until now with the view on P and T invariance also automatically satisfy the other requirements. It is this point which has made the previously discussed HW covariants very awkward in their P and T properties: HEPP and WILLIAMS consider only irreducible spinor functions which are obtained by coupling the spinor indices of all in- and outgoing particles (7). The covariants have then certainly complicated crossing properties and therefore get wildly mixed by P and T transformations.

The problem of finding PES has been attacked in the first paper of this series (hereafter referred to as I) (11) for the simple case of the scattering of massive particles with spin configurations (12)

$$(s, 0) \to (s, 0) , \qquad (0, 0) \to (1, 0) , \qquad (\tfrac{1}{2}, 0) \to (\tfrac{3}{2}, 0) .$$

The program is continued here for the case

$$(s, \tfrac{1}{2}) \to (s, \tfrac{1}{2}).$$

In Sect. **2** we define the spinor expansion of the scattering matrix and impose the physical conditions upon the basic spinors. In Sect. **3** we give the PES for $(\frac{1}{2}, \frac{1}{2}) \to (\frac{1}{2}, \frac{1}{2})$ scattering in order to understand the construction principle which is then used in Sect. **4** to solve the general problem $(s, \frac{1}{2}) \to (s, \frac{1}{2})$. The calculations necessary for the proofs are often quite involved. We shall try to keep the description here as compact as possible and give details elsewhere.

2. – Spinor amplitudes.

The scattering matrix for the process $p' + q' \to p'' + q''$ with spins $(s' s'_3, \sigma' \sigma'_3) \cdot (s'' s''_3, \sigma'' \sigma''_3)$ and masses (M', μ') (M'', μ'') is used to define a covariant spinor amplitude by the relation

$$(1) \qquad \{p'' A'' q'' B'' | T | p' A' q' B'\} = \langle p'' s''_3 q'' \sigma''_3 | T | p' s'_3 q' \sigma'_3 \rangle \cdot \\ \cdot D^{s''*}_{s''_3 A''}(L(p''))\, D^{\sigma''*}_{\sigma''_3 B''}(L(q''))\, D^{s'}_{s'_3 A'}(L(p'))\, D^{\sigma'}_{\sigma'_3 B'}(L(q')) ,$$

(11) K. HARDENBERG, K. H. MÜTTER and W. R. THEIS: *Nuovo Cimento*, to be published (1968).

(12) D. N. WILLIAMS (impublished).

where $D^s(\Lambda)$ is the usual spin-s representation of the Lorentz group $D^{[s,0]}$ and $L(p)$ denotes the pure Lorentz transformation from momentum p to rest ([13]). The spinor amplitude has then the simple Lorentz transformation property

$$\{p''A''q''B''|T|p'A'q'B'\} = \{\Lambda p''\bar{A}''\Lambda q''\bar{B}''|T|\Lambda p'\bar{A}'\Lambda q'\bar{B}'\}\cdot \cdot D^{s''*}_{\bar{A}''A''}(\Lambda)\, D^{\sigma''*}_{\bar{B}''B''}(\Lambda)\, D^{s'}_{\bar{A}'A'}(\Lambda)\, D^{\sigma'}_{\bar{B}'B'}(\Lambda) \tag{2}$$

and is presently believed to be analytic in the whole 12-dimensional complex space spanned by any three independent linear combinations of the four-momenta of the system ([10]). Singularities are supposed to occur on the Lorentz invariant surfaces

$$\begin{cases} s=(p'+q')^2\geqslant s_0\,, & t=(p'-p'')^2\geqslant t_0\,, \quad u=(p'-q'')^2\geqslant u_0\,,\\ p'^2\geqslant M_0'^2\,, \quad p''^2\geqslant M_0''^2\,, & q'^2\geqslant \mu_0'^2\,, \quad q''^2\geqslant \mu_0''^2\,, \end{cases} \tag{3}$$

where s_0, t_0, u_0, M_0', M_0'', μ_0', μ_0'' are the lowest physical thresholds in the corresponding channels. For such an amplitude HEPP and WILLIAMS have given an expansion

$$\{p''A''q''B''|T|p'A'q'B'\} = \sum_m F_m K_m(p''A''q''B'', p'A'q'B')\,, \tag{4}$$

where F_m are functions of only the Lorentz invariants containing at most the physical singularities (3). The requirements of K_m being PES can then be put into the following form:

a) P-invariance:

$$K_m(p''A''q''B'', p'A'q'B') = \pi_m K_m(\tilde{p}''\bar{A}''\tilde{q}''\bar{B}'', \tilde{p}'\bar{A}'\tilde{q}'\bar{B}')\cdot \cdot D^{s''*}_{\bar{A}''A''}(L^2(p''))\, D^{\sigma''*}_{\bar{B}''B''}(L^2(q''))\, D^{s'}_{\bar{A}'A'}(L^2(p'))\, D^{\sigma'}_{\bar{B}'B'}(L^2(q'))\,, \tag{5}$$

where we have denoted the parity transformed momenta by $\tilde{p}''$, etc.

b) T-invariance:

$$K_m(p''A''q''B'', p'A'q'B') = \tau_m K_m(\tilde{p}'\bar{A}'\tilde{q}'\bar{B}', \tilde{p}''\bar{A}''\tilde{q}''\bar{B}'')\cdot \cdot D^{s''*}_{\bar{A}''A''}(i\sigma_2)\, D^{\sigma''*}_{\bar{B}''B''}(i\sigma_2)\, D^{s'}_{\bar{A}'A'}(i\sigma_2)\, D^{\sigma'}_{\bar{B}'B'}(i\sigma_2)\,. \tag{6}$$

([13]) We shall use the conventions of H. JOOS: *Fortschr. Phys.*, **10**, 65 (1962).

c) Real analyticity:

$$K_m(p''A''q''B'', p'A'q'B')^* = K_m(p'A'q'B', p''A''q''B'') . \tag{7}$$

d) $s \leftrightarrow u$ crossing:

$$K_m(p''A'' - q'B', p'A' - q''B'') = \gamma_m K_m(p''A''q''\bar{B}'', p'A'q'\bar{B}') \cdot \\ \cdot D^{\sigma''*}_{\bar{B}''B''}(L^{-2}(q'')i\sigma_2)\, D^{\sigma'}_{\bar{B}'B'}(L^{-2}(q')\,i\sigma_2) . \tag{8}$$

e) Factorization:

$$K_m(p''A''q''B'', p'A'q'B') = V^{\mu_2\dots\mu_k}(p''A''p'A')\,\bar{V}_{\mu_1\dots\mu_k}(q''B''q'B') . \tag{9}$$

Thus the only amplitudes contributing to the expansion (4) are those where π_m is the product of the intrinsic parities of the scattering particles. In addition, elastic amplitudes have to satisfy $\tau_m = 1$ and $s \leftrightarrow u$ crossing symmetric ones

$$F_m(s, t, u) = \gamma_m F_m(u, t, s) .$$

3. – Parity eigenstandard covariants for $(\frac{1}{2}\,\frac{1}{2}) \to (\frac{1}{2}\,\frac{1}{2})$ scattering.

For this case HW's standard covariant basis can be written in terms of three arbitrary independent linear combination of the external momenta z_1, z_2, z_3 and the axial vector associated with them

$$w_\mu = i\varepsilon_{\mu\nu\lambda\varkappa} z_1^\nu z_2^\lambda z_3^\varkappa \equiv i\varepsilon_\mu(z_1, z_2, z_3) \tag{10}$$

in the form (see Appendix A)

$$\begin{array}{cccc} & z_1\bar{z}_1 & z_1\bar{z}_2 & z_1\bar{z}_3 \\ & z_2\bar{z}_1 & z_2\bar{z}_2 & z_2\bar{z}_3 \\ & z_3\bar{z}_1 & z_3\bar{z}_2 & z_3\bar{z}_3 \\ & w\bar{z}_1 & w\bar{z}_2 & w\bar{z}_3 \\ \sigma\bar{\sigma} & i\varepsilon(\sigma\bar{\sigma}z_2z_3) , & i\varepsilon(\sigma\bar{\sigma}z_3z_1) , & i\varepsilon(\sigma\bar{\sigma}z_1z_2) . \end{array}$$

Here σ_μ are the Pauli matrices while the spinors

$$(z_i\sigma)_{A''A'} , \qquad (z_i\sigma)_{B''B'} , \tag{11}$$

have been denoted, for brevity, by z_i, $\bar z_i$ etc. the bar indicating that the spinor carries the spin indices of the particles q'', q' ([13]). Under parity σ_μ, $\bar\sigma_\nu$ transform as

$$(12)\qquad \begin{cases} \sigma_\mu \to \dfrac{p''\sigma}{M''}\,\sigma^\mu\,\dfrac{p'\sigma}{M'}\,, \\[2ex] \bar\sigma_\nu \to \dfrac{q''\sigma}{\mu''}\,\sigma^\nu\,\dfrac{q'\sigma}{\mu'}\,. \end{cases}$$

Because of this we find it useful to take for z_i the following combinations of momenta:

$$(13)\qquad \begin{cases} z_1 = Q = \dfrac{1}{2}\left(\dfrac{p'}{M'} + \dfrac{p''}{M''}\right), \\[2ex] z_2 = P = \left(\dfrac{p'}{M'} - \dfrac{p''}{M''}\right), \\[2ex] z_3 = q = \dfrac{1}{2}\left(\dfrac{q'}{\mu'} + \dfrac{q''}{\mu''}\right). \end{cases}$$

With this definition the covariants Q, P, and $\bar q$ immediately become eigenstates of parity with eigenvalues $+$, $-$, and $+$, respectively. In the equal-mass case, $\bar P$ can also be seen to be of negative parity. For nonequal masses the defect can easily be corrected by introducing the momentum

$$(14)\qquad p = \frac{q''}{\mu''} - \frac{q'}{\mu'} = \frac{M''+M'}{\mu''+\mu'}\,P - 2\,\frac{M''-M'}{M''+M'}\,Q - 2\,\frac{\mu''-\mu'}{M''+M'}\,q$$

and substituting $\bar P \to \bar p$ everywhere. For symmetry purposes it is also convenient to use $i\varepsilon(\sigma\bar\sigma pq)$ instead of $i\varepsilon(\sigma\bar\sigma Pq)$ ([14]). Hence an equally good spinor basis is

$$(15)\qquad \begin{cases} \underline{Q\bar q}\,, & \underline{P\bar q}\,, & q\bar q\,, & W\bar q\,, \\ \underline{Q\bar p}\,, & \underline{P\bar p}\,, & q\bar p\,, & W\bar p\,, \\ Q\bar Q\,, & P\bar Q\,, & q\bar Q\,, & W\bar Q\,, \\ \sigma\bar\sigma\,, & i\varepsilon(\sigma\bar\sigma pq)\,, & i\varepsilon(\sigma\bar\sigma qQ)\,, & i\varepsilon(\sigma\bar\sigma QP)\,, \end{cases}$$

where the underlined covariants are parity eigenstates. They represent in the usual Fermi coupling scheme the covariants 1×1, $1\times\gamma_5$, $\gamma_5\times 1$ and $\gamma_5\times\gamma_5$, respectively $(\bar u(p''A'')O_1u(p'A')\bar u(q''B'')O_2u(q'B')$ are abbreviated by $O_1\times O_2)$. In order to find the remaining 12 covariants we first make use of the six

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([14]) Without introducing kinematic singularities.

other Fermi couplings

$$(16)\quad \begin{cases} \gamma^\mu \times \gamma_\mu = \sigma^{+\mu}\bar{\sigma}^+_\mu\,, \\ \gamma^\mu \times \gamma_5\gamma_\mu = -\sigma^{+\mu}\bar{\sigma}^-_\mu\,, \\ \gamma_5\gamma^\mu \times \gamma_\mu = -\sigma^{-\mu}\bar{\sigma}^+_\mu\,, \\ \gamma_5\gamma^\mu \times \gamma_5\gamma_\mu = \sigma^{-\mu}\bar{\sigma}^-_\mu\,, \\ \sigma^{\mu\nu} \times \sigma_{\mu\nu} = -\frac{1}{4}\tau^{\mu\nu+}\bar{\tau}^+_{\mu\nu} = -\frac{1}{4}\tau^{\mu\nu-}\bar{\tau}^-_{\mu\nu}\,, \\ \sigma^{\mu\nu} \times \gamma_5\sigma_{\mu\nu} = -\frac{1}{4}\tau^{\mu\nu+}\bar{\tau}^-_{\mu\nu} = -\frac{1}{4}\tau^{\mu\nu-}\bar{\tau}^+_{\mu\nu}\,, \end{cases}$$

where $\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$ and $\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the γ-matrices, $\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$

$$(17)\qquad \sigma^\pm_\mu = \sigma_\mu \pm \frac{p''\sigma}{M''}\,\sigma^\mu\,\frac{p'\sigma}{M'}\,,$$

and $\tau^\pm_{\mu\nu}$ denotes the spinors

$$(18)\qquad \tau^\pm_{\mu\nu} = \frac{p''\sigma}{M''}\,(\sigma^\mu\sigma_\nu)_- \pm (\sigma_\mu\sigma^\nu)_-\,\frac{p'\sigma}{M'}\,,\qquad (\sigma^\mu\sigma_\nu)_- = \sigma^\mu\sigma_\nu - \sigma^\nu\sigma_\mu\,.$$

(Clearly, the corresponding $\bar{\tau}^\pm_{\mu\nu}$, $\bar{\sigma}^\pm_\mu$ are obtained by substituting $p''M''$, $p'M' \to \to q''\mu''$, $q'\mu'$.) These Fermi couplings serve to replace of the basis states (15) ([14]). This can most easily be seen by forming

$$(19)\quad \begin{cases} \sigma^+\bar{\sigma}^+ = 8\,\dfrac{M''M'}{\mu''+\mu'}\,W\bar{Q} + P[\text{set (15)} - W\bar{Q}]\,, \\ \dfrac{1}{32}\,\tau^+\bar{\tau}^+ = -\left(1 + \dfrac{M''-M'}{M''+M'}\,\dfrac{\mu''-\mu'}{\mu''+\mu'}\right)q\bar{Q} - \left(\dfrac{M''-M'}{M''+M'} + \dfrac{\mu''-\mu'}{\mu''+\mu'}\right)\cdot \\ \qquad \cdot\, i\varepsilon(\sigma\bar{\sigma}qQ) + P[\text{set (15)} - (W\bar{Q},\, qQ,\, i\varepsilon(\sigma\bar{\sigma}qQ))]\,, \end{cases}$$

$$(20)\quad \begin{cases} \dfrac{1}{32}\,\tau^+\bar{\tau}^+ = \left(\dfrac{M''-M'}{M''+M'} + \dfrac{\mu''-\mu'}{\mu''+\mu'}\right)q\bar{Q} + \left(1 + \dfrac{M''-M'}{M''+M'}\,\dfrac{\mu''-\mu'}{\mu''+\mu'}\right)\cdot \\ \qquad \cdot\, i\varepsilon(\sigma\bar{\sigma}qQ) + P[\text{set (15)} - (W\bar{Q},\, q\bar{Q},\, i\varepsilon(\sigma\bar{\sigma}qQ))]\,, \\ \frac{1}{2}(\sigma^+\bar{\sigma}^+ + \sigma^+\bar{\sigma}^-) = i\varepsilon(\sigma\bar{\sigma}QP) + P[\text{set (15)} - (W\bar{Q},\, q\bar{Q},\, i\varepsilon(\sigma\bar{\sigma}qQ),\, i\varepsilon(\sigma\bar{\sigma}QP))]\,, \end{cases}$$

$$(21)\quad \begin{cases} \frac{1}{2}(\sigma^+\bar{\sigma}^+ + \sigma^-\bar{\sigma}^+) = i\varepsilon(\sigma\bar{\sigma}pq) + \\ \qquad + P[\text{set (15)} - (W\bar{Q},\, q\bar{Q},\, i\varepsilon(\sigma\bar{\sigma}qQ),\, i\varepsilon(\sigma\bar{\sigma}Q\,P),\, i\varepsilon(\sigma\bar{\sigma}pq))]\,, \\ \frac{1}{4}(\sigma^+\bar{\sigma}^+ + \sigma^+\bar{\sigma}^- + \sigma^-\bar{\sigma}^+ + \sigma^-\bar{\sigma}^-) = \sigma\bar{\sigma}\,, \end{cases}$$

where P [set (15) — $W\bar{Q}$] denotes collectively a linear combination of basis spinors (15) with all coefficients being polynomials in the Lorentz invariants in which $W\bar{Q}$ does not occur. For this reason we can first substitute $W\bar{Q}\to\sigma^+\bar{\sigma}^+$ from (19) [14], then, since on the one hand the determinant of the system (20) is

$$1+x^2y^2-x^2-y^2 \qquad \text{with } x=\frac{M''-M'}{M''+M'} \qquad y=\frac{\mu''-\mu'}{\mu''+\mu'}\,,$$

and therefore >0 for massive particles, and on the other hand the already substituted covariant $W\bar{Q}$ does not occur in the remaining polynomial P [set (15) $-W\bar{Q}$, $q\bar{Q}$, $i\varepsilon(\sigma\bar{\sigma}qQ)$] on the right-hand side, we can replace $(\tau^+\bar{\tau}^+,\ \tau^+\bar{\tau}^-)\to\to(q\bar{Q},\ i\varepsilon(\sigma\bar{\sigma}qQ))$ [14]. Finally, eq. (21) permits us to eliminate $i\varepsilon(\sigma\bar{\sigma}QP)$, $i\varepsilon(\sigma\bar{\sigma}pq)$, and $\sigma\bar{\sigma}$ by virtue of $\sigma^+\bar{\sigma}^-$, $\sigma^-\bar{\sigma}^+$ and $\sigma^-\bar{\sigma}^-$ [14]. After this only the basis vectors $W\bar{q}$, $W\bar{p}$, $q\bar{q}$, $q\bar{p}$, $P\bar{Q}$, $Q\bar{Q}$ remain to be parity diagonalized. For this we note that the relations

$$\text{(22)}\qquad \left\{\begin{aligned} &\left.\begin{aligned} q^+\bar{q}&=W\bar{q}\\ q^+\bar{p}&=W\bar{p}\end{aligned}\right\} + P[\text{set }(15)-(W\bar{Q},\,q\bar{Q},\,i\varepsilon(\sigma\bar{\sigma}qQ),\,i\varepsilon(\sigma\bar{\sigma}QP),\\ &\qquad\qquad\qquad\qquad i\varepsilon(\sigma\bar{\sigma}pq),\,\sigma\bar{\sigma},\,W\bar{q},\,W\bar{p})]\,,\\ &q^-\bar{q}=2q\bar{q}-q^+\bar{q}\,,\\ &q^-\bar{p}=2q\bar{p}-q^+\bar{q}\end{aligned}\right.$$

can be used to eliminate the first four of them. The remaining two certainly can be expanded as

$$\text{(23)}\qquad P\bar{Q}=\tfrac{1}{2}(P\bar{Q}^++P^-\bar{Q}^-)\,,\qquad Q\bar{Q}=\tfrac{1}{2}(Q\bar{Q}^++Q\bar{Q}^-)\,,$$

but only two of the four covariants on the right-hand side are independent of the 14 others already constructed.

The sixteen independent ones among these eighteen PES are found by the relations derived in Appendix B:

$$\text{(24)}\qquad \frac{1}{M'+M''}\,\frac{1}{\mu'\mu''}\Big\{(\mu'+\mu'')(M''\mu''+M'\mu')Q\bar{Q}^+ + (\mu''-\mu')(M''\mu''-M'\mu')\cdot$$
$$\cdot Q\bar{Q}^- -(\mu'+\mu'')(M''\mu''-M'\mu')\frac{1}{2}P\bar{Q}^+ -(\mu''-\mu')(M''\mu''+M'\mu')\frac{1}{2}PQ^-\Big\}=$$
$$=P[\text{eighteen PES}-(Q\bar{Q}^\pm,\,P\bar{Q}^\pm)]\,,$$

$$\text{(25)}\qquad \frac{1}{\mu'+\mu''}\,\frac{1}{\mu'\mu''}\Big\{(\mu'+\mu'')(M''\mu''+M'\mu')\frac{1}{2}P\bar{Q}^+ + (\mu''-\mu')(M''\mu''-M'\mu')\cdot$$
$$\cdot\frac{1}{2}P\bar{Q}^- -(\mu'+\mu'')(M''\mu''-M'\mu')Q\bar{Q}^+ -(\mu''-\mu')(M''\mu''+M'\mu')QQ^-\Big\}=$$
$$=P[\text{eighteen PES}-(Q\bar{Q}^\pm,\,P\bar{Q}^\pm)]\,,$$

which show that we can drop $Q\bar{Q}^+$ and $P\bar{Q}^+$. It is easy to verify that the basis obtained in this fashion automatically consists of eigenstates of T and crossing. It satisfies the Hermiticity condition if everywhere iP and ip are used instead of P, p. Also it is trivial to see that the factorization condition is fulfilled. Thus we have found a basis of PES. Their intrinsic parities are given in Table I.

TABLE I.

	$Q\bar{p}$	$Q\bar{q}$	$P\bar{p}$	$P\bar{q}$	$q^\pm\bar{q}$	$q^\pm\bar{p}$	$P\bar{Q}^-$	$Q\bar{Q}^-$	$\sigma^\pm\bar{\sigma}^\pm$	$\sigma^\pm\bar{\sigma}^\mp$	$\tau^+\bar{\tau}^\pm$
π_m	$-$	$+$	$+$	$-$	$\pm$	$\mp$	$+$	$-$	$+$	$-$	$\pm$
τ_m	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$+$	$\pm$
γ_m	$+$	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$\pm$	$\mp$	$+$

4. – Parity eigenstandard covariants for $(s, \frac{1}{2}) \to (s, \frac{1}{2})$ scattering.

With the insights gained in the last Section we are now ready to prove the following.

Theorem: The following $4(2s+1)^2$ spinor covariants constitute a basis set of PES for $(s, \frac{1}{2}) \to (s, \frac{1}{2})$ scattering:

$$
(26)\quad
\left\{
\begin{array}{llll}
 & & m & \text{number of covariants} \\
K^{1m}_{[i,j]} = \mathbf{S}(Q)^i(P)^j \left\{ \begin{array}{l} \bar{q} \\ \bar{p} \\ \bar{Q}_- \\ (q^+\bar{\sigma}^-_\mu + q^-\sigma^+_\mu)\,\bar{\sigma}^{\pm\mu} \end{array} \right. & \begin{array}{l} 1 \\ 2 \\ 3 \\ 4{,}5 \end{array} & \begin{array}{l} \left.\begin{array}{l} \\ i+j=s \\ \end{array}\right\} \\ i+j=s-1 \end{array} & \begin{array}{l} 3(2s+1), \\ 2(2s-1), \end{array} \\
K^{2m}_{[i,j]} = \mathbf{S}(Q)^i(P)^j \left\{ \begin{array}{l} q^\pm\bar{q} \\ q^\pm\bar{p} \\ \sigma^-_\mu\bar{\sigma}^{\pm\mu} \\ \sigma^+_\mu\bar{\sigma}^{-\mu} \\ \sigma^+_\mu\bar{\sigma}^{+\mu} \end{array} \right. & \begin{array}{l} 1{,}2 \\ 3{,}4 \\ 5{,}6 \\ 7 \\ 8 \end{array} & \begin{array}{l} \left.\begin{array}{l} \\ \\ i+j=s-\frac{1}{2} \\ \\ \end{array}\right\} \\ i=s-\frac{1}{2} \end{array} & \begin{array}{l} 14s \\ 1 \end{array} \\
K^{3m}_{[i,j,k]} = \mathbf{S}(Q)^i(P)^j(qq)^k_-(q^+\sigma^+_\mu - q^-\sigma^-_\mu)\cdot & & & \\
\qquad\cdot \left\{ \begin{array}{l} \bar{\sigma}^{\pm\mu} \\ q^\pm\bar{\sigma}^{\pm\mu} \\ q^+q^-\bar{\sigma}^{\pm\mu} \end{array} \right. & \begin{array}{l} 1{,}2 \\ 3{,}4\ 5{,}6 \\ 7{,}8 \end{array} & \left.\begin{array}{l} i+j+k=s-1 \\ i+j+k=s-\frac{3}{2} \\ i+j+k=s-2 \end{array}\right\} & 2(2s-1)^2, \\
K^{4m}_{[i,j,k]} = \mathbf{S}(Q)^i(P)^j(qq)^k_-\tau^-_{\mu\nu} \left\{ \begin{array}{l} \bar{\tau}^{\pm\mu\nu} \\ q^\pm\bar{\tau}^{\pm\mu\nu} \\ q^+q^-\bar{\tau}^{\pm\mu\nu} \end{array} \right. & \begin{array}{l} 1{,}2 \\ 3{,}4\ 5{,}6 \\ 7{,}8 \end{array} & \left.\begin{array}{l} i+j+k=s-\frac{1}{2} \\ i+j+k=s-1 \\ i+j+k=s-\frac{3}{2} \end{array}\right\} & 2(2s)^2.
\end{array}
\right.
$$

Here the indices i, j run through $0, \frac{1}{2}, 1, \ldots, s$ while k is restricted to the integers $0, 1, 2, \ldots, s$ (or $s-\frac{1}{2}$). For brevity we have denoted $D^{[i,0]}(Q\sigma)$ by $(Q)^i$ and used $\mathbf{S}$ to symmetrize all indices of the particles with spin s (see Appendix A for more details). Finally, $(qq)_-$ is a short form for $\frac{1}{2}(q^+q^+ - q^-q^-)$ (see Appendix B). The proof of the theorem proceeds by induction. For $s=\frac{1}{2}$ the PES (26) become indeed the same as those in Table I. Consider than the $4(2s+1)^2$ Hepp basis covariants for the case of spin $s>\frac{1}{2}$ (see Appendix A):

$$(27)\qquad Q^{sm}_{[i,j,k]} = \mathbf{S}(Q)^i(P)^j(q)^k\sigma_\mu \left\{ \begin{array}{lrll} & m & & \\ q^\mu\bar{Q} & 1 & k= & -\frac{1}{2}, 0, \frac{1}{2}, \ldots \\ q^\mu\bar{p} & 2 & k= & -\frac{1}{2}, 0, \frac{1}{2}, \ldots \\ q^\mu\bar{q} & 3 & k= & -\frac{1}{2}, 0, \frac{1}{2}, \ldots \\ W^\mu\bar{p} & 4 & k= & 0 \\ W^\mu\bar{q} & 5 & k= & 0, \frac{1}{2}, 1, \ldots \\ W^\mu\bar{Q} & 6 & k= & 0, j=0 \\ i\varepsilon^\mu(\bar{\sigma}, \bar{p}, q) & 7 & k= & 0, \frac{1}{2}, \ldots \\ i\varepsilon^\mu(\bar{\sigma}, q, Q) & 8 & k= & 0, \frac{1}{2}, \ldots \\ i\varepsilon^\mu(\bar{\sigma}, Q, P) & 9 & k= & 0, \frac{1}{2}, \ldots \\ \bar{\sigma}^\mu & 10 & k= & 0, \frac{1}{2}, \ldots \end{array} \right.$$

$$i+j+k = s-\tfrac{1}{2}, \qquad i, j = 0, \tfrac{1}{2}, \ldots,$$

where $(q)^{-\frac{1}{2}}(q)^{\frac{1}{2}} = 1$. A large subset of these covariants can immediately be made PES. For this we distinguish three classes of covariants:

a) $i>\frac{1}{2}$. In this case we can factor out one $(Q)^{\frac{1}{2}}$ as:

$$(28)\qquad Q^{sm}_{[i,j,k]} = \mathbf{S}(Q)^{\frac{1}{2}} Q^{s-\frac{1}{2}\,m}_{[i-\frac{1}{2},j,k]}\,.$$

But $Q^{s-\frac{1}{2}}_{[i-\frac{1}{2},j,k]}$ may be transformed to the spin $(s-\frac{1}{2})$ — PES $K^{s-\frac{1}{2},p,m}_{[i-\frac{1}{2},j,k]}$ according to (26) and the factor $(Q)^{\frac{1}{2}}$ just brings them to the form $K^{s,p,m}_{[i,j,k]}$. In this way we obtain $4(2s)^2$ PES of spin s.

b) $i=0, j>\frac{1}{2}$. Here we take one factor $(P)^{\frac{1}{2}}$ out to write

$$(29)\qquad Q^{sm}_{[0,j,k]} = \mathbf{S}(P)^{\frac{1}{2}} Q^{s-\frac{1}{2}\,m}_{[0,j-\frac{1}{2},k]}$$

and treat $Q^{s-\frac{1}{2}}_{[0,j-\frac{1}{2},k]}$ in the same way as above. This generates $4(4s-1)$ new PES $K^{s,p,m}_{[0,j,k]}$. Thus only eight more PES are missing to span the space. A little more work is necessary to recover them. Consider the remaining Hepp covariants with:

c) $i=0 \quad j=0 \quad k=s-\frac{1}{2}>0$.

They can be written in the form

$$Q^{sm}_{[0,0,s-\frac{1}{2}]} = \mathbf{S}(q)^{\frac{1}{2}} Q^{s-\frac{1}{2}\, m}_{[0,0,s-1]} \qquad m \neq 4, 6 \,. \tag{30}$$

If we expand $Q^{s-\frac{1}{2}\, m}_{[0,0,s-1]}$ in the PES $K^{s-\frac{1}{2},p,m}_{[i,j,k]}$ and set $q=\frac{1}{2}(q^+ + q^-)$ we can express (30) in terms of $2\cdot 4(2s)^2$ PES

$$\mathbf{S} q^{\pm} K^{s-\frac{1}{2},p,m}_{[i,j,k]} \,. \tag{31}$$

To find the eight independent ones notice that $i>\frac{1}{2}$ or $j>\frac{1}{2}$ terms lie in the space treated in *a*) and *b*) (since we can pull out one Q or P and absorb $q^{(\pm)}$ in K). Hence they must be contained in the set of 16 PES:

$$\mathbf{S} q^{\pm} K^{s-\frac{1}{2},p,m}_{[0,0,k]} \,. \tag{32}$$

These are explicitly given:

i) For integer $s \geqslant 2$ by

$$\mathbf{S} q^{\pm} \begin{cases} (qq)_-^{s-2} q^{\pm}(q^+ \sigma^+_\mu - q^- \sigma^-_\mu)\bar\sigma^{\pm\mu}, \\ (qq)_-^{s-1} \tau^-_{\mu\nu} \bar\tau^{\pm\mu\nu}, \\ (qq)_-^{s-2} q^+ q^- \tau^-_{\nu\mu} \bar\tau^{\pm\mu\nu} \,. \end{cases} \tag{33}$$

ii) For half-integer $s \geqslant \frac{5}{2}$ by

$$\mathbf{S} q^{\pm} \begin{cases} (qq)_-^{s-\frac{3}{2}} (q^+ \sigma^+_\mu - q^- \sigma^-_\mu)\bar\sigma^{\pm\mu}, \\ (qq)_-^{s-\frac{5}{2}} q^+ q^- (q^+ \sigma^+_\mu - q^- \sigma^-_\mu)\bar\sigma^{\pm\mu}, \\ (qq)_-^{s-\frac{3}{2}} q^{\pm} \tau^-_{\mu\nu} \bar\tau^{\pm\mu\nu} \,. \end{cases} \tag{34}$$

iii) For $s=1$ by

$$\mathbf{S} q^{\pm}\{q^{\pm}\bar q,\; q^{\pm}\bar p,\; \sigma^{\pm}_\mu \bar\sigma^{\pm\mu},\; \tau^-_{\mu\nu} \bar\tau^{\pm\mu\nu}\} \,. \tag{35}$$

iv) For $s=\frac{3}{2}$ by

$$\mathbf{S} q^{\pm} \begin{cases} (q^+ \sigma^-_\mu + q^- \sigma^+_\mu)\bar\sigma^{\pm\mu}, \\ (q^+ \sigma^+_\mu - q^- \sigma^-_\mu)\bar\sigma^{\pm\mu}, \\ (qq)_- \bar\tau^-_{\mu\nu} \bar\tau^{\pm\mu\nu},\; q^+ q^- \tau^-_{\mu\nu} \bar\tau^{\pm\mu\nu} \,. \end{cases} \tag{36}$$

In the first two cases we can make use of the result of ref. ([11]) that q^+q^+ and q^-q^- can be expanded in terms of

$$(qq)_- = \tfrac{1}{2}(q^+q^+ - q^-q^-),\ Q^1,\ P^1,\ Qq^+,\ Pq^-$$

with only polynomial coefficients (see Appendix B). This allows us to separate out the eight independent covariants (the other are contained in the sets *a*) and *b*) in case i)

$$(37)\qquad \begin{cases} \mathsf{S}(qq)_-^{s-1}(q^+\sigma_\mu^+ - q^-\sigma_\mu^-)\bar\sigma^{\pm\mu}\,, \\ \mathsf{S}(qq)_-^{s-2}q^+q^-(q^+\sigma_\mu^+ - q^-\sigma_\mu^-)\bar\sigma^{\pm\mu}\,, \\ \mathsf{S}(qq)_-^{s-1}q^\pm\tau_{\mu\nu}^-\bar\tau^{\pm\mu\nu}\,, \end{cases}$$

and in case ii)

$$(38)\qquad \begin{cases} \mathsf{S}(qq)_-^{s-\frac{3}{2}}q^\pm(q^+\sigma_\mu^+ - q^-\sigma_\mu^-)\bar\sigma^{\pm\mu}\,, \\ \mathsf{S}(qq)_-^{s-\frac{1}{2}}\tau_{\mu\nu}^-\bar\tau^{\pm\mu\nu}\,, \\ \mathsf{S}(qq)_-^{s-\frac{3}{2}}q^+q^-\tau_{\mu\nu}^-\bar\tau^{\pm\mu\nu}\,, \end{cases}$$

which are indeed the remaining covariants described in (26).

It remains to discuss the cases $s=1$ and $s=\frac{3}{2}$. First we note that the covariant $\mathsf{S}qq(\bar q, \bar p)$ can be used to build the four PES

$$(39)\qquad \mathsf{S}(qq)_-q\bar q\,,\qquad \mathsf{S}(qq)_-\bar p\qquad \text{and}\qquad \mathsf{S}q^+q^-\bar q\,,\qquad \mathsf{S}q^+q^-\bar p$$

by the same argument as the one used above in the cases i) and ii). In the other six covariants of the $s=1$ case,

$$\mathsf{S}q\{W\bar q,\ q\bar Q,\ \sigma\bar\sigma,\ qi\varepsilon(\sigma\bar\sigma, \bar p q),\ qi\varepsilon(\sigma\bar\sigma QP),\ qi\varepsilon(\sigma\bar\sigma Qq)\}\,,$$

we can perform the spin-$\frac{1}{2}$ substitutions (19), ..., (21) to obtain the 12 PES:

$$(40)\qquad \mathsf{S}q^\pm\sigma_\mu^\pm\bar\sigma^{\pm\mu}\,,\qquad \mathsf{S}q^\pm\tau_{\mu\nu}^-\bar\tau^{\pm\mu\nu}\,,$$

which are not obviously contained in the set of 28 PES described in *a*) and *b*). Since only eight among the 16 PES (39) (40) can be independent of the 28 others we have to find eight linear relations. They are described in the Appendix B. In order to save space we give these relations only for the case of equal masses $M''=M'$, $\mu''=\mu'$. (The general case produces involved but trivial mass factors as in the case of $(\frac{1}{2}\frac{1}{2}) \to (\frac{1}{2}, \frac{1}{2})$ scattering.) They are:

(41) $$M^2\mu^2(8\mu(qq)_-\bar{p}^- - Mq^+\tau^+_{\mu\nu}\bar{\tau}^{\mu\nu-}) = P\,(28 \text{ PES of } a) \text{ and } b)) \,,$$

(42) $$M^2\mu^2 q^+ q^- \bar{p}^- = P\,(28 \text{ PES of } a) \text{ and } b)) \,,$$

(43) $$M^2\mu\Big[-\mu^2(qq)_-\bar{q}^+ + \frac{\mu^2}{2}(q^+\sigma^+_\mu - q^-\sigma^-_\mu)\bar{\sigma}^{+\mu} + M^2 q^+\sigma^+_\mu\bar{\sigma}^{+\mu} + \frac{M\mu}{16}\tau^+_{\mu\nu}\bar{\tau}^{+\mu\nu} + \\ + \frac{M\mu}{2}(Q, q)(q^-\sigma^+_\mu + q^+\sigma^-_\mu)\bar{\sigma}^{-\mu}\Big] = P(28 \text{ PES of } a) \text{ and } b)) \,,$$

(44) $$M\mu\Big[M\mu^2 q^+ q^- \bar{q}^+ + \frac{Mt}{2}(q^+\sigma^+_\mu - q^-\sigma^-_\mu)\bar{\sigma}^{-\mu} - 2M^3 q^+\sigma^+_\mu\bar{\sigma}^{-\mu} + \frac{t\mu}{16} q^-\tau^+_{\mu\nu}\bar{\tau}^{+\mu\nu} - \\ - M(\mu^2 + 2(Qq))(q^-\sigma^+_\mu + q^+\sigma^-_\mu)\bar{\sigma}^{+\mu}\Big] = P(28 \text{ PES of } a) \text{ and } b)) \,,$$

(45) $$M^2\mu(q^+\sigma^-_\mu - q^-\sigma^+_\mu)\bar{\sigma}^{\pm\mu} = P\,(28 \text{ PES of } a) \text{ and } b)) \,,$$

(46) $$M^4\mu q^+\sigma^+_\mu\bar{\sigma}^{\pm\mu} - M^2\mu\frac{t}{4}(q^+\sigma^+_\mu - q^-\sigma^-_\mu)\bar{\sigma}^{\pm\mu} = P(28 \text{ PES of } a) \text{ and } b)) \,.$$

Hence all the 16 covariants (39) and (40) are spanned by the eight PES:

$$(q^+\sigma^-_\mu + q^-\sigma^+_\mu)\bar{\sigma}^{\pm\mu} \,, \qquad (q^+\sigma^+_\mu - q^-\sigma^-_\mu)\bar{\sigma}^{\pm\mu} \,, \qquad q^\pm\tau^+_{\mu\nu}\bar{\tau}^{\pm\mu\nu} \,,$$

which indeed obey the formula (26).

In the $s = \frac{3}{2}$ case the four PES of the first row of eq. (36) turn out to be superfluous by the relations

(47) $$M^4\mu^2 q^+(q^+\sigma^-_\mu + q^-\sigma^+_\mu)\bar{\sigma}^{\pm\mu} + M^2\mu^2\frac{t}{2}q^-(q^+\sigma^+_\mu - q^-\sigma^-_\mu)\bar{\sigma}^{\pm\mu} = \\ = P(56 \text{ PES of } a) \text{ and } b)) \,,$$

(48) $$-M^4\mu^2 q^-(q^+\sigma^-_\mu + q^-\sigma^+_\mu)\bar{\sigma}^{\pm\mu} + 2M^2\mu^2\left(M^2 - \frac{t}{4}\right)q^+(q^+\sigma^+_\mu - q^-\sigma^-)\bar{\sigma}^{\pm\mu} = \\ = P(56 \text{ PES of } a) \text{ and } b)) \,,$$

which are obtained by multiplying (45) and (46) with q^+ and q^- and combining them.

This completes the proof.

Also in this general case it is obvious that all covariants can be written in the factorized form (9).

5. – Conclusion.

By keeping the spinor indices of in- and outgoing particles, a rather straightforward construction of PES has been possible. The methods developed in this work can certainly be extended to the more general cases $(s, 0) \to (s + \Delta S, \sigma)$ and $(S, \frac{1}{2}) < (s + \Delta S, \frac{1}{2} + \Delta\sigma)$ which make up all spin configurations, experimentalists will be able to investigate for quite some time. Equipped with this knowledge of the S-matrix expansion all threshold consequences can be derived in a straightforward fashion. Certainly, the PES (26) are a little clumsier to handle for larger spins than the helicity basis. But in the Reggeization procedure the work invested in deriving (26) pays well off by naturally generating minimal pole families, automatically fulfilling all kinematical restrictions, and facilitating the simultaneous treatment of s and u channels.

There is some problem yet as to how one should include zero-mass particles in this framework. This problem will be investigated in future work.

* * *

The authors are grateful to Profs. F. PENZLIN, G. SOMMER, O. STEINMANN and W. THEIS for useful discussions.

APPENDIX A

Hepp-Williams basis.

For the irreducible basis spinor functions of n vectors $(z_1 \dots z_n)$ of rank $[r, s]$ Hepp has given the following expressions:

α) $r = s + 2t > s$

$$Q_{\varkappa}^{[r,s]}(z)_{\alpha\dot{\beta}} = \underset{\alpha\dot{\beta}}{\mathsf{S}} \left[\prod_{i=1}^{s} (z_{\varkappa_l}) \prod_{m=1}^{t} (z_{\varkappa_{s+2m-1}} \times z_{\varkappa_{s+2m}}) \right]_{\alpha\dot{\beta}} . \tag{A.1}$$

β) $r = s - 2t < s$

$$Q_{\varkappa}^{[r,s]}(z)_{\alpha\dot{\beta}} = \underset{\alpha\dot{\beta}}{\mathsf{S}} \left[\prod_{l=1}^{r} ((z_{\varkappa_l}) \prod_{m=1}^{t} (z_{\varkappa_{r+2m-1}} \times z_{\varkappa_{r+2m}})^* \right]_{\alpha\dot{\beta}} . \tag{A.2}$$

γ) $r = s \neq 0$

$$Q_{\varkappa,1}^{[r,r]}(z)_{\alpha\dot{\beta}} = \underset{\alpha\dot{\beta}}{\mathsf{S}} \prod_{l=1}^{r} (z_{\varkappa_l})_{\alpha_l\dot{\beta}_l} , \tag{A.3}$$

$$Q_{\varkappa,2}^{[r,r]}(z)_{\alpha\dot{\beta}} = \underset{\alpha\dot{\beta}}{\mathsf{S}} \prod_{l=1}^{r-1} (z_{\varkappa_l})_{\alpha_l\dot{\beta}_l} (z_{\varkappa_r} \wedge z_{\varkappa_{r+1}} \wedge z_{\varkappa_{r+2}})_{\alpha_r\dot{\beta}_r} . \tag{A.4}$$

δ) $r = s = 0$

(A.5) $$Q^{[0,0]}_{0.1}(z) = 1\ , \qquad Q^{[0,0]}_{\varkappa.2}(z) = \varepsilon_{\mu\nu\varrho\sigma} z^{\mu}_{\varkappa_1} z^{\nu}_{\varkappa_2} z^{\varrho}_{\varkappa_3} z^{\sigma}_{\varkappa_4}\ ,$$

where $\underset{\alpha\dot\beta}{S}$ means complete symmetrization with respect to the indices α and $\dot\beta$ separately,

$$\underset{\alpha\dot\beta}{\mathbf{S}} T_{\alpha_1\ldots\alpha_r\dot\beta_1\ldots\dot\beta_s} = \frac{1}{r!\,s!} \sum_{P,Q} T_{\alpha_{P(1)}\ldots\alpha_{P(r)}\dot\beta_{Q(1)}\dot\beta_{Q(r)}}\ .$$

$\varkappa_i$ are all combinations of z_i giving different covariants, and $z_1 \times z_2$, ... are the spinors

(A.6) $$z_1 \times z_2 = \underset{\alpha}{\mathbf{S}} (z_1\, i\sigma_2\, z_2^*)_{\alpha_1\alpha_2}\ ,$$

while

(A.7) $$(z_1 \wedge z_2 \wedge z_3)_{\alpha\dot\beta} = \varepsilon_{\mu\nu\lambda\varkappa} \sigma^{\mu}_{\alpha\dot\beta} z_1^{\nu} z_2^{\lambda} z_3^{\varkappa}\ .$$

For the special case of 2×2 scattering we have $n = 3$ and we find with

(A.8) $$w_\mu = i\varepsilon_{\mu\nu\lambda\varkappa} z_1^{\mu} z_2^{\nu} z_3^{\varkappa}$$

for $r = s$ the basis vectors:

(A.9) $$Q^{[ss]m}_{[i.j.k]\alpha\dot\beta} = \underset{\alpha\dot\beta}{\mathbf{S}} (z_1)^i (z_2)^j (z_3)^k \sigma_\mu \begin{cases} z_3^{\mu} & m = 1,\ k = -\tfrac{1}{2}, 0, \ldots \\ w^{\mu} & m = 2,\ k = 0, \tfrac{1}{2}, 1, \ldots \end{cases},$$

where

(A.10) $$(z_1)^i = D^{[i.0]}(z_1)\ , \qquad z_3^{\frac{1}{2}} z_3^{-\frac{1}{2}} = 1$$

and

(A.11) $$\underset{\alpha\dot\beta}{\mathbf{S}} (z_1)^i (z_2)^j (z_3)^k = \left(\frac{(s+\alpha)!(s-\alpha)!}{(2s)!}\right)^{\frac{1}{2}} \left(\frac{(s+\beta)!(s-\beta)!}{(2s)!}\right)^{\frac{1}{2}} \cdot \\ \cdot \sum_{\substack{\alpha_i+\alpha_j+\alpha_k=\alpha \\ \dot\beta_i+\dot\beta_j+\dot\beta_k=\dot\beta}} (z_1)^i_{\alpha_i\dot\beta} (z_2)^j_{\alpha_j\dot\beta_j} (z_3)^k_{\alpha_k\dot\beta_k}\ .$$

For $r = s + 1$ we obtain as a basis

(A.12) $$Q^{[s+1s]m}_{[ijk]\alpha\dot\beta} = (s+1\ \alpha | s\ \alpha'\ 1\ \alpha'')\, Q^{[ss]1}_{[ijk]\alpha'\dot\beta} \begin{cases} (z_2 \times z_3)_{\alpha''} & m = 1 & & i+j+k = s\ , \\ (z_3 \times z_1)_{\alpha''} & m = 2 & j = 0 & i+k = s\ , \\ (z_1 \times z_2)_{\alpha''} & m = 3 & & i+j+k = s\ , \end{cases}$$

similarly for $s = r + 1$

(A.13) $$Q^{[r,r+1]m}_{[ijk]\alpha\dot\beta} = (r+1\ \dot\beta | s\ \dot\beta'\ 1\ \dot\beta'')\, Q^{[rr]1}_{[ijk]\alpha\dot\beta'} \begin{cases} (z_2 \times z_3)^{\varkappa}_{\dot\beta''} & m = 1 & & i+j+k = s\ , \\ (z_3 \times z_1)^{\varkappa}_{\dot\beta''} & m = 2 & j = 0 & i+k = s\ , \\ (z_1 \times z_2)^{\varkappa}_{\dot\beta''} & m = 3 & & i+j+k = s\ . \end{cases}$$

Because of its irreducibility this basis cannot be used directly for our purpose. We need a reducible basis transforming as the product of two ingoing particles $[s, 0]\times[\frac{1}{2}, 0]$ and two outgoing ones $[0, s]\times[0, \frac{1}{2}]$. This space is spanned by the irreducible basis vectors of the spaces

$$[s+\tfrac{1}{2}, s+\tfrac{1}{2}], \quad [s+\tfrac{1}{2}, s-\tfrac{1}{2}], \quad [s-\tfrac{1}{2}, s+\tfrac{1}{2}] \quad \text{and} \quad [s-\tfrac{1}{2}, s-\tfrac{1}{2}].$$

In Sect. **3** eq. (27) we gave a basis for this space. We can show that this basis does not lead to kinematic singularities of the invariant amplitudes by expanding all vectors of the HW basis (A.9)-(A.13) in terms of our basis (27). In fact, it is easy to verify that

$$\text{(A.14)} \qquad Q^{[s+\frac{1}{2},s+\frac{1}{2}]m}_{[ijk]\alpha\dot{\beta}} = \left(s+\tfrac{1}{2}\,\alpha|s\,A'\,\tfrac{1}{2}\,B'\right)\left(s+\tfrac{1}{2}\,\dot{\beta}|s\,A''\,\tfrac{1}{2}\,B''\right)\cdot Q^{sm'}_{[i'j'k']}(p''\,A'', q''\,B''; p'\,A'q'B');$$

with the following index relations:

m	i	j	k	m'	i'	j'	k'
1			$\geqslant 0$	3	i	j	$k-\frac{1}{2}$
2			$\geqslant \frac{1}{2}$	5	i	j	$k-\frac{1}{2}$
1		$\geqslant \frac{1}{2}$	$-\frac{1}{2}$	2	i	$j-\frac{1}{2}$	$k=-\frac{1}{2}$
2		$\geqslant \frac{1}{2}$	0	4	i	$j-\frac{1}{2}$	$k=0$
1	$s+\frac{1}{2}$	0	$-\frac{1}{2}$	1	$i-\frac{1}{2}=s$	$j=0$	$k=-\frac{1}{2}$
2	s	0	0	6	$i-\frac{1}{2}=s-\frac{1}{2}$	$j=0$	$k=0$

while

$$\text{(A.15)} \qquad Q^{[s-\frac{1}{2},s-\frac{1}{2}]m}_{[ijk]\alpha\dot{\beta}} = \sum_{i',j',k',m'} G(i\,j\,k\,m; i'\,j'\,k'\,m')\left(s-\tfrac{1}{2}\,\alpha|s\,A'\,\tfrac{1}{2}\,B'\right)\cdot \left(s-\tfrac{1}{2}\,\beta|s\,A''\,\tfrac{1}{2}\,B''\right)Q^{sm'}_{[i',j',k']}(p''\,A''\,q''\,B'', p'\,A'\,q'\,B'),$$

where the coefficients G are given by

$$G(i\,j\,k\,1; i'\,j'\,k'\,m') = \delta_{ii'}\delta_{jj'}\delta_{kk'+\frac{1}{2}}\delta_{m'10},$$

$$G(i\,j\,k\,2; i'\,j'\,k'\,m') = \delta_{ii'}\delta_{jj'}\delta_{kk'-\frac{1}{2}}\delta_{m'9} - \delta_{ii'-\frac{1}{2}}\delta_{jj'}\delta_{kk'}\delta_{m'7} + \delta_{ii'-\frac{1}{2}}\delta_{jj'-\frac{1}{2}}\delta_{kk'}\delta_{m'8}.$$

Similarly one expands $Q^{[s-\frac{1}{2},s+\frac{1}{2}]m}_{[ijk]\alpha\dot{\beta}}$ and $Q^{[s+\frac{1}{2},s-\frac{1}{2}]m}_{[ijk]\alpha\dot{\beta}}$ by using the definitions (A.12) and (A.13).

APPENDIX B

Relations among PES.

Superfluous PES occurring in the diagonalization of parity on the standard covariants are eliminated by the following method. One substitutes $i\varepsilon(\sigma QP)$, $i\varepsilon_\mu(\bar\sigma, Q, P)$ or $i\varepsilon_\mu(\bar\sigma, p, q)$ for σ_μ, $\bar\sigma_\mu$ in some suitable covariants. This creates new covariants which can be expanded in terms of the standard basis with at most polynomial coefficients. After this the standard covariants are replaced by PES. The work is facilitated by the fact that one is only interested in expressing a very small number of PES as a polynomial combination of the others. Thus one only has to peel out the coefficients of these few PES.

1) *Spin* $(\frac{1}{2}\,\frac{1}{2}) \to (\frac{1}{2}\,\frac{1}{2})$ *scattering.* Here we start out with the substitution $\bar\sigma_\mu \to i\varepsilon_\mu(\bar\sigma, Q, p)$ in $P\bar q$ and $\sigma_\mu \to i\varepsilon_\mu(\sigma QP)$, $\bar\sigma_\mu = i\varepsilon_\mu(\bar\sigma, p, q)$ in $\sigma\bar\sigma$. This gives the relation

$$\text{(B.1)} \qquad Pi\varepsilon(\bar\sigma QPq) = i\varepsilon(\sigma QPq)\bar P + iP^2\varepsilon(\sigma\bar\sigma Qq) + i(Pq)\varepsilon(\sigma\bar\sigma QP)\ ,$$

$$\text{(B.2)} \qquad i\varepsilon_\mu(\sigma QP)\,i\varepsilon^\mu(\bar\sigma Pq) = -P^2(Qq)\sigma\bar\sigma + (Qq)P\bar P + P^2 q\bar Q - (Pq)P\bar Q\ .$$

Introducing the eighteen PES (cf. eqs. (15), (19)-(23)) we are led to the relations (24) and (26), which permit us to eliminate $Q\bar Q^+$ and $P\bar Q^+$.

2) *Spin* $(\frac{1}{2}\,1) \to (\frac{1}{2}\,1)$ *scattering.* In order to avoid astronomical formulae we give the relations only for equal masses $M' = M''$, $\mu' = \mu''$. One uses the same method as above to derive the equations

$$\text{(B.3)} \qquad Wq\bar P = -iP^2\varepsilon(\sigma\bar\sigma Qq) + P(28 \text{ HW covariants of } a) \text{ and } b))\ ,$$

$$\text{(B.4)} \qquad Wi\varepsilon(\sigma\bar\sigma Qq) = Q^2 q^1 \bar P + P(28 \text{ HW covariants of } a) \text{ and } b))\ ,$$

$$\text{(B.5)} \qquad Wi\varepsilon(\sigma\bar\sigma Pq) = -P^2 q^1 \bar Q + P^2(Qq)q\sigma\bar\sigma + P(28 \text{ HW covariants of } a) \text{ and } b))\ ,$$

$$\text{(B.6)} \qquad Wq\bar Q = Q^2 qi\varepsilon(\sigma\bar\sigma Pq) + (Qq)qi\varepsilon(\sigma\bar\sigma QP) + P(28 \text{ HW covariants of } a) \text{ and } b))\ ,$$

$$\text{(B.7)} \qquad W\sigma\bar\sigma = iq\varepsilon(\sigma\bar\sigma QP) + P(28 \text{ HW covariants of } a) \text{ and } b))\ ,$$

$$\text{(B.8)} \qquad Wi\varepsilon(\sigma\bar\sigma QP) = -P^2 Q^2 q\sigma\bar\sigma + P(28 \text{ HW covariants of } a) \text{ and } b))\ ,$$

where we have explicitly shown only the contribution of eight standard covariants of interest for the derivation of the relations while the remaining ones are lumped together in the polynomial expression P(28 HW covariants of a) and b)). The relations (41)-(46) result from eqs. (B.3)-(B.8) substituting HW covariants by the 28 PES of a) and b) and 16 PES of eq. (35).

RIASSUNTO (*)

Si costruisce una base spinoriale, covariante secondo Lorentz, per lo scattering di configurazioni di spin $(s, \frac{1}{2}) \to (s, \frac{1}{2})$, per particelle dotate di massa producenti ampiezze scalari libere da singolarità cinematiche, analitiche reali, ed autostati di P, T e dell'incrocio $s \leftrightarrow u$.

(*) *Traduzione a cura della Redazione.*

Инвариантные функции и дискретные симметрии. - II

Резюме (*). — Конструируется Лорентц-ковариантный спинорный базис для рассеяния спиновых конфигураций $(s, \frac{1}{2}) \to (s, \frac{1}{2})$ для массивных частиц, посредством образования скалярных амплитуд, которые являются свободными от кинематических сингулярностей, вещественными, аналитическими, и собственных состояний для P, T и $s \leftrightarrow u$ кроссинга.

(*) *Переведено редакцией.*