One-loop critical exponents for Ginzburg-Landau theory with Chern-Simons term.*

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Abstract

The critical exponents of Ginzburg-Landau theory (scalar QED) with a Chern-Simons term in the action are calculated at the one-loop level as functions of the statistics determining parameter θ. The calculation is performed in $D = 3$ dimensions which emphasizes the infrared aspects of the critical phenomena, as suggested by Parisi. Ordinary scalar QED is obtained in the limit $\theta \to 0$, in which case the Chern-Simons term generates a topological mass and serves merely as an infrared regulator. In general, the term deforms the statistics of the complex field slightly into the direction of fermions. Only for large θ there is an infrared-stable fixed point in the renormalization flow in the case of a single complex field.

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With the discovery of integer and fractional quantum Hall effects and high-T_c superconductivity, physics in 2+1 dimensions has attracted considerable attention. An intriguing property of the reduced dimensionality is the fact that it allows for the appearance of a topological term in the action of the vector potential, a so-called Chern-Simons term. This term is at the root of various special effects. It produces a gauge-invariant photon mass and, most spectacularly, changes the statistics of the particles to which it is coupled into fractional statistics (an impossibility in a 3+1-dimensional local quantum field theory).

In this note we would like to explore two different aspects of a Chern-Simons term in three spatial dimensions. On the one hand, we take advantage of the fact that such a term renders the photon massive. In former days there where two principal mechanism by which this happened. One is the so-called Meissner-Higgs mechanism, where the photon aquires a mass by absorbing a massless Goldstone particle. The other is Debeye screening which occurs in a charged plasma. Both mechanisms are important in many physical systems and the same thing may be true for the new mechanism.

The second aspect of a Chern-Simons term is the deformation of the statistics of the field it is coupled to. We want to explore the consequences of such a deformation with respect to an old problem in the description of the superconducting phase transition in three space dimensions. The Ginzburg-Landau theory does not allow for the standard estimate of critical exponents in 3 dimensions via the renormalization group since the symmetric phase of the model is plagued by infrared divergencies, due to the masslessness of the photon. To circumvent this problem one can formulate the theory in arbitrary dimension D and construct an expansion for the critical exponents in powers of the deviation $\epsilon = D - D_c$ from the critical dimension D_c, which is the dimension where the coupling constant is dimensionless and the critical exponents take there mean-field values. The critical dimension of the Ginzburg-Landau model is $D_c = 4$. Close to the critical dimension, the ϵ-expansion has the advantage that it automatically provides a small parameter, viz. ϵ, in which the fixed point and the corresponding critical exponents can be computed as a power series. So, for ϵ small, this procedure is systematic and consistent. This is, of course, no longer true when ϵ is set to, for example, 1. In that case, the power series in ϵ turns into an asymptotic series which only after a resummation gives reliable results. It is one of the pleasant surprises of nature that in a pure ϕ^4-theory, despite the fact that ϵ ceases to be small, one-loop calculations have yielded results which are of the correct order of magnitude.

A second possibility to circumvent the infrared problems of massless photons is to consider an artificially enlarged theory with n components of the complex scalar field in which one can study the limit $n \to \infty$ where a mean-fild theory exists and perform a perturbation around that yielding results as an expansion is $1/n$.

Both methods have been employed in the context of the Ginzburg-Landau theory by Halperin, Lubensky and Ma [1]. Their main result was that it is impossible to calculate the critical exponents in a low-order ϵ-expansion due to the absence of an infrared-stable fixed point. Only for an unphysical large number of field components ($n > 365.9$) does there exist such a fixed point. They interpreted this result as signaling a first-order transition for $n < 365.9$ and ϵ small. This is in accord with the results obtained by Coleman and Weinberg [2] who studied electrodynamics of *massless* scalar mesons in four dimensions and discovered that at the one-loop level the photon becomes massive. A study of their effective action shows a precocious onset of the Higgs mechanism with a sudden appearance of a finite photon mass out of the symmetric phase. This is typical for a first-order transition.

The critical exponents corresponding to the fixed point for $n > 365.9$ are according to Halperin, Lubensky and Ma [1] given by

$$\eta = -18\,\epsilon\, n^{-1}, \tag{1}$$

$$\begin{aligned}\nu^{-1} &= 2 - \frac{1}{2}\epsilon\,(n+8)^{-1}[n + 2 - 216\,n^{-1} + (n+2)n^{-1}(n^2 - 360n - 2160)^{1/2}] \\ &\approx 2 - \epsilon + \epsilon\, 96\, n^{-1} + O(n^{-2}),\end{aligned} \tag{2}$$

to first order in ϵ.

In the large-n calculation in fixed ($D = 3$) dimension these authors obtained a fixed point with critical exponents

$$\eta = -4.053\, n^{-1} + O(n^{-2}), \quad \nu = 1 - 9.72\, n^{-1} + O(n^{-2}). \tag{3}$$

On the basis of this result they conjectured that also in $D = 3$ there exists a critical value n_c such that for $n > n_c$ the transition is second order, and first order for $n < n_c$. The authors recognized that this conclusion could not be reliable since there exists an experimantally well-studied system of a smectic liquid crystal whose transition to the nematic phase can be described by a Ginzburg-Landau theory but which has regimes of both first and second order. In fact, it was possible to show by a duality transformation of the Ginzburg-Landau model to a pure ϕ^4 theory on a lattice that there exists a tricritical point characterized by a certain ratio of penbetration depth versus coherence length [3] below which the transition is of first order. This theoretical conclusion has meanwhile been corroborated by Monte Carlo simulations [4]. The failure of the Halperin, Lubensky and Ma calculation in the continuum indicates the failure of the ϵ-expansion at the large value $\epsilon = 1$.

The circumstance that a Chern-Simons term imparts a (gauge-invariant) mass to the photon allows us to use this term as an infrared regulator and carry out a direct computation of the critical exponents in three dimensions, making the large-n limit superfluous. At the end of the calculation the θ parameter, which multiplies the Chern-Simons term, may be set to zero to obtain the results pertaining to the usual Ginzburg-Landau model.

The topologically massive Ginzburg-Landau model is defined by the Hamiltonian

$$\begin{aligned} H_0 = {} & \left|(\partial_\mu - ie_0 A^0_\mu)\phi^0\right|^2 + m_0^2|\phi^0|^2 + \lambda_0|\phi^0|^4 \\ & + \frac{1}{4}(F^0_{\mu\nu})^2 + \frac{1}{2\zeta_0}(\partial_\mu A^0_\mu)^2 + i\frac{\theta}{2}e_0^2\epsilon_{\mu\nu\lambda}A^0_\mu\partial_\nu A^0_\lambda. \end{aligned} \tag{4}$$

Here, $\mu, \nu, \lambda = 1, 2, 3$ are space indices, $\epsilon_{\mu\nu\lambda}$ is the antisymmetric Levi-Civita symbol in three dimensions and ϕ^0 is a complex scalar field which we parametrize as $\phi^0 = \frac{1}{\sqrt{2}}(\phi^0_1 + i\phi^0_2)$. We have added a gauge-fixing term to the Maxwell term, with ζ_0 the gauge-fixing parameter, and the last term is the Chern-Simons term. Without this term (4) is the usual Ginzburg-Landau model. It has been shown by Semenoff and Sodano [5] that in the low-energy limit, which is the relevant domain for us, the theory (4) describes fields with exotic spin and statistics. Specifically, the spin is given by $1/4\pi\theta$.

Calculations of critical properties in 3 dimensions where first perormes in a pure scalar field theory by Parisi [6]. Near the critical point the system has only one relevant length scale, viz. the correlation length m^-1 which diverges at this point. This implies that m is the only mass scale available to convert dimensionful coupling constants to dimensionless ones. The Ginzburg-Landau model has two (bare) coupling constants: λ_0 and the electric charge e_0 which have mass dimension $4-D$ and $(4-D)/2$, respectively, so that the corresponding dimensionless coupling constants are $g_0 = \lambda_0/m^{4-D}$ and $\alpha_0 = e_0^2/m^{4-D}$. On dimensional grounds it follows that a perturbative expansion in powers of the coupling constants λ_0 and e_0^2 is, in effect, an expansion in powers of the dimensionless quantities g_0 and α_0. Since $m \to 0$ at the critical point, g_0 and α_0 tend to infinity and a perturbative expansion in terms of these bare parameters breaks down at the transition point. To find expansion parameters which remain finite at the critical point Parisi uses the fact that according to the experimentally known scaling laws of statistical mechanics the connected two-point correlation function for the scalar field

$$G_0(\mathbf{k}) =< \phi^0(\mathbf{k})\phi^0(-\mathbf{k}) > \tag{5}$$

behaves near the critical point as

$$G_0(0) \sim m^{\eta-2} \tag{6}$$

with η being a critical exponent. A non-zero η implies that the mass-dimension of te bare field ϕ^0 has shifted from the canonical value $\frac{1}{2}(D-2)$ to

$$d_\phi = \frac{1}{2}(D-2+\eta). \tag{7}$$

To comply with the scaling law (6), $G_0(\mathbf{k})$ can be written for small momenta as

$$G_0(\mathbf{k}) = \frac{Z_\phi}{k^2+m^2+O(k^4)}, \tag{8}$$

with the dimensionless factor Z_ϕ the so-called field renormalization constant, which near the critical point behaves as m^η. In terms of this factor the critical exponent η is given by the function

$$\gamma := m \frac{\partial}{\partial m} \ln(Z_\phi) \tag{9}$$

evaluated at the critical point. The factor Z_ϕ may be cancelled from the Green function by introducing a renormalized field

$$\phi := Z_\phi^{-1/2} \phi^0. \tag{10}$$

The scaling laws of statistical mechanics tell us in addition that the one-particle irreducible n-point correlation function, or vertex function, behaves near the critical point as

$$\Gamma_0^{(n)} \sim m^{D - n d_\phi}. \tag{11}$$

A dimensionless expansion parameter which, in contrast to $g_0 = \lambda_0/m^{D-4}$, remains finite at the critical point is now easily constructed, namely

$$g := m^{D-4} \lambda. \tag{12}$$

with $\lambda := \Gamma^{(4)}$ the renormalized 4-point function which is related to the bare vertex function $\Gamma_0^{(4)}$ via $\Gamma^{(4)} = Z_\phi^2 \Gamma_0^{(4)}$, since it involves 4 scalar fileds. Close to the critical point where $g_0 = \lambda_0/m^{D-4} \to \infty$, g tends to a finite constant g^*. Indeed, $g^* \sim m^{D-4} m^{2\eta} m^{D-d_\phi} \sim 1$. A similar construction can be given for the electric coupling constant e_0.

The fact that the thus constructed coupling parameters remain finite at the critical point allows for a perturbation expansion of the critical exponents. To this end one writes the Hamiltonian (4) $H_0 = H + \delta H$ as a sum of the renormalized Hamiltonian H

$$\begin{aligned} H = & \frac{1}{2} (\partial_\mu \phi_a)^2 + \frac{1}{2} m^2 \phi_a^2 + \frac{1}{2} e^2 A_\mu^2 \phi_a^2 - e \epsilon_{ab} A_\mu \phi_a \partial_\mu \phi_b + \frac{\lambda}{4} \left(\phi_a^2\right)^2 \\ & + \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\zeta} (\partial_\mu A_\mu)^2 + i \frac{\theta}{2} e^2 \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \end{aligned} \tag{13}$$

and counter terms δH

$$\begin{aligned} \delta H = & \frac{1}{2} (Z_\phi - 1) (\partial_\mu \phi_a)^2 + \frac{1}{2} (Z_\phi m_0^2 - m^2) \phi_a^2 + \frac{1}{2} e^2 (Z_\phi - 1) A_\mu^2 \phi_a^2 \\ & - e (Z_\phi - 1) \epsilon_{ab} A_\mu \phi_a \partial_\mu \phi_b + \frac{\lambda}{4} (Z_\lambda - 1) \left(\phi_a^2\right)^2 + \frac{1}{4} (Z_A - 1) F_{\mu\nu}^2, \end{aligned} \tag{14}$$

whose form takes advantage of the gauge invariance of the theory. The first term contains only the renormalized fields and parameters

$$
\begin{aligned}
&A_\mu = Z_A^{-1/2} A_\mu^0, \quad e = Z_A^{1/2} e_0 \\
&\phi_a = Z_\phi^{-1/2} \phi_a^0, \quad \zeta = Z_A^{-1} \zeta_0 \\
&\lambda = Z_\lambda^{-1} Z_\phi^2 \lambda_0.
\end{aligned}
\tag{15}
$$

In (13) and (14), the field labels $a, b = 1, 2$ and ϵ_{ab} is the two-dimensional Levi-Civita symbol. The statistics determining parameter θ is dimensionless and will not be renormalized. This is why it does not require a Z factor. It may be considered as a free parameter of the theory. We shall be working in the gauge $\zeta = 0$, thus effectively setting $\partial_\mu A_\mu$ to zero in (13).

The Feynman rules we obtain from (13) and (14) are the usual ones for scalar QED apart from the photon propagator, which due to the presence of the Chern-Simons term becomes

$$
= \frac{1}{k^2 + \theta^2 e^4} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \theta e^2 \epsilon_{\mu\nu\lambda} \frac{k_\lambda}{k^2} \right), \tag{16}
$$

showing the topologically acquired photon mass $m_A = \theta e^2$. When working with a massive theory, we may chose vanishing external momenta as a renormalization point. The renormalization conditions are then:

$$
\begin{aligned}
&\Gamma_{11}^{(2)}(k^2 = 0) = m^2, \\
&\frac{\partial}{\partial k^2} \Gamma_{11}^{(2)}(k^2)|_{k^2=0} = 1, \\
&\Gamma_{11,11}^{(4)}(k^2 = 0) = 6\lambda \\
&\Gamma_{11}^{(2,1)}(k^2 = 0) = 1 \\
&\frac{\partial}{\partial k^2} \Pi_{\mu\mu}^{(2)}(k^2)|_{k^2=0} = 2,
\end{aligned}
\tag{17}
$$

where the Γ's are the renormalized scalar field correlation functions, with the indices indicating the type of the external fields. The correlation function $\Gamma^{(2,1)}$ stands for the two-point function with one mass insertion, and $\Pi_{\mu\nu}^{(2)}$ is the (renormalized) gauge field correlation function. The conditions (17) fix the counter terms appearing in (14). The advantage of chosing the Lorentz gauge ($\zeta = 0$) is that the diagrams depicted in Fig. 1 are zero for zero external momenta, which is our renormalization point. It is straightforward to calculate the other one-loop diagrams. The important results are

diagram	$D = 3$	$D = 4 - \epsilon$	
	$\frac{2}{3\pi}\frac{e^2}{m+\theta e^2}k^2$	$\frac{3}{8\pi^2}\frac{e^2}{\epsilon}k^2$	(18a)
			(18b)
	$-\frac{1}{24\pi}\frac{e^2}{m}k^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$	$-\frac{1}{24\pi^2}\frac{e^2}{\epsilon}k^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$	(18c)
	$-\frac{1}{2\pi}\frac{\lambda}{m}$	$-\frac{1}{32\pi^2}\frac{\lambda}{\epsilon}$	(18d)
	$\frac{15}{2\pi}\frac{\lambda^2}{m}$	$\frac{15}{2\pi^2}\frac{\lambda^2}{\epsilon}$	(18e)
	0	$\frac{9}{4\pi^2}\frac{e^4}{\epsilon},$	(18f)
			(18g)

where for comparison we included also the results obtained in an ϵ-expansion for ordinary scalar QED. In the latter case we did not introduce an arbitrary mass parameter to render the coupling constants dimensionless. In the diagrams with four external legs we included a factor of three to account for the s,t and u channel.

The fact that the last diagram is zero is a remarkable property of topologically generated mass. In the ϵ-expansion of the ordinary Ginzburg-Landau theory, this diagram is proportional to e^4 and this is responsible for the non-existence of a fixed point. It would be determined by the solution of a quadratic equation in e^2 but the sign of the diagram is such that apart from the trivial solution $e^2 = 0$ the equation has no solution, and thus no non-trivial fixed point. In the presence of the Chern-Simons term, however, the vanishing of this diagram guarantees, as we shall demonstrate below, that there exists always a non-trivial fixed point.

The evaluation of the diagrams in the table with the normalization conditions (17) yield the following values for the various Z factors to one-loop order:

$$\begin{aligned} Z_\phi &= 1 + \frac{2}{3\pi}\frac{1}{\alpha^{-1}+\theta}, \\ Z_A &= 1 - \frac{\alpha}{24\pi}, \\ Z_{\phi^2} &= 1 + \frac{g}{2\pi} - \frac{2}{3\pi}\frac{1}{\alpha^{-1}+\theta} \\ Z_\lambda &= 1 + \frac{5}{4\pi} g. \end{aligned} \tag{19}$$

The factor Z_{ϕ^2} shows up in the counter term corresponding to the fourth diagram in Table 1:

$$: Z_\phi Z_{\phi^2} - 1, \tag{20}$$

with the broken line indicating a mass insertion. From the counter terms (19) we now calculate the critical exponents in the standard way and find

$$\begin{aligned} \gamma &= m\frac{\partial}{\partial m}\ln(Z_\phi)|_{\lambda_0,e_0} = -\frac{2}{3\pi}\frac{\alpha^{-1}}{(\alpha^{-1}+\theta)^2} \\ \beta(\alpha) &= m\frac{\partial}{\partial m}\alpha|_{\lambda_0,e_0} = -\alpha + \frac{\alpha^2}{24\pi} \\ \bar{\gamma} &= m\frac{\partial}{\partial m}\ln(Z_{\phi^2})|_{\lambda_0,e_0} = -\frac{g}{2\pi} + \frac{2}{3\pi}\frac{\alpha^{-1}}{(\alpha^{-1}+\theta)^2} \\ \beta(g) &= m\frac{\partial}{\partial m} g|_{\lambda_0,e_0} = -g + \frac{5}{4\pi}g^2 - \frac{4}{3\pi}\frac{\alpha^{-1}}{(\alpha^{-1}+\theta)^2} g. \end{aligned} \tag{21}$$

The set of equations yields apart from the trivial fixed point, the infrared-stable fixed point

$$\alpha^* = 24\pi, \quad g^* = \frac{4\pi}{5}\left[1 + \frac{32}{(1+24\pi\theta)^2}\right], \tag{22}$$

with the corresponding critical exponents

$$\eta = \gamma(g^*, \alpha^*) = -\frac{16}{(1+24\pi\theta)^2} \tag{23}$$

and

$$\nu^{-1} = 2 + \bar{\gamma}(g^*, \alpha^*) = \frac{8}{5}\left[1 + \frac{2}{(1+24\pi\theta)^2}\right] \tag{24}$$

Eq. (22) shows that topologically massive scalar QED has always a fixed point at the one-loop order. However, the value of the critical exponent η is not always physical. On general grounds η and ν should fulfill the inequalities $\eta > 2 - D$ and $\nu > 1$. The latter condition is fulfilled for every value of θ, but the former only for values of the statistics determining parameter θ larger than the threshold value

$$\theta_{\rm th} = 1/8\pi. \tag{25}$$

As a result, η varies from 0 ($\theta = \infty$) to -1 ($\theta = 0$) and ν from $\frac{5}{8} = .625$ ($\theta = \infty$) to $\frac{5}{9} \approx .556$ ($\theta = \infty$). This restriction on θ is an analog of the condition $n > 365.9$ on the number of field components found in [1]. It forbids us to set $\theta = 0$ and go to the standard Ginzburg-Landau model which we really would like to do.

According to Semenoff and Sodano [5] the theory decribes fields with spin given by $1/4\pi\theta$. Thus the threshold value (25) corresponds to a spin-2 field implying that in an application to particles with fractional statistics we are restricted to spins smaller than two.

Reliable results are obtained in the limit $\theta \to \infty$ in which case the theory goes over into a pure spin-0 $\lambda\phi^4$ filed theory where we recover the known critical exponents

$$\eta \to 0, \quad \nu^{-1} = \frac{8}{5} = 2 - \frac{n+2}{n+8}, \tag{26}$$

where n is the number of real field components.

It follows from (24) that the value of the critical exponent ν at finite θ is smaller than that at $\theta = \infty$. This fact surprises us for the following reason. Anyons, having spin, experience an effective repulsion (centrifugal barrier) which is absent in the case of spinless particles. We, therefore, expect that the effect of turning on the spin of the field by letting $1/\theta > 0$, is to increase the coupling constant g. This is indeed the case as can be seen from Eq. (22) giving the value of g at the critical point. For a pure $\lambda\phi^4$ theory this would imply a larger value for the critical exponent ν. In the charged case, however, the function $\bar{\gamma}$, which according to (24) determines ν has a second term [see (21)] that for increasing $1/\theta$ moves in a direction opposite to that of the first term. In fact, this second term dominates, resulting in the positive sign in front of the second term in (24) and a smaller value for ν when $1/\theta$ increases. We suspected a sign error but could not detect any.

We have seen that ordinary scalar QED, corresponding to taking $\theta = 0$, could not be recovered from the topologically massive theory, because it leads to a value $\eta < -1$, which is unphysical. In order to be able to set $\theta = 0$ we generalize the theory to one with $n/2$ complex fields, with n an even integer. This leads to the folowing changes in the results given in Table 1. The second diagram has to be multiplied with $n/2$, the third with $(n+2)/4$ and the fourth diagram obtains a factor $(n+8)/10$. The resulting critical point is

$$\alpha^* = \frac{48\pi}{n}, \quad g^* = \frac{8\pi}{n+8}\left[1 + \frac{64\,n}{(n+48\pi\theta)^2}\right], \tag{27}$$

with the critical exponents

$$\eta = -\frac{32\,n}{(n+48\pi\theta)^2} \tag{28}$$

and

$$\nu^{-1} = 2 - \frac{n+2}{n+8} - \frac{n-4}{n+8}\,\frac{32\,n}{(n+48\pi\theta)^2}. \tag{29}$$

Eq. (27) shows that, as is usual the case, sufficient large values of n render α^* and g^* small enough so as to make a perturbative expansion in these parameters reliable. In the limit $\theta \to \infty$ we recover, of course, the well-known results for the pure O(n)-symmetric $\lambda\phi^4$ theory:

$$\eta \to 0, \quad \nu \to \frac{n+8}{n+14}. \tag{30}$$

On the other hand, for $\theta = 0$ we find the values

$$\eta = -\frac{32}{n}, \quad \nu = \frac{n(n+8)}{n^2 - 18\,n + 128} = 1 + \frac{26}{n} + O(\frac{1}{n^2}), \tag{31}$$

the first of which makes sense only if $n > 32$. The threshold value $n_{\rm c} = 32$ for the number of components of the scalar field should be compared to the much larger value $n_{\rm c} \approx 365.9$ obtained by Halperin, Lubensky and Ma [1] in the ϵ-expansion. We also note that our critical exponents (31) differ considerable from those obtained by these authors in the large n-limit for $D = 3$, Eq. (3).

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