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Finite-Size and Temperature Properties of Matter and Radiation Fluctuations in Closed Friedmann Universe

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Abstract

We investigate the finite-size effects upon the free energy of fluctuating matter and radiation as a function of temperature in a slowly evolving Friedmann universe. The equation of state turns out to have the form $P = \rho/3\rho$ for radiation with arbitrary coupling. The finite-size effects are negligible small for the standard model of the universe.
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1 Introduction

The global topology of space-time may have effects upon the local properties of the universe such as energy density and pressure of all fluctuating fields. Consider, for example, three possible topologically distinct Friedmann universes, closed, open flat and open hyperbolic. In contrast to the open flat and hyperbolic universes, the closed universe has a finite size and possess a discrete spectrum of matter and radiation fluctuations. As a result, its partition function contains an additional term, a difference between a spectral sum and an integral, which can be determined by the Euler-Maclaurin formula. In a flat space-time, such differences arise from the energy spectrum of electromagnetic waves between two conducting plates and give rise to an attraction known as the Casimir effect [1] discussed extensively in the literature (see the reviews in [2-3]).

An important quantum effect in a closed Friedmann universe is that of particle creation [4-6]. It is a dynamic effect and depends sensitively on the speed of evolution. This will be ignored here assuming the evolution to be sufficiently slow, to get pure finite-size effects (see e.g. [7,8]).

The purpose of this note is to calculate the finite-size properties of matter and radiation fuctuations in a closed universe. The difference between a field energy in the infinite and finite universe will be called Casimir energy of that field.

Until now, Casimir energies have been investigated only for zero temperature [3-6, 9, 10] (vacuum case). In this paper we shall derive it for any temperature. Although it is clear that in the systems with an entropy $s >> 1$, as in our universe, the finite-size corrections should be negligible small and no Casimir effect is observable. This will be seen explicitly from the high-temperature limit formulas obtained in our paper.

For the description of both fluctuating matter and radiation we consider generically a free massive scalar field with an arbitrary coupling to the gravity. If the mass is set equal to zero the result can be applied to fluctuating radiation. The free energy of the scalar field is obtained by performing the functional integral over the Fourier components of free field [11,12]. In the static universe, there is no problem to do this since the oscillator frequencies are time independent.

The regularization of the infinite sums is performed using various standard methods existing in the literature [13-17,3-5]. The expressions for the energy density and the pressure follow from the free energy by the thermodynamic rules.

Multidimensional cosmology is an important example, where static space calculation of the Casimir effect can be applied since models with one or more static compact inner spaces represent an interesting class of the solutions with spontaneous compactification [18]. The vacuum Casimir energy in this case may have a considerably effect on the evolution of an external universe we live in. Finite-temperature Casimir effect takes place if there is thermalization in compactified dimensions.

2 Free scalar field in a slowly evolving Robertson-Walker-Friedmann universe

The action of a real scalar field on the background of an arbitrary gravitational field is [5,6]

$$S = \int d^{D+1}x|g|^{1/2}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right] \tag{2.1}$$

with the signature of the $D+1$-dimensional space-time metric $g_{\mu\nu}$ being $-++\ldots+$. For a massive scalar field the most general harmonic potential

$V(\varphi)$ is

$$V(\varphi) = \frac{\xi R \varphi^2}{2} + \frac{m^2 \varphi^2}{2} \tag{2.2}$$

where R is the scalar curvature of a spacetime, m is a mass of scalar field φ and ξ is the coupling constant. A free particle has $\xi = 0$ [12]. Without additional work we treat here the general case of arbitrary ξ. The value $\xi = (D-1)/4D$ makes the massless scalar action conformally invariant.

We choose a Robertson-Walker-Friedmann (RWF) universe as a background and assume that it evolves so slowly that it can be assumed to be static for the purpose of our calculation. Later in Appendix B we shall specify precisely the condition under which the slowness assumption is valid. Thus, the metric is taken to be

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2 dl^2 \tag{2.3}$$

where a is scale factor and dl^2 is the metric of a D-dimensional space of constant curvature. The space dimensionality treated in this paper will be $D = 3$ with the geometry given by

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta = dr^2 + f^2(r)(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.4}$$

where $f(r)$ can be $\sin r, r, \sinh r$ for the different spaces of constant curvature with $\chi = +1, 0, -1$, respectively. The Casimir effect takes place only for $\chi = +1$. (There are also special types of the Ricci-flat spaces ($\chi = 0$) and the spaces of the negative constant curvature ($\chi = -1$) which have the limited volume. It is clear that the Casimir effect takes place for these geometries also.) The static universe with positive χ is also called *Einstein universe*. There is, of course, no problem in generalizing the results to the case of arbitrary D.

The free energy of the system with the temperature $k_B T \equiv 1/\beta$ is

$$F = -k_B T \ln Z \tag{2.5}$$

where the partition function Z may be defined by the formula

$$Z = \int \mathcal{D}\varphi e^{-S^{\rm e}} \tag{2.6}$$

where $S^{\rm e}$ is the euclidean action obtained from the Lorentzian one (2.1) by the substitution $t \to -i\tau$. The functional integral is performed over all fields φ which are periodic in the imaginary time τ with period $\hbar\beta$. For the static metric (2.3) with the spatial part (2.4), the action reads

$$S^{\rm e} \;=\; \frac{1}{2}\int d\tau d^3x \gamma^{1/2} a^3 [(\partial_\tau \varphi)^2 + \frac{1}{a^2}\gamma^{\alpha\beta}(\partial_\alpha\varphi)(\partial_\beta\varphi) + M^2\varphi^2], \tag{2.7}$$

where we have introduced the effective mass

$$M^2 = \frac{6\chi\xi}{a^2} + m^2. \tag{2.8}$$

To calculate the path integral (2.6), we expand the scalar field φ in to the eigenfunctions of the Laplace-Beltrami operator $\Delta_2^{(3)}$ [see formula (A.1) in Appendix A]. Restricting our attention to the case of positive constant curvature with $\chi = +1$, we take the eigenfunctions in the form

$$\Phi_J(\mathbf{x}) \equiv Q^n_{lm}(\mathbf{x}) = \Pi^n_l(r) Y_{lm}(\theta,\phi), \tag{2.9}$$

where $Y_{lm}(\theta,\phi)$ are the scalar spherical harmonics (see Appendix A). The expansion is

$$\varphi(x) = \frac{1}{2}\sum_J [\varphi_J(\tau)\Phi_J(\mathbf{x}) + \text{c.c.}], \tag{2.10}$$

The coefficient functions satisfy periodic boundary conditions:

$$\varphi_J(\tau = 0) = \varphi_J(\tau = \hbar\beta) = 0. \tag{2.11}$$

Substituting (2.10) into (2.7) and using the orthogonormality relations for the spherical harmonics given in Appendix A we find the euclidean action

$$S^{\mathrm{e}} = \sum_J \frac{1}{2} \int d\tau [|\dot{\varphi}_J|^2 + \omega_n^2 |\varphi_J|^2], \tag{2.12}$$

where the dot denotes the differentiation with respect to τ and $\omega_n^2 = M^2 + (n^2-1)/a^2, n = 1,2,3\ldots$ are the eigenfrequencies. The decomposition has reduced the functional integral for the partition function (2.6) to a product of simple path integrals of harmonic oscillators. It is then easy to calculate the total free energy [11,12]

$$F = k_B T \sum_n n^2 \ln\left(1 - e^{-\frac{\hbar\omega_n}{k_B T}}\right) + \sum_n n^2 \frac{\hbar\omega_n}{2}, \tag{2.13}$$

the frequencies being

$$\omega_n^2 = \frac{6\chi\xi}{a^2} + m^2 + \frac{n^2-\chi}{a^2} = m^2 + \frac{n^2 + (6\xi-1)\chi}{a^2}, \tag{2.14}$$

The multiplier n^2 in the sums has its origin in the degeneracy of the eigenvalues in the isotropic spaces. The second term is the divergent zero-point energy. In a realistic theory of the universe, the divergence must be canceled by the presence of an equal number of Fermi fields whose masses $\sum m_i^{\mathrm{F}}$ have to satisfy, together with those of all Bose fields $\sum m_i^{\mathrm{B}}$, certain sum rules ($\sum_i m_i^{\mathrm{F}2} = \sum_i m_i^{\mathrm{B}2}$). When calculating the contribution of a single Bose field only, the expression may be regularized by any standard method. This will be done in the next section. In the case of an open universe with $\chi = 0$ or -1, the sums in (2.13) are replaced by integrals [4,5] and the standard regularization amounts to dropping the last integral [11].

3 The finite-size effects on the field fluctuations

To perform the sum in the free energy (2.13) we rewrite the expression as

$$F = k_B T \sum_n n^2 \ln \left[2 \sinh \frac{\hbar \omega_n}{2 k_B T} \right] \tag{3.1}$$

and the frequency (2.14) as

$$\omega_n^2 = m^2 + \frac{n^2 - n_c^2}{a^2} \tag{3.2}$$

with the constant

$$n_c^2 = (1 - 6\xi)\,\chi, \qquad \chi = 1. \tag{3.3}$$

The parameter ξ is usually assumed to lie in the interval [6] $0 \leq \xi \leq \frac{1}{6}$. Thus, $0 \leq n_c \leq 1$ and $n_c = 0$ for the conformal coupling ($\xi = 1/6$), $n_c = 1$ for the minimal coupling ($\xi = 0$).

Due to its simplicity, we first consider a massive scalar field with conformal coupling ($n_c = 0$). Then we have from (3.2)

$$\omega_n^2 = m^2 + \frac{n^2}{a^2}. \tag{3.4}$$

Using the dimensionless frequencies

$$\tilde{\omega}_n^2 = m^2 a^2 + n^2, \tag{3.5}$$

we rewrite the formula (3.1) as

$$F = k_B T \sum_{n=1}^{\infty} n^2 \ln \left[2 \sinh \frac{\hbar \tilde{\omega}_n}{2\Theta} \right], \tag{3.6}$$

with the reduced temperature parameter

$$\Theta \equiv k_B T \cdot a. \tag{3.7}$$

It measures the temperature in units of $\hbar/ak_B$ (using $c=1$). To isolate the finite-size effects, we add and subtract the $a\to\infty$-limit and write

$$F = (F - F_\infty) + F_\infty \equiv F_{\rm fs} + F_\infty. \tag{3.8}$$

The $a\to\infty$-limit F_∞

$$F_\infty = k_B T \int_0^\infty dn n^2 \ln\left[2\sinh\frac{\hbar\tilde{\omega}}{2\Theta}\right] \tag{3.9}$$

is divergent. After a standard zeta function regularization (which makes $\int_0^\infty dn n^2 \tilde{\omega}_n = 0$) this becomes

$$\begin{aligned} F_{\infty,\rm ren} &= k_B T \int_0^\infty dn n^2 \ln\left[1 - \exp\left(-\sqrt{\left(\frac{m\hbar}{k_B T}\right)^2 + \left(\frac{\hbar n}{\Theta}\right)^2}\right)\right] \qquad (3.10)\\ &= -\frac{a^3}{\hbar^3}\frac{(k_B T)^4}{3}\int_0^\infty dx x^4 \frac{1}{\sqrt{\left(\frac{m\hbar}{k_B T}\right)^2 + x^2}\left[\exp\left(\sqrt{\left(\frac{m\hbar}{k_B T}\right)^2 + x^2}\right) - 1\right]}. \end{aligned}$$

The finite-size effects are contained in the finite sum-minus-integral expression

$$F_{\rm fs} = k_B T\left[\sum_{n=1}^\infty n^2 \ln\left(2\sinh\frac{\hbar\tilde{\omega}_n}{2\Theta}\right) - \int_0^\infty dn n^2 \ln\left(2\sinh\frac{\hbar\tilde{\omega}_n}{2\Theta}\right)\right]. \tag{3.11}$$

Here it is convenient to use the Abel-Plana summation formula [3-5, 19]

$$\left[\sum_{n=1} f(n) - \int_0^\infty f(n)dn\right] = \frac{1}{2}f(0) + i\int_0^\infty \frac{f(i\nu) - f(-i\nu)}{[\exp(2\pi\nu) - 1]}d\nu, \tag{3.12}$$

which is correct if $f(\nu)$ is regular for $\mathrm{Re}\nu \geq 0$ (on the imaginary axis, $f(\nu)$ may have poles and branch points which are passed during the integration on the right, i.e., with $\mathrm{Re}\nu > 0$). Then

$$F_{\rm fs} = k_B T i \int_0^\infty \frac{f(i\nu) - f(-i\nu)}{[\exp(2\pi\nu) - 1]}d\nu \tag{3.13}$$

with

$$
\begin{aligned}
f(n) &= n^2 \ln\left(2\sinh\frac{\hbar}{2\Theta}\sqrt{m^2a^2+n^2}\right) \qquad (3.14)\\
&= n^2 \ln\left\{\frac{\hbar}{\Theta}\sqrt{m^2a^2+n^2}\prod_{p=1}^{\infty}\left[1+\left(\frac{\hbar}{2\pi p\Theta}\right)^2\left(m^2a^2+n^2\right)\right]\right\}.
\end{aligned}
$$

The function $f(n)$ has logarithmic branch points on the imaginary axis with constant discontinuities starting at $n = \left[m^2a^2 + (2\pi p\Theta/\hbar)^2\right]^{1/2}$ for $p = 0, 1, 2\ldots$. The integral (3.13) becomes therefore a sum

$$
\begin{aligned}
F_{\rm fs} &= k_BT2\pi\int_0^\infty dnn^2 \sum_{p=0}^{\infty}{}'\theta\left[n^2 - m^2a^2 - \left(\frac{2\pi p\Theta}{\hbar}\right)^2\right]\frac{1}{\exp(2\pi n)-1}\\
&= k_BT2\pi\sum_{p=0}^{\infty}{}'\int_{n_p}^\infty dnn^2\frac{1}{\exp(2\pi n)-1}, \qquad (3.15)
\end{aligned}
$$

where the integrals start at

$$
n_p \equiv \left[m^2a^2 + \left(\frac{2\pi p\Theta}{\hbar}\right)^2\right]^{1/2} \qquad (3.16)
$$

and the prime on the sum indicates that the term with $p = 0$ should be counted with the weight 1/2. Going back from the momentum quantum number n to "physical" wave vectors $k = n/a$ we see that (3.15) corresponds to a Planck distribution form with the effective temperature $T_{\rm eff} = \hbar/ak_B$. The appearance of such an effective temperature is typical for the Casimir effect [10,3].

The total renormalized free energy is

$$
\begin{aligned}
F_{\rm ren} &= F_{\rm fs} + F_{\infty,{\rm ren}} = \frac{k_BT}{4\pi^2}\sum_{p=0}^{\infty}{}'\int_{2\pi n_p}^\infty \frac{x^2dx}{e^x-1} \qquad (3.17)\\
&\quad -\frac{a^3}{\hbar^3}\frac{(k_BT)^4}{3}\int_0^\infty \frac{x^4dx}{\sqrt{\left(\frac{m\hbar}{k_BT}\right)^2+x^2}\left[\exp\left(\sqrt{\left(\frac{m\hbar}{k_BT}\right)^2+x^2}\right)-1\right]}.
\end{aligned}
$$

The internal energy density of the fluctuations is found from

$$\rho = \frac{1}{V}U = \frac{1}{2\pi^2 a^3}\frac{\partial(\beta F_{\rm ren})}{\partial\beta} = \rho_{\rm fs} + \rho_{\infty,\rm ren}, \qquad (3.18)$$

where $V = 2\pi^2 a^3$ is the volume of the closed RWF universe. Explicitly:

$$\rho_{\rm fs} = 8\pi^2\frac{(k_BT)^4}{\hbar^3}\sum_{p=1}^{\infty}\frac{p^2\left(\left(\frac{m\hbar}{2\pi k_BT}\right)^2+p^2\right)^{1/2}}{\exp\left(\frac{4\pi^2\Theta}{\hbar}\cdot\sqrt{\left(\frac{m\hbar}{2\pi k_BT}\right)^2+p^2}\right)-1} \qquad (3.19)$$

$$\rho_{\infty,\rm ren} = \frac{1}{2\pi^2}\frac{(k_BT)^4}{\hbar^3}\int_0^\infty\frac{x^2\cdot\sqrt{\left(\frac{m\hbar}{k_BT}\right)^2+x^2}dx}{\exp\left(\sqrt{\left(\frac{m\hbar}{k_BT}\right)^2+x^2}\right)-1}. \qquad (3.20)$$

It is convenient to deduce from (3.18) - (3.20) the energy density in the high-temperature limit $k_BT >> \hbar/a$ $(\Theta/\hbar >> 1)$. It is clear that the main contribution comes from the term $p = 1$.

To find the pressure of the fluctuation we use the formula

$$P = -\frac{\partial F_{\rm ren}}{\partial V} = -\frac{1}{6\pi^2 a^2}\frac{\partial F_{\rm ren}}{\partial a} = P_{\rm fs} + P_{\infty,\rm ren}, \qquad (3.21)$$

where

$$P_{\rm fs} = \frac{1}{3}\rho_{\rm fs} + \frac{2}{3}\frac{(k_BT)^4}{\hbar^3}\left(\frac{m\hbar}{k_BT}\right)^2{\sum_{p=0}^{\infty}}'\frac{\left(\left(\frac{m\hbar}{2\pi k_BT}\right)^2+p^2\right)^{1/2}}{\exp\left(\frac{4\pi^2\Theta}{\hbar}\cdot\sqrt{\left(\frac{m\hbar}{2\pi k_BT}\right)^2+p^2}\right)-1}, \qquad (3.22)$$

$$P_{\infty,\rm ren} = \frac{1}{6\pi^2}\frac{(k_BT)^4}{\hbar^3}\int_0^\infty\frac{x^4dx}{\sqrt{\left(\frac{m\hbar}{k_BT}\right)^2+x^2}\left[\exp\left(\sqrt{\left(\frac{m\hbar}{k_BT}\right)^2+x^2}\right)-1\right]}. \qquad (3.23)$$

In high-temperature limit $\Theta/\hbar \gg 1$ we should keep the terms with $p = 0, 1$ in (3.22). They give the main contribution to P.

An alternative form

The formula (3.17) is not yet useful in the low-temperature limit $\Theta/\hbar \ll 1$. For this purpose we use the Poisson summation formula [12-15] according to which

$$\sum_{p=-\infty}^{\infty} \sigma(p) = 2\pi \sum_{p=-\infty}^{\infty} c(2\pi p) \tag{3.24}$$

if $\sigma(p)$ and $c(p)$ are connected by the Fourier transform

$$c(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sigma(x) e^{-i\alpha x} dx. \tag{3.25}$$

In our case

$$\sigma(p) = \int_{2\pi n_p}^{\infty} \frac{x^2 dx}{e^x - 1} \tag{3.26}$$

so we can rewrite the (3.15) as

$$F_{\text{fs}} = \frac{k_B T}{4\pi^2} \sum_{p=0}^{\infty}{}' \sigma(p) = \frac{k_B T}{2\pi} \sum_{p=0}^{\infty}{}' c(2\pi p). \tag{3.27}$$

The expression for $c(0)$ is easily obtained

$$c(0) = \frac{\hbar}{4\pi^3 \Theta} \int_{2\pi ma}^{\infty} \frac{\sqrt{x^2 - (2\pi ma)^2} x^2 dx}{e^x - 1} \tag{3.28}$$

and (3.28) is proportional to the free energy of the vacuum fluctuations F_{vf}:

$$F_{\text{vf}} = \sum_{n=1}^{\infty} n^2 \frac{\hbar\omega_n}{2} - \int_0^{\infty} n^2 \frac{\hbar\omega_n}{2} dn = \frac{\hbar}{a} \frac{1}{(2\pi)^4} \int_{2\pi ma}^{\infty} \frac{\sqrt{x^2 - (2\pi ma)^2} x^2 dx}{e^x - 1}, \tag{3.29}$$

where for the regularization $F_{\rm vf}$ we use the Abel-Plana summation formula (3.12). Eqn. (3.27) takes the form

$$F_{\rm fs} = F_{\rm vf} + \frac{k_B T}{2\pi} \sum_{p=1}^{\infty} c(2\pi p), \tag{3.30}$$

where for $p \neq 0$

$$c(\alpha) = \frac{1}{\pi\alpha} \left[-\frac{d^2}{dz^2} + (2\pi m a)^2 \right] \int_{2\pi m a}^{\infty} \frac{\sin\left(z\sqrt{x^2 - (2\pi m a)^2} \right)}{e^x - 1} dx|_{z=\frac{\alpha\hbar}{4\pi^2\Theta}}. \tag{3.31}$$

Thus, the alternative expression for the total renormalized free energy is $F_{\rm ren} = F_{\rm fs} + F_{\infty,\rm ren}$ with $F_{\rm fs}$ being defined by (3.30), (3.31).

This is the most convenient expression for dealing with the masless case which will be done below. Being interested in the low-temperature limit, it is preferable to find yet another representation of the same expression.

The low-temperature limit for non-zero mass

For $m \neq 0$, the integral (3.31) cannot be calculated exactly and it is more easy to obtain the low-temperature limit for $F_{\rm fs}$ directly from the formula (3.11), which can be written in the form

$$F_{\rm fs} = k_B T \sum_{1}^{\infty} n^2 \ln \left[1 - \exp\left(-\frac{\hbar\tilde{\omega}_n}{\Theta} \right) \right] + F_{\rm vf} - F_{\infty,\rm ren}. \tag{3.32}$$

Adding to this $F_{\infty,\rm ren}$ we have in the low-temperature limit

$$\begin{aligned} F_{\rm ren} &= k_B T \sum_{1}^{\infty} n^2 \ln \left[1 - \exp\left(-\frac{\hbar\tilde{\omega}_n}{\Theta} \right) \right] + F_{\rm vf} \\ &\approx -k_B T \exp\left(-\frac{\hbar\tilde{\omega}_1}{\Theta} \right) + F_{\rm vf}. \end{aligned} \tag{3.33}$$

Substituting (3.33) into formula (3.18) we obtain

$$\rho = \frac{\hbar}{2\pi^2 a^4} \sum_{1}^{\infty} \frac{n^2 \tilde{\omega}_n}{\exp\left(\frac{\hbar\tilde{\omega}_n}{\Theta}\right) - 1} + \rho_{\rm vf}$$

$$\approx \frac{\hbar\tilde{\omega}_1}{2\pi^2 a^4}\exp\left(-\frac{\hbar\tilde{\omega}_1}{\Theta}\right)+\rho_{\rm vf}, \tag{3.34}$$

where $\rho_{\rm vf}$ is the well-known expression for the energy density for the vacuum fluctuations of the massive scalar field [3-6,9,10]:

$$\rho_{\rm vf}=\frac{\hbar}{a^4\pi(2\pi)^5}\int_{2\pi ma}^{\infty}\frac{\sqrt{x^2-(2\pi ma)^2}x^2dx}{e^x-1}. \tag{3.35}$$

For the pressure we have from (3.21) and (3.33)

$$\begin{aligned} P &= \frac{\hbar}{6\pi^2\alpha^4}\sum_1^{\infty}\frac{n^4}{\tilde{\omega}_n\left[\exp\left(\frac{\hbar\tilde{\omega}_n}{\Theta}\right)-1\right]}+P_{\rm vf} \\ &\approx \frac{\hbar}{6\pi^2 a^4\tilde{\omega}_1}\exp\left(-\frac{\hbar\tilde{\omega}_1}{\Theta}\right)+P_{\rm vf}, \end{aligned} \tag{3.36}$$

where

$$P_{\rm vf}=\frac{1}{3}\rho_{\rm vf}+\frac{(ma)^2\hbar}{24\pi^2a^4}\int_{2\pi ma}^{\infty}\frac{x^2dx}{\sqrt{x^2-(2\pi ma)^2}(e^x-1)}. \tag{3.37}$$

The integral in (3.37) is convergent one.

The massless case

The formulas obtained above are simplified considerably in the case of radiation ($m=0$). The formula (3.17) for $F_{\rm ren}$ reads

$$F_{\rm ren}=\frac{k_BT}{4\pi^2}\sum_{p=0}^{\infty}{}'\int_{\frac{4\pi^2\Theta}{\hbar}p}^{\infty}\frac{x^2dx}{e^x-1}-\frac{a^3}{h^3}\frac{\pi^4}{45}(k_BT)^4. \tag{3.38}$$

The energy density of the fluctuations is

$$\rho=8\pi^2\frac{(k_BT)^4}{\hbar^3}\sum_{p=1}^{\infty}\frac{p^3}{\exp\left(\frac{4\pi^2p\Theta}{\hbar}\right)-1}+\frac{\pi^2}{30}\frac{(k_BT)^4}{\hbar^3}. \tag{3.39}$$

The second term here is usual black-body energy density. The first term contains the finite-size effects. In principle, it opens up the possibility to obtain information on global topology of our universe by measuring the microwave radiation. Unfortunately, as it will be shown later, the corrections to the black-body energy-density are exponentially small at the present moment for the standard model of the hot Friedmann universe. If we demand the constancy of the total energy carried by the fluctuations during the evolution of the universe, i.e., $\rho \cdot a^4 = \text{const}$ [20], then we have from (3.39)

$$k_B T \cdot a = \Theta = \text{const}. \tag{3.40}$$

This is the usual relation between the temperature of the radiation and the scale factor.

It is convenient to deduce from (3.39) the energy density in the high-temperature limit $k_B T \gg \hbar/a$, which in the present units amounts to $\Theta/\hbar \gg 1$:

$$\rho \approx 8\pi^2 \frac{\hbar}{a^4} \left(\frac{\Theta}{\hbar}\right)^4 \exp\left(-4\pi^2 \frac{\Theta}{\hbar}\right) + \frac{\pi^2}{30} \frac{\hbar}{a^4} \left(\frac{\Theta}{\hbar}\right)^4. \tag{3.41}$$

To find the pressure of the fluctuations we use the formula (3.21) and obtain

$$P = \frac{\rho}{3}, \tag{3.42}$$

where ρ is defined by formula (3.39). This is the usual equation of state for radiation.

As was stressed above, the formulas of the type (3.39) are not yet useful in the low-temperature limit $\Theta/\hbar << 1$. For this purpose we use the expression (3.30) where in the massless case:

$$F_{\text{vf}} = \frac{1}{240} \frac{\hbar}{a} \tag{3.43}$$

and [15]

$$c(\alpha) = -\frac{1}{\pi\alpha} \left\{ 4\pi^3 \frac{\left[1 + \exp\left(-\frac{\alpha\hbar}{2\pi\Theta}\right)\right] \exp\left(-\frac{\alpha\hbar}{2\pi\Theta}\right)}{\left[1 - \exp\left(-\frac{\alpha\hbar}{2\pi\Theta}\right)\right]^3} - \frac{(4\pi^2\Theta)^3}{\alpha^3\hbar^3} \right\}. \tag{3.44}$$

The expression (3.43) is the well-known free energy of the vacuum fluctuations of the massless scalar field in the closed Friedmann universe [3-6,9,10]. As a result we have the alternative formula for the free energy

$$F_{\rm ren} = \frac{1}{240}\frac{\hbar}{a} - k_B T \sum_{n=1}^{\infty} \frac{1}{n} \frac{\left[1 + \exp\left(-\frac{n\hbar}{\Theta}\right)\right] \exp\left(-\frac{n\hbar}{\Theta}\right)}{\left[1 - \exp\left(-\frac{n\hbar}{\Theta}\right)\right]^3}. \tag{3.45}$$

Then the low-temperature limit $\Theta/\hbar << 1$ becomes simply

$$F_{\rm ren} \approx \frac{1}{240}\frac{\hbar}{a} - k_B T \exp\left(-\frac{\hbar}{\Theta}\right). \tag{3.46}$$

This could have been derived directly from (3.32), (3.33). Substituting (3.45) into (3.18) we obtain the alternative form for the energy density

$$\rho = \frac{\hbar}{480\pi^2}\frac{1}{a^4} + \frac{\hbar}{2\pi^2}\frac{1}{a^4} \sum_{n=1}^{\infty} \frac{\left[1 + 4\exp\left(-\frac{n\hbar}{\Theta}\right) + \exp\left(-\frac{2n\hbar}{\Theta}\right)\right] \exp\left(-\frac{n\hbar}{\Theta}\right)}{\left[1 - \exp\left(-\frac{n\hbar}{\Theta}\right)\right]^4} \tag{3.47}$$

which converges fast for low temperatures. It is easy to see from this formula that the requirement of a constant total energy gives again the condition (3.40) $k_B T \cdot a = \text{const}$. In the low-temperature limit $k_B T << \hbar/a$ we obtain the approximation

$$\rho \approx \frac{\hbar}{480\pi^2}\frac{1}{a^4} + \frac{\hbar}{2\pi^2}\frac{1}{a^4} \exp\left(-\frac{\hbar}{\Theta}\right). \tag{3.48}$$

For the pressure of the fluctuations we can get with the help of (3.21) the equation of the state (3.42) where ρ is defined now by formula (3.47).

Non-conformal coupling

Let us consider finally the scalar field with an arbitrary coupling $0 \le \xi \le 1/6$. We study only the massless case, for simplicity. The generalization of the formulas obtained to the case $m \neq 0$ is straightforward.

The frequency in this case is $\omega_n^2 = (n^2 - n_c^2)/a^2$ and after rotation in complex n_c-plain $n_c \to i\nu_c$ is reduced to

$$\omega_n^2 = \frac{\nu_c^2}{a^2} + \frac{n^2}{a^2} \equiv m^2(a) + \frac{n^2}{a^2} \tag{3.49}$$

or

$$\tilde{\omega}_n^2 = a^2 \cdot \omega_n^2 = \nu_c^2 + n^2, \tag{3.50}$$

where $m(a) = \nu_c/a$ plays the role of the scalar field mass which depends on the scale factor. Now we can use directly the formulas derived for the massive scalar field with conformal coupling. At the end, we rotate ν_c back to its proper imaginary value $\nu_c \to -in_c$.

The expression for the non-regularized free energy takes the form

$$F = k_B T \sum_{n=1}^{\infty} n^2 \ln \left[2 \sinh \frac{\hbar \sqrt{\nu_c^2 + n^2}}{2\Theta} \right]. \tag{3.51}$$

Performing the regularization similar to (3.6)-(3.15) we have finally

$$\begin{aligned} F_{\text{fs}} &= k_B T 2\pi \int_0^\infty dn n^2 \sum_{p=0}^{\infty}{}' \theta \left[n^2 + n_c^2 - \left(\frac{2\pi p \Theta}{\hbar} \right)^2 \right] \frac{1}{\exp(2\pi n) - 1} \\ &= k_B T \pi \left(1 + 2p_c \right) \int_0^\infty \frac{n^2 dn}{\exp(2\pi n) - 1} \\ &\quad + k_B T 2\pi \sum_{p=p_c+1}^{\infty} \int_{\sqrt{\left(\frac{2\pi p \Theta}{\hbar}\right)^2 - n_c^2}}^{\infty} dn n^2 \left[\exp(2\pi n) - 1 \right]^{-1} \end{aligned} \tag{3.52}$$

where

$$p_c = \left[\frac{\hbar n_c}{2\pi\Theta} \right] \tag{3.53}$$

is the largest integer $\leq \hbar n_c/2\pi\Theta)$. It is clear from the formulas (3.52), (3.53), that p_c can be treated as infrared cut-off.

The renormalized $a \to \infty$ limit of (3.51) is

$$F_{\infty,\text{ren}} = -\frac{a^3}{3\hbar^3}(k_B T)^4 \int_0^\infty \frac{x^4 dx}{\sqrt{\left(\frac{\hbar\nu_c}{\Theta}\right)^2 + x^2}\left[\exp\left(\sqrt{\left(\frac{\hbar\nu_c}{\Theta}\right)^2 + x^2}\right) - 1\right]}$$
$$\overset{a\to\infty}{\longrightarrow} -\frac{\pi^4}{45}\frac{a^3}{\hbar^3}(k_B T)^4. \tag{3.54}$$

This is the usual black-body free energy, a result which was predictable since the mass term $m(a)$ tends to zero for $a \to \infty$. Thus, $F_{\infty,\text{ren}}$ has the same form for any coupling ζ.

The renormalized expression for the free energy can be found as the sum

$$F_{\text{ren}} = F_{\text{fs}} + F_{\infty,\text{ren}}. \tag{3.55}$$

Then, with the help of the formulas (3.18) and (3.21), we obtain for the energy density ρ

$$\rho = 8\pi^2 \frac{(k_B T)^4}{\hbar^3} \sum_{p=p_c+1}^{\infty} \frac{p^2\left(p^2 - \left(\frac{n_c\hbar}{2\pi\Theta}\right)^2\right)^{1/2}}{\exp\left(\frac{4\pi^2\Theta}{\hbar}\cdot\sqrt{p^2 - \left(\frac{n_c\hbar}{2\pi\Theta}\right)^2}\right) - 1} + \frac{\pi^2}{30}\frac{(k_B T)^4}{\hbar^3} \tag{3.56}$$

For the pressure we find the usual equation of state for the radiation: $P = 1/3\rho$. The requirement of a constant total energy during the evolution of the universe gives again the condition (3.40).

The formulas (3.52), (3.56) are useful to estimate the high-temperature limit $\Theta/\hbar >> 1$. The main contribution in the sums is given by the term with $p = p_c + 1$. The presence of the "mass" in this model does not permit an exact calculation of the coefficients $c(\alpha)$ in (3.31). To obtain an estimate in the low-temperature limit $\Theta/\hbar << 1$, we write, by analogy with (3.33), the expression for the F_{ren} in the form

$$F_{\text{ren}} = k_B T \sum_{n=n^*+1}^{\infty} n^2 \ln\left[1 - \exp\left(-\frac{\hbar\tilde{\omega}_n}{\Theta}\right)\right] + F_{\text{vf}}, \tag{3.57}$$

where we introduced the infrared cut-off similar to (3.53): $n^* = [n]$ and

$$n^* = \{ \begin{array}{ll} 0, & n_c < 1 \\ 1, & n_c = 1. \end{array} \tag{3.58}$$

The energy $F_{\rm vf}$ is the free energy of the vacuum fluctuations

$$F_{\rm vf} = \frac{\hbar}{a}\frac{1}{(2\pi)^4}\int_0^\infty \frac{\sqrt{x^2+(2\pi n_c)^2}x^2dx}{e^x-1}. \tag{3.59}$$

The corresponding energy density is

$$\rho_{\rm vf} = \frac{\hbar}{a^4\pi(2\pi)^5}\int_0^\infty \frac{\sqrt{x^2+(2\pi n_c)^2}x^2dx}{e^x-1} \tag{3.60}$$

with the pressure satisfying $P_{\rm vf} = \frac{1}{3}\rho_{\rm vf}$. In the low-temperature limit $\Theta/\hbar << 1$, we find

$$\rho = \frac{1}{2\pi^2a^3}\frac{\partial}{\partial\beta}(\beta F_{\rm ren}) \approx \frac{\hbar(n^*+1)^2\tilde{\omega}^*_{n+1}}{2\pi^2a^4}\exp\left(-\frac{\hbar\tilde{\omega}^*_{n+1}}{\Theta}\right) + \rho_{\rm vf} \tag{3.61}$$

and the pressure satisfies once more the equation of state $P = \rho/3$. The requirement of a constant total energy $\rho \cdot a^4 =$ leads again to the condition (3.40).

It is remarkable that the finite-size effects do not change this formula.

Application to our universe

It is interesting to estimate the value of the reduced temperature Θ for our universe. If we assume that the standard model of the hot universe [20], which most cosmologists believe describes the evolution of the now observable universe, we take for the present state the value $a \sim 12,48 \cdot 10^{27}$cm. Using the temperature of the microwave radiation $T \sim 2,7^0$K, we find $\Theta/\hbar \sim 1,5 \cdot 10^{29}$. Approximately the same sign of Θ is estimated for the relict neutrinos and gravitons [20]. Thus, at the present state of the evolution, our universe for these radiations is in high-temperature limit and the finite-size quantum effects are certainly unobservable. This estimate of the reduced temperature shows us that, in the standard model of the hot universe, it is impossible to use formula (3.39) to answer the question about the global topology of our universe on the basis of the observed microwave radiation. The static-universe approximation was used in this paper to extract pure finite-size effects. As is shown in Appendix B, the slow-evolution approximation is good at the present time for the standard model of the hot universe. Obviously, the approximation breaks down for $a \to 0$.

In conclusion, we stress that the finite temperature effects may become relevant if the thermalization was achieved during radiation dominated era. During the inflation and presumably also in the compactified dimensions of a multidimensional universe, there is presumably no thermalization and therefore no finite temperature Casimir effect.

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Appendix A: Scalar spherical harmonics

In this appendix we describe the main properties of the scalar harmonics on the three-sphere S^3 with the metric (2.4). More detailed information about their properties and eigenfunctions in the case of the hyperbolic and flat 3-spaces of constant curvature it is possible to find in the [5,6,12,21,22].

The Laplace-Beltrami operator for the metric (2.4) is

$$\Delta_2^{(3)} = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\gamma} \gamma^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right) \tag{A.1}$$

and its eigenfunctions are the spherical harmonics Φ_J

$$\Delta_2^{(3)} \Phi_J = -(n^2 - 1)\Phi_J \tag{A.2}$$

with collective index $J = \{n, l, m\}$ where $n = 1, 2, 3, \ldots; l = 0, \ldots, n-1;$ $|m| \leq l$. The functions Φ_J can be expressed in terms of the usual spherical harmonics Y_{lm} by the next way

$$\Phi_J \equiv \Pi_l^n(r) Y_{lm}(\theta, \phi), \tag{A.3}$$

where Π_l^n are the "Fock" harmonics [25]

$$\Pi_l^n(r) = \sin^l r \frac{d^{l+1}(\cos nr)}{d(\cos r)^{(l+1)}}. \tag{A.4}$$

The orthonormality reaction is

$$\int d^3x \sqrt{\gamma} \Phi_J^*(x) \Phi_{J'}(x) = \delta_{JJ'} = \delta_{nn'}\delta_{ll'}\delta_{mm'}, \tag{A.5}$$

where Φ_J^* is the complex conjugation of Φ_J and $\Phi_J^* = (-1)^m \Phi_{\bar{J}}, \bar{J} = \{n, l, -m\}$. Note also the equality holds [5]

$$\int d^3x \sqrt{\gamma} \gamma^{\alpha\beta} \partial_\alpha \Phi_J \partial_\beta \Phi_{J'} = (n^2 - 1)\delta_{JJ'}. \tag{A.6}$$

The information given in this appendix about the spherical harmonics is all that is needed to get (2.12).

Appendix B: Stabilization of the Friedmann universe

In Chapters 2 and 3, the time evolution was assumed to be sufficiently slow to justify the calculation of the Casimir effect in a static metric. The standard model of the universe is not a static one [20]. The scale factor a is a time-dependent function and the scalar field in the case of conformal coupling is reduced to the harmonic oscillator with a time dependent frequency $\omega(\eta)$ [5,6]

$$\omega^2(\eta) = m^2 a^2(\eta) + n^2, \tag{B.1}$$

where η is conformal time which is related to the synchronous time t by the formula $ad\eta = dt$. The slow-evolution approximation is good if the adiabatic parameter δ of the oscillators [5] satisfies the condition

$$\delta \equiv (1/\omega^2) d\omega/d\eta = (m^2 a/\omega^3) da/d\eta << 1. \tag{B.2}$$

It is clear that in our present universe this parameter is extremely small. As an estimate we obtain, for particles with electron mass m_e in a typical cosmological model [20] compatible with presend-day astronomical observations, the value $\delta \leq 2 \cdot 10^{-39}$.

Static solutions are important also in multidimensional cosmology where models with compact static inner spaces represent solutions with spontaneous compactification [18].

In this Appendix we show that the quantum fluctuations of scalar field are capable of stabilizing the solutions of Einstein's equations and yielding a static universe. Thus, in this case, the initial assumption of a slowly evolving universe becomes completely self-consistent.

The Einstein equations with a cosmological constant Λ in the case of the RWF metric and in the presence of a perfect-fluid stress-energy tensor are [20]

$$\begin{aligned} \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 &= -\frac{\chi}{a^2}+\frac{\Lambda}{3}+\frac{8\pi}{3}\rho, \\ \frac{2}{a}\frac{d^2a}{dt^2} &= -\frac{1}{a^2}\left(\frac{da}{dt}\right)^2-\frac{\chi}{a^2}+\Lambda-8\pi P, \end{aligned} \quad \text{(B.3)}$$

where ρ and P are density of mass-energy and pressure of the perfect fluid. It is easy to see that by taking

$$\begin{aligned} \rho &= \frac{1}{8\pi}\left(\frac{3\chi}{a^2}-\Lambda\right), \\ P &= \frac{1}{8\pi}\left(\Lambda-\frac{\chi}{a^2}\right), \end{aligned} \quad \text{(B.4)}$$

the universe can have a steady state. The parameters of the universe are

$$\begin{aligned} \Lambda &= 4\pi(\rho+3P), \\ a^2 &= \frac{\chi}{4\pi(\rho+P)}. \end{aligned} \quad \text{(B.5)}$$

Inserting ρ and P from the earlier results, the equations can be solved self-consistently.

It was shown in Chapter 3 that with the parameters of the standard model of the universe the quantum fluctuations are at present time in the high-temperature limit and the Casimir effect gives an exponential small correction to the energy density and pressure of the fluctuations. Thus, for a quantitative estimate of ρ and P it is enough to take for the regularized expression of (2.13) the formula (which is valid in all three cases $\chi = 0, \pm 1$)

$$\begin{aligned} F_{\text{ren}} &= k_B T \int_0^\infty n^2 dn \ln\left(1 - e^{-\frac{\hbar\omega_n}{k_B T}}\right) \\ &= -\frac{\hbar}{3} \int_0^\infty \frac{n^3 \frac{\partial \omega_n}{\partial n} dn}{\exp\left(\frac{\hbar\omega_n}{k_B T}\right) - 1} \end{aligned} \tag{B.6}$$

where ω_n is defined by (2.14). For the mass-energy density, we have in a closed universe

$$\rho = \frac{1}{2\pi^2 a^3} \frac{\partial(\beta F_{\text{ren}})}{\partial \beta} = \frac{\hbar}{2\pi^2 a^3} \int_0^\infty \frac{n^2 \omega_n dn}{\exp\left(\frac{\hbar\omega_n}{k_B T}\right) - 1}. \tag{B.7}$$

It is not difficult to calculate P using formulas (3.21), (B.6) and the connection $\partial\omega_n/\partial a = -(n/a)\partial\omega_n/\partial n$:

$$P = -\frac{\hbar}{6\pi^2 a^2} \int_0^\infty \frac{n^2 \frac{\partial \omega_n}{\partial a} dn}{\exp\left(\frac{\hbar\omega_n}{k_B T}\right) - 1} = -\frac{1}{2\pi^2 a^3} F_{\text{ren}}. \tag{B.8}$$

Consider two particular cases: high- and low-temperature limits, in which the temperature of the universe $k_B T$ is much or much smaller than the mass of scalar particles. Note through that the parameter $\Theta/\hbar >> 1$ in all cases.

I. High-temperature limit: $k_B T >> \hbar m$.

This is the limit of ultra-relativistic particles: $\omega_k \approx k = n/a$ where the equation of state is $P = \frac{\rho}{3}$. For the energy density we get (in restored dimension) from (B.7)

$$\rho\left(\frac{g}{cm^3}\right) = \frac{\pi^2}{30} \frac{(k_B)^4}{\hbar^3 c^5} \tag{B.9}$$

in agreement with (3.41) (up to the exponentially small Casimir correction).

The standard model of the universe gives for the present state the estimates [20] $\Lambda \leq 10^{-56}\text{cm}^{-2}$ and $a \geq 10^{28}\text{cm}$. From formulas (B.4) and (B.5) we have $\rho = \frac{\Lambda}{8\pi}$ and $a^2 = \frac{3}{2}\frac{1}{\Lambda}$. If we take now for ρ the expression (B.9) then we obtain the parameters $\Lambda \sim 10^{-56}\text{cm}^{-2}$ and $a \sim 10^{28}\text{cm}$ for $T \sim 40^0\text{K}$. Thus, if ultra-relativisitic particles exist at the present time in the thermodynamical equilibrium state with the temperature $T \sim 40^0\text{K}$, they can stabilize the universe. The temperature $T \sim 40^0\text{K}$ gives the upper limit for the mass of the particles to consider them as ultra-relativistic ones: $m < 10^{-8} m_e$, where m_e is the electron mass. Such super-light particles are predicted in some types of unified theories, supersymmetry and supergravity [23]. Of course, this consideration is rather rough. We have shown here only the possibility in principle of a stabilization of our universe.

II. Low-temperature limit: $k_B T << \hbar m$.

In this case $\omega_k \approx k^2/2m + m$ and for the mass-energy density we have

$$\begin{aligned} \rho &= \frac{3\sqrt{2\pi}}{8\pi^2}\hbar m^4 e^{-\frac{\hbar m}{k_B T}} \left(\frac{k_B T}{\hbar m}\right)^{5/2} + \frac{\sqrt{2\pi}}{4\pi^2}\hbar m^4 e^{-\frac{\hbar m}{k_B T}} \left(\frac{k_B T}{\hbar m}\right)^{3/2} \\ &\equiv \rho_1 + \rho_0, \end{aligned} \tag{B.10}$$

where ρ_1 is the kinetic energy density and ρ_0 is the rest mass density and $\rho_1 << \rho_0$ in this limit.

For the pressure we can get the relation $P = (2/3)\rho_1$ which coincides with the equation of state for Fermi and Bose gases of elementary particles [24]. As $\rho_0 >> \rho_1, P$ we can omit ρ_1 and P in the Einstein equations

(B.4) and for the parameters of the static universe we get the formulas $\Lambda = 4\pi\rho_0, a^2 = \chi/(4\pi\rho_0)$ which for $\chi = +1$ coincides with the formulas for the Einstein universe in the matter dominated era. It is impossible to get from formulas (B.10) and (B.5) the more or less real parameters of the observable universe at present time. The point is that formula (B.10) relates to particles (and antiparticles) which are in thermodynamical equilibrium state and their energy density and number of particles (antiparticles) for $k_B T << \hbar m$ is exponentially small. The parameters of real universe under low temperatures are defined by usual neutral matter which is not in equilibrium state already [25].

If the scale parameter a is such that $\delta << 1$ we can use also the above formulas for ρ and P change also. If a $\rightarrow a_s, \rho \rightarrow \rho_s$ and $P \rightarrow P_s$ where a_s, ρ_s and P_s are connected with each other by formulas (B.5), then the stabilization of the universe near a_s will take place. But this state is a metastable one [20].

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