ELSEVIER

19 September 1996

PHYSICS LETTERS B

Physics Letters B 384 (1996) 213-217

Nambu-Goto string without tachyons between a heavy and a light quark – real interquark potential at all distances

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Received 26 March 1996; revised manuscript received 7 May 1996
Editor: P.V. Landshoff

Abstract

We point out that in infinite spacetime dimensions, the singularity in the interquark potential at small distances disappears if a Nambu-Goto string is anchored at one end to an infinitely heavy quark, at the other end to an infinitely light quark. This suggests that if such quarks are placed at the ends, some unphysical features such as tachyon states are absent also in finite dimensions.

PACS: 11.17.+y; 12.38.Aw; 12.40.Aa

1. It is generally believed that some modification of the Nambu-Goto string model will eventually become a fundamental theory, capable of explaining the forces between quarks in a simpler way than quantum chromodynamics (QCD). Indeed, the correct large-distance confinement behavior [1–3] is automatically obtained, by construction, whereas that in QCD can only be found by arduous lattice simulations [4,5]. Also the first quantum correction to this behavior, the universal Lüscher term [1,6], is found immediately. It is a one-loop contribution to the string energy and corresponds to the black-body energy of the small oscillations, coinciding with the Casimir energy at $T = 0$.

Certainly, it cannot be hoped that the Nambu-Goto string is anywhere close to to the real color-electric flux tube between quarks since it is incapable of reproducing the $1/R$-singularity at small R caused by the asymptotic freedom of gluons. Some essential modification accounting for the finite diameter of the flux tube, in particular its transition into a spherical bag at small quark separations will be necessary. A first attempt in this direction was taken some time ago by adding an asymptotically-free curvature stiffness term [7], but this term introduced other problems. In particular, the true stiffness constant of the flux tube appears the opposite string [8].

In spite of the essential differences between a Nambu-Goto string and a flux tube between quarks, the question arises how the unphysical properties of a Nambu-Goto string change if quarks are placed at the ends. The purpose of this note is to point out that in one extremal configuration, at least the singularity of the string potential disappears, indicating the absence of tachyons in that case.

To obtain a first idea about all properties of a fully fluctuating string it is useful to investigate the limit

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PII S0370-2693(96)00791-5

of infinite spacetime dimension D, where a saddle-point approximation to the functional integral yields exact results via one-loop calculations. This limit exhibits immediately an important unphysical feature of the Nambu-Goto string model [1]: a complex string potential at distances smaller than a critical radius R_c, where the interquark potential vanishes. The existence of such a critical radius is attributed to tachyonic states in the string spectrum [9].

This and other unphysical properties are found in the so-called *static* interquark potential, where the string is anchored to immobile infinitely heavy quarks. In this limit, the eigenfrequencies ω_n of the string are integer multiples of π/R with $n = 1,2,\ldots$, where R is the distance between the quarks. The associated Casimir energy

$$E_{\mathrm{C}} = \frac{D-2}{2}\sum_{n=1}^{\infty}\omega_n = \frac{\pi(D-2)}{2R}\sum_{n=1}^{\infty} n \qquad (1)$$

is summed with the help of Riemann's zeta function $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ to

$$E_{\mathrm{C}} = \frac{\pi(D-2)}{2R}\zeta(-1) = -\frac{\pi(D-2)}{24R}, \qquad (2)$$

yielding the well-known Lüscher term. The finiteness is the result of an analytic continuation of the zeta function by which the formally divergent sum $\zeta(-1) = \sum_{n=1}^{\infty} n$ is turned into the convergent sum $-(1/2\pi^2)\zeta(2) = -(1/2\pi^2)\sum_{m=1}^{\infty} 1/m^2 = -1/12$ via the formula $\zeta(z) = \pi^{z-1}2^z\sin(z\pi/2)\Gamma(1-z)\zeta(1-z)$. The same Casimir energy is obtained for a string with free ends where the nonzero eigenfrequencies are the same.

We shall see below that in the limit $D \to \infty$ the Casimir energy determines the interquark potential completely, yielding

$$V(R) = M_0^2 R\sqrt{1+\frac{\bar{R}^2}{R^2}}, \quad \bar{R}^2 = \frac{2RE_{\mathrm{C}}}{M_0^2}. \qquad (3)$$

Inserting (2), we find the potential calculated by Alvarez [1]:

$$V_{\mathrm{Alvarez}} = M_0^2 R\sqrt{1-\frac{R_c^2}{R^2}}, \quad R_c^2 = \frac{\pi(D-2)}{12M_0^2}. \qquad (4)$$

The quantity M_0^2 is the string tension.

The same potential is found for strings with free ends due to the same Casimir energy (1).

On the basis of this observation one might expect that the string potential depends only weakly on the quark masses. This expectation, however, is false. The string properties depend on the quark masses at the ends of the string in an essential way, so essential in fact, that an important unphysical property of the static string potential disappears if one of the quark masses is zero, the other infinite.

First attempts to investigate this problem have been undertaken in Refs. [11,12]. These former works failed to find the interesting result to be presented her. The first did not investigate the most interesting situation of asymmetric mass configuration. The second used an unphysical regularization procedure. In this note, the former authors have joined efforts and derived a result which may help constructing strings between quarks without unphysical properties.

2. If a Nambu-Goto string has point-like quarks of masses m_1, m_2 at the ends moving along the worldlines $C_a, a = 1,2$, the action reads [10] ($\hbar = c = 1$):

$$\mathcal{A} = -M_0^2\iint d^2\xi\sqrt{g} - \sum_{a=1}^{2} m_a\int_{C_a} ds_a, \qquad (5)$$

where $g = \det(g_{\alpha\beta})$ is the determinant of the string metric. If the string coordinates are parametrized by $x^\mu(\xi)$, then $g_{\alpha\beta} = \partial_\alpha x^\mu\,\partial_\beta x_\mu$, $\alpha,\beta = 0,1$. For calculating the interquark potential from such an action one conveniently uses the Gauss parametrization:

$$x^\mu(t,r) = (t,r,\boldsymbol{u}(t,r)), \quad 0 \le r \le R,$$
$$\mu = 0,1,\ldots,D-1, \qquad (6)$$

where the vector field $\boldsymbol{u}(t,r) = (x^2(t,r),\ldots, x^{D-1}(t,r))$ describes the transverse displacements of the string in D dimensions. Then $g_{\alpha\beta} = \delta_{\alpha\beta}+\partial_\alpha\boldsymbol{u}\partial_\beta\boldsymbol{u}$, with $\boldsymbol{uu} \equiv \sum_{j=2}^{D-1} u^j u^j$. The fluctuation spectrum is found from the linearized equations of motion and boundary conditions:

$$\Box\mathbf{u} = 0, \qquad (7)$$

$$m_1\ddot{\boldsymbol{u}} = M_0^2\boldsymbol{u}', \quad r = 0, \qquad (8)$$

$$m_2\ddot{\boldsymbol{u}} = -M_0^2\boldsymbol{u}', \quad r = R. \qquad (9)$$

Here dots and primes denote the derivatives with respect t and r, respectively, and $\Box \equiv \partial^2/\partial t^2 - \partial^2/\partial r^2$. The general solution to these equations has the form

$$u^j(t,r) = i\frac{\sqrt{2}}{M_0^2}\sum_{n\neq 0} e^{-i\omega_n t}\frac{\alpha_n^j}{\omega_n}u_n(r),$$
$$j = 2,\ldots,D-1, \tag{10}$$

where the amplitudes α_n^j satisfy the usual rule of the complex conjugation, $\alpha_n = \alpha*_{-n}$. The unnormalized eigenfunctions $u_n(r)$ are

$$u_n(r) = \cos\omega_n r - \omega_n\frac{m_1}{M_0^2}\sin\omega_n r, \tag{11}$$

and the eigenfrequencies ω_n satisfy the secular equation

$$\tan\omega R = \frac{M_0^2(m_1+m_2)\omega}{m_1 m_2\omega^2 - M_0^4}. \tag{12}$$

The Hamiltonian operator reads

$$H = \sum_n\sum_{j=2}^{D-1}\omega_n a_n^{j\dagger}a_n^j + E_C, \tag{13}$$

where E_C is the Casimir energy

$$E_C = \frac{D-2}{2}\sum_n \omega_n. \tag{14}$$

The creation and annihilation operators satisfy the usual commutation rules

$$[a_n^i, a_m^{j\dagger}] = \delta^{ij}\delta_{nm}. \tag{15}$$

The Casimir energy [13,14] renders the Lüscher correction to the interquark potential [6].

As in all field theories [14], the Casimir energy E_C diverges for large n, and a renormalization is necessary to obtain physical results. If both masses are infinite or zero, the roots in Eq. (12) are $n\pi/R$ with integer n, and the sum over eigenvalues is made finite with the help of the zeta function in (2).

The interesting alternative situation which drew our attention to a possible disease-curing effect of different masses at the ends of strings is the limiting case, $m_1 = \infty$ and $m_2 = 0$, in which one end is fixed, the other free. Such a string approximates mesons consisting of one heavy and one light quark bound together by a color-electric flux tube. In this limit, the boundary conditions (8) and (9) simplify to

$$u(t,0) = 0, \quad u'(t,R) = 0, \tag{16}$$

and the secular Eq. (12) assumes the form

$$\cos\omega R = 0, \tag{17}$$

which is solved by string eigenfrequencies ω_n which are half-integer multiples of π/R: $\omega_n = (n+1/2)\pi/R$ for $n = 0, 1, \ldots$. In this case the Casimir energy is given by the formal sum

$$E_C = \frac{D-2}{2}\sum_{n=0}^{\infty}\omega_n = \frac{\pi(D-2)}{2R}\sum_{n=0}^{\infty}(n+1/2)$$
$$= \frac{\pi(D-2)}{2R}\zeta(-1,1/2) = \frac{\pi(D-2)}{48R}, \tag{18}$$

where [15] $\zeta(z,1/2) = \sum_{n=0}^{\infty}(n+1/2)^{-z} = (2^z-1)\zeta(z)$. In contrast to the previous case, the Casimir energy has now a *positive* sign, and half the magnitude, and (3) yields the interquark potential

$$V = M_0^2 R\sqrt{1+\frac{1}{2}\frac{R_c^2}{R^2}}, \quad R_c^2 = \frac{\pi(D-2)}{12M_0^2}. \tag{19}$$

This is an important result. Since the Casimir energy determines completely the interquark potential to be (3), a string with these boundary conditions is physical for all distances R in the limit $D \to \infty$. Fig. 1 compares the new string potential which is physical for all distances R with Alvarez' potential which is real only for $R > R_c$.

This observation raises the question whether there might be an entire regime of asymmetric quark mass configurations for which the potential remains physical and we must study the general case of both masses being finite. Then the roots in Eq. (12) have the large-n behavior

$$\omega_n \simeq n\frac{\pi}{R} + \frac{M_0^2(m_1+m_2)}{m_1 m_2}\frac{1}{n\pi} + O(n^{-3}). \tag{20}$$

and the formal zeta function regularization can no longer be applied (since $\sum_{n=1}^{\infty} n^{-1} = \zeta(1) = \infty$), calling for a different and more physical subtraction procedure.

There exists a simple analytic expression for the subtracted Casimir energy. To find it we introduce the dimensionless frequency sum $S \equiv (12R/\pi)\sum_n \omega_n$ and rewrite it as

$$S = -\frac{6R}{\pi^2 i}\int d\omega\omega\frac{d}{d\omega}\log\left[\cos(\omega R)M_0^2(m_1+m_2)\omega - \sin(\omega R)(m_1m_2\omega^2 - M_0^4)\right] - (R\to\infty)\,. \tag{21}$$

The derivative of the logarithm contains the solutions of the secular Eq. (12) as poles with unit residue. The contour of integration encloses the positive ω-axis in the clockwise sense. After opening up the contour and integrating along the imaginary frequency axis $\omega = iy$, a partial integration leads to

$$S = \frac{6R}{\pi^2}\int_{-\infty}^{\infty} dy\log\left[\cosh(yR)M_0^2(m_1+m_2)y + \sinh(yR)(m_1m_2y^2 + M_0^4)\right] - (R\to\infty)\,. \tag{22}$$

For a comparison of the behavior of the quark potential for various quark mass configurations it is useful to go over to the dimensionless distance variable $\rho \equiv R/R_c$ and to reduced quantities $\rho_{1,2} \equiv R_{1,2}/R_c$ where $R_{1,2}$ are length parameters associated with the quark masses defined by

$$R_{1,2} \equiv \frac{\pi(D-2)}{12m_{1,2}}\,. \tag{23}$$

With the integration variable $z = yR$, we can rewrite S as

$$S(\rho) = \frac{12}{\pi^2}\int_0^{\infty} dz\log\left[1 - e^{-2z}h(z,\rho)\right],$$

$$h(z,\rho) = \frac{z^2 - (\rho_1+\rho_2)\rho z + \rho_1\rho_2\rho^2}{z^2 + (\rho_1+\rho_2)\rho z + \rho_1\rho_2\rho^2}\,. \tag{24}$$

For $m_1 = \infty$, i.e., $\rho_1 = 0$, $S(\rho)$ is a simple function of $\rho_2\rho$ which runs from $S = -1$ for $\rho_2\rho = 0$ to $S = 1/2$ for $\rho_2\rho = \infty$. In terms of $S(\rho)$, the interquark potential acquires the general form

$$\frac{V}{M_0^2R_c} = \rho\sqrt{1 + \frac{S(\rho)}{\rho^2}}\,. \tag{25}$$

In Fig. 1 we have plotted the potential for $\rho_1 = 0$ and different $\rho_2 = 0,\ 1/5,\ 1,\ 2,\ 10,\ 100,\ \infty$. The plot

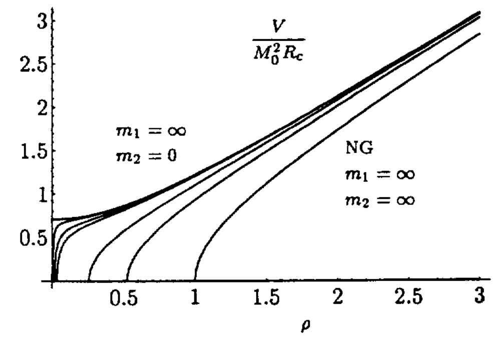

Fig. 1. Dependence of the dimensionless interquark potential on boundary conditions in string model. The upper curve shows the potential (19) corresponding to the extremely asymmetric boundary condition of one string end being fixed, the other free. The lowest curve presents Alvarez' result (4) for strings with both ends fixed or free. The lengths are measured in units of Alvarez' critical radius R_c. The remaining curves show the potential for one infinite and one finite quark mass m_2 corresponding to the reduced mass parameter $\rho_2 = 0,\ 1/5,\ 1,\ 2,\ 10,\ 100,\ \infty$ (from lowest to highest curves).

shows that, unfortunately, only the limit $m_2 = 0$ is associated with a real for all R. For a small but finite m_2, the function $S(\rho)$ always becomes negative if the radius R is much smaller than m_2/M_0^2.

3. Let us verify that the interquark potential is indeed determined by the Casimir energy as stated in Eq. (3). The potential $V(R)$ between massive quarks separated by a distance R is defined by the functional integral [6,17,16]

$$e^{-TV(R)} = \int[Du]\,e^{-\mathcal{A}_E[u]},\quad T\to\infty\,, \tag{26}$$

where $\mathcal{A}_E$ is the euclidean action (2.1).

$$\mathcal{A}_E = M_0^2\int_0^T dt\int_0^R dr\sqrt{\det(\delta_{\alpha\beta} + \partial_\alpha \boldsymbol{u}\,\partial_\beta \boldsymbol{u})} + \sum_{a=1}^{2} m_a\int_0^T dt\sqrt{1 + \dot{\boldsymbol{u}}^2(t,r_a)}\,. \tag{27}$$

We want to calculate the leading term for $D\to\infty$. As usual, we make the action harmonic in the string positions by introducing an auxiliary composite fields $\sigma_{\alpha\beta}$ and constrain it to be equal to $\partial_\alpha\boldsymbol{u}\,\partial_\beta\boldsymbol{u}$ by means of a Lagrange multiplier $\alpha^{\alpha\beta}$. By a similar manipulation, also the end-point actions can be made harmonic. After some manipulations, the functional integral (26) becomes Gaussian in $\boldsymbol{u}$ and can be performed with the result

$$e^{-TV(R)} = \int [D\alpha][D\sigma]\, e^{-\mathcal{A}_E[\alpha,\sigma]},$$

$$T \to \infty\,, \tag{28}$$

where

$$\mathcal{A}_E = M_0^2 \int_0^T dt \int_0^R dr \left[\sqrt{\det(\delta_{\alpha\beta}+\sigma_{\alpha\beta})} - \tfrac{1}{2}\alpha^{\alpha\beta}\sigma_{\alpha\beta}\right] + \frac{D-2}{2}\operatorname{Tr}\ln(-\partial_\alpha \alpha^{\alpha\beta}\partial_\beta)\,. \tag{29}$$

The boundary term in (27) is taken into account via the eigenvalues of the differential operator $-\partial_\alpha\alpha^{\alpha\beta}\partial_\beta$ in the action (29). As in Ref. [1], the functional integral is determined by the stationary point of (29) at which the matrices α and σ are diagonal. This simplifies the functional trace in (29) which becomes

$$\frac{D-2}{2}\operatorname{Tr}\ln(-\partial_\alpha\alpha^{\alpha\beta}\partial_\beta) = \frac{D-2}{2}T\sum_n \int_{-\infty}^{+\infty}\frac{dq_0}{2\pi}\ln\left(\alpha^{00}q_0^2+\alpha^{11}\omega_n^2\right) = T\sqrt{\frac{\alpha^{11}}{\alpha^{00}}}\,E_C\,. \tag{30}$$

Extremizing (29) with respect to $\sigma_{00}, \sigma_{11}, \alpha^{00}, \alpha^{11}$ yields indeed the string potential (3), as stated above. As in Alvarez' calculation, we can verify that the boundary conditions at the massive end points which are in general not compatible with the constant values of $\sigma_{00}, \sigma_{11}, \alpha^{00}, \alpha^{11}$ do not cause any error.

4. It will be interesting to see whether the results derived in this note are present also for a finite dimension D. If this is so, then at least the limiting asymmetric quark mass configuration may be free of some of the unphysical features of present-day string models.

Finally we remark that a dependence of the interquark potential on the quark masses at the ends was observed before in different ways [4,5]. In quantum field theory, the influence of different boundary conditions upon the Casimir effect has also been explored [14] resulting in energies of opposite signs.

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