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Broken Scale Invariance and $\sigma\pi\pi$, $A\sigma\pi$ and σAA Vertices.

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The assumption that the trace of the energy momentum tensor $\theta(x) \equiv \theta_\mu{}^\mu(x)$ is a good interpolating field of the (hypothetical) σ-meson [1] has recently turned out to be a powerful tool in relating the properties of this meson to the dimensional content of the Hamiltonian density [2-5].

Let $\theta_{00}(x)$ consist of a sum of an $SU_2 \times SU_2$ singlet $\bar{\theta}_{00}(x)$ and a term $\theta_4(x)$, which, together with the divergence of the axial vector current $\partial_\mu A^\mu(x)$, forms a representation $(\frac{1}{2}, \frac{1}{2})$ of chiral $SU_2 \times SU_2$. This means that the axial current $A_0(x)$ has the following commutation rules with $\theta_4(x)$ and $\partial_\mu A^\mu(x)$:

$$[A_0(x), \partial_\mu A^\mu(y)]_{x_0=y_0} = -i\theta_4(x)\,\delta^3(x-y)\,, \tag{1}$$

$$[A_0(x), \theta_4(y)]_{x_0=y_0} = i\partial_\mu A^\mu(x)\,\delta^3(x-y) \tag{2}$$

Let furthermore $\theta_4(x)$ be a scalar operator of dimension d and assume that all parts in $\bar{\theta}_{00}(x)$ having a dimension different from four are Lorentz and chiral scalars.

Then it can be shown [4] that $\theta_4(x)$ appears in the trace of the energy-momentum tensor $\theta(x)$ in the form $(4-d)\theta_4(x)$. As a consequence, the commutator (2) leads to

$$[A_0(\overset{\circ}{x}), \theta(y)]_{x_0=y_0} = i(4-d)\,\partial_\mu A^\mu(x)\,\delta^3(x-y) \tag{3}$$

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(1) H. A. KASTRUP: *Phys. Rev.*, **150**, 1183 (1966), and references therein.
(2) S. P. DE ALWIS and P. J. O'DONNELL: Toronto preprint (1970).
(3) H. KLEINERT and P. H. WEISZ: CERN preprint, to be published in *Nucl. Phys.*.
(4) M. GELL-MANN: Hawaii Summer School (1969).
(5) H. KLEINERT and P. H. WEISZ: CERN preprint, to be published in *Nucl. Phys.*.

and can be used to write a Ward identity relating the three-point function

$$\tau_\mu(q, p) \equiv i\int \mathrm{d}x\,\mathrm{d}y \exp[-i(qx - py)]\langle 0|T(\theta(x) A_\mu(y)\,\partial^\nu A_\nu(0))|0\rangle , \tag{4}$$

$$\tau(q^2, p^2, (q-p)^2) \equiv \int \mathrm{d}x\,\mathrm{d}y \exp[-i(qx - py)]\langle 0|T(\theta(x)\,\partial_\mu A^\mu(y)\,\partial_\nu A^\nu(0))|0\rangle , \tag{5}$$

to the propagators

$$\Delta(q) \equiv \int \mathrm{d}x \exp[-iqx]\langle 0|T(\partial_\mu A^\mu(x)\,\partial_\nu A^\nu(0))|0\rangle , \tag{6}$$

$$\Delta_{\theta\theta_4}(q) \equiv \int \mathrm{d}x \exp[-iqx]\langle 0|T(\theta(x)\,\theta_4(0))|0\rangle , \tag{7}$$

by

$$p^\mu \tau_\mu(q, p) = -\tau(q^2, p^2, (q-p)^2) + (4-d)\,\Delta(q-p) - \Delta_{\theta\theta_4}(q) . \tag{8}$$

We perform a maximal smoothness parametrization (6):

$$\tau(q^2, p^2, (q-p)^2) = -\Delta(p)\,\Delta(q-p)\,\Gamma(q^2, p^2, (q-p)^2) , \tag{9}$$

$$\tau_\mu(q, p) = \Delta(p)\,\Delta(q-p)\,\frac{p_\mu}{m_\pi^2}\,\Gamma(q^2, p^2, (q-p)^2) - \Delta_{\mu\nu}(p)\,\Delta(q-p)\,\Gamma^\nu(q, p) , \tag{10}$$

with

$$\Gamma(q^2, p^2, k^2) = \frac{m_\sigma^2}{q^2 - m_\sigma^2}\,[\Gamma_0 m_\pi^2 + \Gamma_1 q^2 + \Gamma_2(p^2 + k^2)]\,\frac{1}{f_\pi^2 m_\pi^2} , \tag{11}$$

$$\Gamma^\nu(q, p) = \frac{1}{q^2 - m_\sigma^2}\left[\alpha p^\nu - 2\beta\,\frac{m_\sigma^2}{m_\pi^2}\,q^\nu\right] C_\mathrm{A}^{-1} , \tag{12}$$

where $\Delta_{\mu\nu}(q)$ is the propagator of the axial vector:

$$\Delta_{\mu\nu}(q) \equiv \int \mathrm{d}m^2\,\frac{q_\mu q_\nu - g_{\mu\nu} m^2}{q^2 - m^2}\,\frac{\varrho_\mathrm{A}(m^2)}{m^4} \tag{13}$$

and

$$C_\mathrm{A} \equiv \int \mathrm{d}m^2\,\frac{\varrho_\mathrm{A}(m^2)}{m^4} . \tag{14}$$

If we assume these integrals to be saturated by a single A_2-meson

$$\varrho_\mathrm{A}(m^2) = m_\mathrm{A}^4/\gamma_\mathrm{A}^2\,\delta(m^2 - m_\mathrm{A}^2) , \tag{15}$$

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(6) H. J. SCHNITZER and S. WEINBERG: *Phys. Rev.*, **64**, 1824 (1968).

we find from eq. (8) the $\sigma\pi\pi$ and $A\sigma\pi$ vertices in terms of two parameters x and β (7):

$$(16)\qquad \begin{cases} \Gamma_0 = -4 + (1+x)d\,, \qquad \Gamma_1 = \left[(4-d)\dfrac{m_\pi^2}{m_\sigma^2} - \beta\right], \qquad \Gamma_2 = \beta - dx\,, \\ \alpha = (2-dx)\dfrac{m_\sigma^2}{m_\pi^2}\,. \end{cases}$$

This result corresponds exactly to what one would obtain from good old hard-pion techniques (8): eqs. (16) contain the maximal information that can be extracted from chiral current commutation rules. Define the coupling constants $g_{\sigma\pi\pi}$ and $g_{A\sigma\pi}$ by

$$\mathscr{L}_{\rho\pi\pi} = g_{\sigma\pi\pi}\frac{m_\sigma}{2}\sigma\pi\pi + g_{A\sigma\pi}A_\mu\pi\,\partial^\mu\sigma$$

and introduce the σ-meson–graviton coupling

$$\langle 0|\theta(0)|\sigma\rangle \equiv \frac{m_\sigma^3}{\gamma}\,.$$

Then equation (16) leads to

$$(17)\qquad \begin{cases} g_{\sigma\pi\pi} = \beta\gamma\left[1 + \left(\dfrac{x}{\beta}d - 2\right)\dfrac{m_\pi^2}{m_\sigma^2}\right], \\ g_{A\sigma\pi} = -2\beta\gamma f_\pi \dfrac{\gamma_A}{m_\sigma}\,. \end{cases}$$

It is the purpose of this letter to point out that the knowledge of the dimension of $\theta_4(x)$ is sufficient to determine the free constants β and x and thus to fix the ratio $g_{A\sigma\pi}/g_{\sigma\pi\pi}$.

Since $\theta_4(x)$ has dimension d it follows from $SU_2\times SU_2$ (eqs. (1) and (2)) that also $\partial_\mu A^\mu$ has dimension d.

If

$$(18)\qquad \mathscr{D}_\varkappa(x) \equiv x^\mu\theta_{\varkappa\mu}(x)$$

denotes the current density of dilatations, this means that

$$(19)\qquad i[\mathscr{D}_0(x), \partial_\mu A^\mu(y)]_{x_0=y_0} = (x\partial + d)\,\partial_\mu A^\mu(x)\,\delta^3(x-y)\,.$$

As a consequence we can derive a Ward identity for the function

$$(20)\qquad \sigma_\varkappa(q,p) \equiv i\int \mathrm{d}x\,\mathrm{d}y\,\exp[-i(qx-py)]\langle 0|T(\mathscr{D}_\varkappa(x)\,\partial_\mu A^\mu(y)\,\partial_\nu A^\nu(0))|0\rangle$$

(7) The parameter x is defined by the ratio of the propagators $\Delta_{\theta\theta}(0)/d\Delta(0) \equiv x$.

(8) Among the vast literature one may conveniently consult R. ARNOWITT, M. H. FRIEDMANN and P. NATH: *Phys. Rev.*, **174**, 2008 (1968).

in the form

$$q^{\varkappa}\sigma_{\varkappa}(q,p) = \tau(q^2, p^2, (q-p)^2) - (d-4-(q-p)\,\partial_q)\,\Delta(q-p) + d\Delta(p)\,. \tag{21}$$

The corresponding low-energy theorem at $q=0$ gives the results (3)

$$\Gamma(0, \mu^2, \mu^2) = 2\mu^2\,, \tag{22}$$

$$\frac{\partial}{\partial p^2}\,\Gamma(0, \mu^2, p^2)\big|_{p^2=\mu^2} = 1-d\,. \tag{23}$$

Inserting Γ, from eq. (11), and our parametrization eq. (16) proves our assertion ($\beta = x = 1$).

The result implies

$$\frac{g_{A\sigma\pi}}{g_{\sigma\pi\pi}} = -2\,\frac{\gamma_A f_\pi}{m_\sigma}\,\frac{1}{1+(d-2)(m_\pi^2/m_\sigma^2)}\,. \tag{24}$$

The important observation is that this ratio is essentially independent of the dimension of the symmetry breaker $\theta_4(x)$. (Since one usually assumes (9) $1 \leqslant d < 4$ and since $m_\pi^2/m_\sigma^2 \ll 1$.)

If one neglects the small d-dependent term we recover the well-known ratio of Gilman and Harari (10).

Notice that the additional information supplied by broken scale invariance on this ratio is nontrivial. Without it the ratio β/x could have been of the order of m_σ^2/m_π^2 causing a strong d-dependence in $g_{\sigma\pi\pi}$.

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(9) K. WILSON: *Phys. Rev.*, **179**, 1499 (1969).

(10) F. GILMAN and H. HARARI: *Phys. Rev.*, **165**, 1821 (1967). Using this ratio and the experimental mass and width $m_\sigma \approx 700$, $\Gamma_\sigma \approx 400$ MeV we find $g_{\sigma\pi\pi}^2/4\pi \approx 1.7$, $g_{A\sigma\pi}^2/4\pi \approx 13.6$, $\Gamma_{A\sigma\pi} \approx 50$ MeV. In saturation schemes of the algebra $SU_2 \times SU_2$ by π, σ, A and ρ mesons one obtains $g_{A\sigma\pi}/g_{\sigma\pi\pi} = 2(m_A/m_\rho)$ (S. WEINBERG: *Phys. Rev.*, **177**, 2613 (1967)). In larger schemes containing also the f-meson (F. BUCCELLA, H. KLEINERT, C. SAVOY, E. CELEGHINI and E. SORACE: *Nuovo Cimento*, **69** A, 133 (1970)) one finds

$$\frac{g_{A\sigma\pi}}{g_{\sigma\pi\pi}} = 2\,\frac{m_A}{m_\sigma}\sqrt{\frac{m_f^2+2m_\sigma^2}{3m_\rho^2}}\,.$$

Since $m_f^2 \approx 3m_\rho^2 \approx 3m_\sigma^2$ this amounts to a $\Gamma_{A\sigma\pi}$ width about $\frac{5}{3}$ larger than that of Gilman and Harari. It is interesting to note that the Veneziano amplitude determines $g_{A\sigma\pi}$ in terms of $g_{A\rho\pi}$ and $h_{A\rho\pi}$ to be

$$g_{A\sigma\pi} = -\frac{g_{A\rho\pi}}{m_\sigma} - \frac{1}{2}\,\frac{m_\rho^2 h_{A\rho\pi}}{m_\sigma}$$

(where $g_{A\rho\pi}$ and $h_{A\rho\pi}$ are defined by $\mathscr{L} = g_{A\rho\pi}\,\varrho_\mu A^\mu \times \pi + h_{A\rho\pi}\,\varrho_\mu \partial^\mu A^\nu \times \partial_\nu \pi$). For reference see C. SAVOY: *Lett. Nuovo Cimento*, **2**, 870 (1969); J. L. ROSNER and H. SUURA: *Phys. Rev.*, **187**, 1905 (1969); P. CARRUTHERS and F. COOPER: *Phys. Rev. D*, **1**, 1232 (1970). Inserting longitudinal and transverse coupling constants

$$g_L \equiv -h_{A\rho\pi} + 2g_{A\rho\pi}\,\frac{m_A^2+m_\rho^2}{m_A^2-m_\rho^2}\,, \qquad g_T \equiv -g_{A\rho\pi}\,\frac{4m_A^2}{(m_A^2-m_\rho^2)^2}$$

we find $g_{A\sigma\pi} \approx \frac{1}{2}(g_T+g_L)(m_\rho^2/m_\sigma)$. GILMAN and HARARI give $g_T \approx 0$, $g_L = 4/f_\pi$, hence $g_{A\sigma\pi} \approx 2m_\rho^2/m_\sigma f_\pi$. Comparing this with their value $g_{\sigma\pi\pi} = m_\sigma/\sqrt{2}\,f_\pi$ one obtains $g_{A\sigma\pi}/g_{\sigma\pi\pi} \approx 2\sqrt{2}\,m_\rho^2/m_\sigma^2$, which is approximately the same ratio as before.

In addition to the $A\sigma\pi$ coupling the more academic vertex σAA is determined as well by our Ward identities. Just for completeness we note that we can write a Ward identity for the three-point function

$$\tau_{\mu\nu}(q, p) = i\int \mathrm{d}x\,\mathrm{d}y \exp[-i(qx - py)]\langle 0|T(\theta(x)\,A_\mu(y)\,A_\nu(0))|0\rangle \tag{25}$$

in terms of the propagator

$$\Delta_\mu(q) = \frac{q_\mu}{m_\pi^2}\,\Delta(q) \tag{26}$$

reading

$$p^\mu \tau_{\mu\nu}(q, p) = \tau_\nu(q; (q-p)) + (4-d)\,\Delta_\nu(q-p)\,. \tag{27}$$

After a maximal-smoothness assumption for the σAA vertex, this yields for the coupling constant $g_{\sigma AA}$ (defined by $\mathscr{L} = \frac{1}{2} g_{\sigma AA} m_A \sigma A_\mu A^\mu$)

$$\frac{g_{\sigma AA}}{g_{\sigma\pi\pi}} = -2\,\frac{\gamma_A^2 f_\pi^2}{m_A m_\sigma}\,\frac{1}{1 + (d-2)(m_\pi^2/m_\sigma^2)}\,. \tag{28}$$

Our results can be compared with the couplings given by the linear σ-model with axial and vector fields introduced à la Yang-Mills. We find

$$g_{\sigma\pi\pi} = -\frac{m_\sigma}{f_\pi}\,\frac{m_\rho^3}{m_A^3}\left[1 + (Z-2)\,\frac{m_\pi^2}{m_\sigma^2}\right] \tag{29}$$

and exactly the same ratios (24) and (28) except that d is replaced by $Z = m_A^2/m_\rho^2$ everywhere [11]. We conclude that if we want to reproduce the results of this model according to our methods we have to assign the dimension $d = Z$ to pion and σ fields when the Yang-Mills fields A_1 and ρ are present.

The dimension Z can be read off the result eq. (29) in another independent way. From eq. (17) (for $\beta = x = 1$) we know that the factor in front of the brackets has to be identified with the graviton-σ coupling γ:

$$\gamma = -\frac{m_\sigma}{f_\pi}\,\frac{m_\rho^3}{m_A^3} = -\frac{m_\sigma}{f_\pi}\,\frac{1}{Z^{\frac{3}{2}}}\,. \tag{30}$$

In the effective Lagrangian this constant is recovered in the following way: One takes the terms that can contribute to $\theta(x)$ linearly in the field $\sigma' = \sigma - \langle 0|\sigma|0\rangle = \sigma - \sigma_0$:

$$\mathscr{L} = \ldots - \frac{\mu_0^2}{2}(\sigma^2 + \pi^2) + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 + m_\pi^2 f_\pi Z^{-\frac{1}{2}}\sigma\,. \tag{31}$$

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(11) These results can be read directly off the Lagrangian eq. (6.3) of S. GASIOROWICZ and D. A. GEFFEN: *Rev. Mod. Phys.*, **41**, 542 (1969) after having the terms $-(\mu_0^2/2)(\sigma^2 + \pi^2) + (\lambda/4)(\sigma^2 + \varphi^2)^2$ in order to go over to the linear σ-model. Note that the magnitude of $g_{\sigma\pi\pi}$ is about half the value following from the Adler-Weisberger sum rule. The reason is clearly that the $\pi\pi$ scattering amplitude following from the chiral Lagrangian is in general not unsubstracted.

If σ and π have dimension Z, this leads to a trace of the energy-momentum tensor (4)

$$\theta(x) = \ldots + (4 - 2Z)\,\mu_0^2(\sigma^2 + \pi^2) - \lambda(1 - Z)(\sigma^2 + \pi^2)^2 - (4 - Z)\,m_\pi^2 f_\pi Z^{-\frac{1}{2}}\sigma\,, \tag{32}$$

which gives, upon inserting $\sigma = \sigma_0 + \sigma'$, a linear term

$$\theta(x) = \ldots - Z(\mu_0^2 - \lambda\sigma_0^2)\,f_\pi Z^{\frac{1}{2}}\sigma' = \ldots - Z^{\frac{3}{2}}\,m_\sigma^2 f_\pi \sigma'\,, \tag{33}$$

comparing with the defining relation

$$\theta(x) = \ldots + \frac{m_\sigma^3}{\gamma}\,\sigma'\,, \tag{34}$$

we obtain indeed the result (30).

Notice that this assignment of dimension Z different from the canonical dimension one (12) has nothing to do with the concept of anomalous dimensions as discussed by WILSON on the basis of exactly scale-invariant theories (13). Wilson's dimension is connected with the Schwinger term in the commutator

$$i\,[\theta_{0i}(x), \varphi(y)]_{x_0=y_0} = \partial_i\varphi(x)\,\delta^3(x-y) - \frac{d}{3}\,\varphi(y)\,\partial_i\,\delta^3(x-y)\,, \tag{35}$$

where $\theta_{\mu\nu}(x)$ is the local energy-momentum tensor with *finite* matrix elements (14). In our phenomenological Lagrangian we are free to assign any dimension d to the pion field. This can be done, at the canonical level, by adding the term

$$-\frac{d}{6}\,(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\,(\sigma^2 + \pi^2) \tag{36}$$

to the canonical energy-momentum tensor. A detailed investigation of this energy-momentum tensor shows that every step in our calculation can be carried through in this model for $d = Z$. If, however, d is chosen differently a term $Z^{-2}(d - Z)(\partial_\mu\pi)^2$ appears in the trace of the energy-momentum tensor destroying the validity of the maximal smoothness parametrization of eq. (11).

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(12) R. A. BRANDT and G. PREPARATA: CERN preprint (1970), TH 1208.

(13) K. WILSON: Stanford preprint (1970), discovered the anomalous dimension in the case of the Thirring model. R. GATTO claims to have found the same phenomena in the scale-invariant $\lambda\varphi^4$ theory (private communication).

(14) C. CALLAN, S. COLEMAN and R. JACKIW: *Ann. of Phys.* (to be published).