Stability of a three-dimensional cubic fixed point in the two-coupling-constant ϕ^4 theory

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For an anisotropic Euclidean ϕ^4 theory with two interactions $[u(\Sigma_{i=1}^M \phi_i^2)^2+v\Sigma_{i=1}^M \phi_i^4]$ the β functions are calculated from five-loop perturbation expansions in $d=4-\varepsilon$ dimensions, using the knowledge of the large-order behavior and Borel transformations. For $\varepsilon=1$, an infrared-stable cubic fixed point for $M\geqslant 3$ is found, implying that the critical exponents in the magnetic phase transition of real crystals are of the cubic universality class. There were previous indications of the stability based either on lower-loop expansions or on less reliable Padé approximations, but only the evidence presented in this work seems to be sufficiently convincing to draw this conclusion. [S0163-1829(97)01946-2]

I. INTRODUCTION

The most elegant approach to phase transitions in many physical systems proceeds via field-theoretic renormalization-group techniques.[1] The best-studied system is the isotropic Heisenberg ferromagnet with M classical spin components. Its critical behavior can be described correctly by an $O(M)$-symmetric vector field theory with a quartic interaction $\Sigma_{i=1}^M(\phi_i^2)^2$.

In a real crystal, such an interaction is never presentably by itself. The crystalline structure gives rise to anisotropies, most prominently of cubic symmetry, which can be represented by an extra field interaction $\Sigma_{i=1}^M \phi_i^4$. This term breaks the $O(M)$ symmetry by favoring magnetizations along the edges or the diagonals of a hypercube in M dimensions. The extended theory interpolates between an $O(M)$ symmetric and a cubic system. It has been pointed out a long time ago[2] that, depending on M, the $O(M)$-symmetric and the cubic fixed point interchange their stability. For $M<M_c$, the $O(M)$-symmetric, *isotropic* fixed point is stable. For $M>M_c$ the isotropy is destabilized and the trajectories of renormalization flow cross over to the *cubic* fixed point. Estimates using calculations up to three loops[2–4] indicated that M_c must lie somewhere between 3 and 4. Resummation procedures based on Padé approximations[5] suggested M_c to lie below 3, thus permitting real crystals to exhibit critical exponents of the cubic universality class. The uncertainty of these estimates have prompted Kleinert and Schulte-Frohlinde[6] to carry the expansions up to five loops. They increased the evidence for $M_c<3$ considerably, again via Padé resummation.

For a simple ϕ^4 theory, the Padé approximation is known to be inaccurate. At present, the most accurate renormalization group functions for that theory have been obtained by combining perturbation expansions with large-order estimates, using a resummation procedure based on Borel transformations.[7–13]

Intending the application of these more powerful resummation methods, the large-order behavior of renormalization-group functions has recently been derived for the M-vector model with cubic anisotropy, by Kleinert and Thoms.[14]

It is the purpose of this paper is to combine these large-order results with the five-loop perturbation expansions[6] using a simple Borel type of resummation algorithm,[7,8] whose power has been exhibited in recent model studies by Kleinert, Thoms, and Janke.[15]

The results to be presented in this paper allow us to conclude with a reasonable certainty that an infrared-stable cubic fixed point exists at the physically most relevant value $M=3$. However, due to the vicinity of the isotropic fixed point, the differences in the critical exponents are very hard to measure experimentally. Going beyond the Padé work in Refs. 5 and 6, we also show explicitly the instability and stability of the isotropic and the cubic fixed point, respectively.

II. RESUMMATION

A. The problem

The object of investigation is a ϕ^4 theory with cubic anisotropy. The corresponding energy functional reads

$$H(\vec{\phi})=\int d^dx\left[\frac{1}{2}\partial_\mu\phi_{Bi}\partial_\mu\phi_{Bi}+\frac{8\pi^2}{3}\left(\frac{u_B}{4}S_{ijkl}+\frac{v_B}{4}\delta_{ijkl}\right)\phi_{Bi}\phi_{Bj}\phi_{Bk}\phi_{Bl}\right], \quad (1)$$

where $\phi_{Bi}(x)$ $(i=1,2,\ldots,M)$ is the bare M-component field in $d=4-\varepsilon$ dimensions, and u_B, v_B are the bare coupling constants. In particular, we shall consider the physically most interesting case of $M=3$, and continue ε to $\varepsilon=1$. The tensors associated with the two interaction terms in Eq. (1) have the following symmetrized form:

$$S_{ijkl}=\frac{1}{3}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),$$

$$\delta_{ijkl}=\begin{cases}1, & i=j=k=l,\\ 0, & \text{otherwise.}\end{cases}$$

The symmetry of the action under reflection $\phi_i\rightarrow-\phi_i$ and under permutations of the M field indices i implies the following form of the vertex functions, persisting to all orders in perturbation theory:

$$\Gamma^{(2)}_{ij} \sim \Gamma^{(2)}\delta_{ij}; \quad \Gamma^{(2,1)}_{ij} \sim \Gamma^{(2,1)}\delta_{ij}, \tag{2}$$

$$\Gamma^{(4)}_{ijkl} \sim \Gamma^{(4)}_{u} S_{ijkl} + \Gamma^{(4)}_{v}\delta_{ijkl}. \tag{3}$$

The symmetry permits us to renormalize the M components of the field $\vec{\phi}$ and the composite field $\frac{1}{2}\vec{\phi}^2$ with only a single renormalization constant Z_ϕ and Z_{ϕ^2}, respectively. The bare field $\vec{\phi}_B$, the composite field $\frac{1}{2}\vec{\phi}_B^2$, and the two coupling constants u_B and v_B are related to the corresponding physical objects by

$$\vec{\phi}_B(x) = Z_\phi^{1/2}\vec{\phi}(x), \quad [\vec{\phi}_B^2](x) = (Z_{\phi^2})^{-1}[\vec{\phi}^2](x),$$

$$u_B = \mu^\varepsilon Z_u (Z_\phi)^{-2} u, \quad v_B = \mu^\varepsilon Z_v (Z_\phi)^{-2} v, \tag{4}$$

where μ is a mass parameter. We employ dimensional regularization with minimal subtraction. The square brackets around $\vec{\phi}^2$ indicate a renormalization of this composite operator. Recall that this renormalization is different from that of the wave function which converts $\vec{\phi}_B$ to $\vec{\phi}$. In fact it is closely related to the mass renormalization of a theory with mass $m \neq 0$. In the minimal subtraction scheme, the corresponding renormalization constants are related by $Z_{\phi^2} = Z_{m^2}(Z_\phi)^{-1}$.

The RG functions are defined in the usual way:

$$\beta^u(u,v) = \mu\partial_\mu u|_{u_B, v_B, \varepsilon},$$

$$\beta^v(u,v) = \mu\partial_\mu v|_{u_B, v_B, \varepsilon},$$

$$\gamma_\phi(u,v) = \mu\partial_\mu \ln Z_\phi|_{u_B, v_B, \varepsilon},$$

$$\gamma_{\phi^2}(u,v) = -\mu\partial_\mu \ln Z_{\phi^2}|_{u_B, v_B, \varepsilon}. \tag{5}$$

The natural parameter for the anisotropy of the system is the ratio $\delta = v/(u+v)$, and the isotropic case corresponds to $\delta = 0$. We shall use the new couplings $g = u + v$ and δ for the calculation of the fixed points from the resummed β functions

$$\beta^g(g,\delta) = \beta^v[u(g,\delta), v(g,\delta)] + \beta^u[u(g,\delta), v(g,\delta)],$$

$$g\beta^\delta(g,\delta) = (1-\delta)\beta^v[u(g,\delta), v(g,\delta)] - \delta\beta^u[u(g,\delta), v(g,\delta)]. \tag{6}$$

The $O(M)$-symmetric and cubic fixed points can be obtained by calculating the simultaneous zeros (g^*, δ^*) of $\beta^u[u(g,\delta), v(g,\delta)]$ and $\beta^v[u(g,\delta), v(g,\delta)]$. For the physically interesting number of field components, $M = 3$, the infrared-stable cubic fixed point is expected to appear very close to the $O(M)$-symmetric one. Since δ is very small in this region, it will be sufficient to restrict the resummation efforts to the g series accompanying each power δ^n, so that the β functions at the cubic fixed point will be approximated by

$$0 = \beta^u[u(g^*,\delta^*), v(g^*,\delta^*)] \approx \sum_{n=0}^{N} B_n^{u(N)}(g^*)\delta^{*n},$$

$$0 = \beta^v[u(g^*,\delta^*), v(g^*,\delta^*)] \approx \delta^* \sum_{n=1}^{N} B_n^{v(N)}(g^*)\delta^{*n-1}, \tag{7}$$

where $B_n^{(N)}(g) \equiv \mathrm{res}[\Sigma_{k=n}{}^N \beta_{kn} g^k]$ indicates resummed g series.

From the five-loop perturbation expansion in Ref. 6, the perturbation coefficients $\beta_{kn}^{u/v}$ are known up to the order $N = 6$. For $\varepsilon = 1$ and the number of field components $M = 3$, the following expansions are known:

$$\sum_{k=0}^{6} \beta^u_{k0} g^k = -g + 3.667g^2 - 7.667g^3 + 47.651g^4 - 437.646g^5 + 4998.62g^6,$$

$$\sum_{k=1}^{6} \beta^u_{k1} g^k = g - 5.333g^2 + 15.667g^3 - 121.767g^4 + 1341.05g^5 - 17821.1g^6,$$

$$\sum_{k=2}^{6} \beta^u_{k2} g^k = 1.667g^2 - 10g^3 + 115.885g^4 - 1664.86g^5 + 27191g^6,$$

$$\sum_{k=3}^{6} \beta^u_{k3} g^k = 2g^3 - 50.074g^4 + 1064.62g^5 - 22916.2g^6,$$

$$\sum_{k=4}^{6} \beta^u_{k4} g^k = 8.305g^4 - 350.528g^5 + 11183.1g^6,$$

$$\sum_{k=5}^{6} \beta^u_{k5} g^k = 47.368g^5 - 2966.14g^6, \quad \beta^u_{66} g^6 = 330.76g^6,$$

and

$$\sum_{k=1}^{6} \beta^v_{k1} g^k = -g + 4g^2 - 10.778g^3 + 75.875g^4 - 776.26g^5 + 9707.36g^6,$$

$$\sum_{k=2}^{6} \beta^v_{k2} g^k = -g^2 + 6.222g^3 - 67.319g^4 + 944.05g^5 - 15030.9g^6,$$

$$\sum_{k=3}^{6} \beta^v_{k3} g^k = -1.111g^3 + 30.211g^4 - 639.243g^5 + 13549.6g^6,$$

$$\sum_{k=4}^{6} \beta^v_{k4} g^k = -6.218g^4 + 233.262g^5 - 7122.94g^6,$$

$$\sum_{k=5}^{6} \beta^v_{k5} g^k = -33.414g^5 + 1973.58g^6, \quad \beta^v_{66} g^6 = -228.19g^6. \tag{8}$$

In addition to the five-loop expansions, the large-order behavior of β^u_{kn} and β^v_{kn} were obtained explicitly in Ref. 14, with the result

$$\beta^{u/v}_{kn} \xrightarrow{k\to\infty} \gamma^{u/v}(n)(-1)^k k!\, k^{(d+5)/2+n}[1+\mathcal{O}(1/k)],\ k\gg n. \tag{9}$$

The resummation algorithm to be employed in this paper will make use of all these informations.

B. The algorithm

As explained in detail in Ref. 15, it is possible to reexpand a divergent perturbation series

$$Z(g,\delta)=\sum_{k=0}^{\infty}\sum_{n=0}^{k} Z_{kn} g^k \delta^n \tag{10}$$

in a special infinite set of Borel summable functions $I_{pn}(g)$ as

$$Z(g,\delta)=\sum_{n=0}^{\infty}\left[\sum_{p=n}^{\infty} a_{pn} I_{pn}(g)\right]\delta^n, \tag{11}$$

so that the approximation

$$\begin{aligned} Z(g,\delta)\approx Z^{(N)}(g,\delta)&=\sum_{n=0}^{N}\left[\sum_{p=n}^{N} a_{pn} I_{pn}(g)\right]\delta^n \\ &=\sum_{n=0}^{N} Z_n^{(N)}(g)\,\delta^n \end{aligned} \tag{12}$$

has the same series as $Z(g,\delta)$ up to the powers $g^N\delta^N$, while reproducing the known large-order behavior of the perturbation expansion (10):

$$Z_{kn} \xrightarrow{k\to\infty} \gamma(n)(-\sigma)^k k!\, k^{\beta_n}[1+\mathcal{O}(1/k)],\quad k\gg n. \tag{13}$$

As shown by Janke and Kleinert in Refs. 7 and 8, the natural choice for the functions $I_{pn}(g)$ are certain confluent hypergeometric functions, where for the case of two coupling constants a second index was introduced in Ref. 15. One possibility was to use

$$\begin{aligned} I_{pn}(g)&=\left(\frac{4}{\sigma g}\right)^{b_0(n)+1}\int_0^1 dw\,\frac{(1+w)w^{b_0(n)+p}}{\Gamma[b_0(n)+1](1-w)^{2b_0(n)+2\alpha+3}} \\ &\quad\times\exp\left[-\frac{4w}{(1-w)^2\sigma g}\right] \end{aligned} \tag{14}$$

with

$$b_0(n)=\beta_n+\frac{3}{2}. \tag{15}$$

Then the coefficients a_{pn} are given by

$$a_{pn}=\sum_{k=n}^{p}\frac{Z_{kn}}{[b_0(n)+1]_k}\left(\frac{4}{\sigma}\right)^k\binom{p+k-1-2\alpha}{p-k}, \tag{16}$$

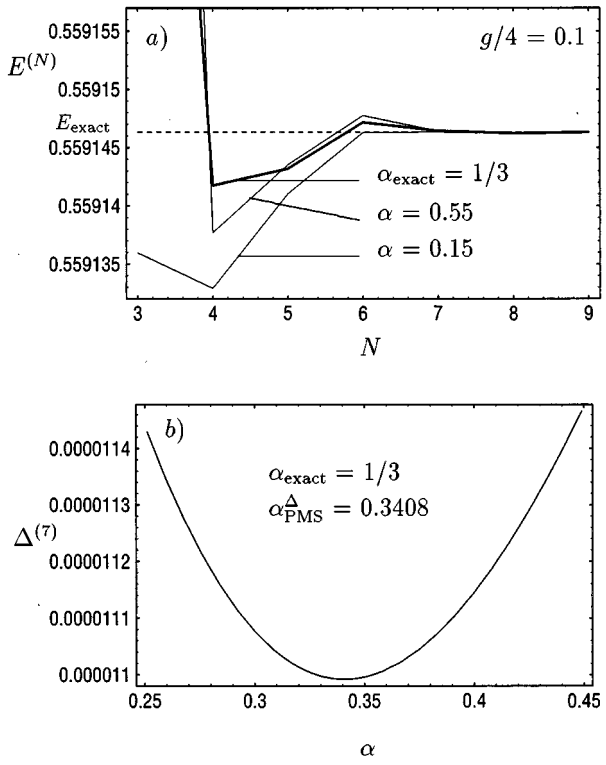

FIG. 1. Convergence of the ground-state energy E of the anharmonic oscillator with $g/4=0.1$. (a) Resummed ground-state energies $E^{(N)}$ are plotted versus the order of approximation N for various strong-coupling parameters α. (b) The α dependence of the function $\Delta^{(7)}$ of Eq. (18). The optimal α value is given by the minimum $\alpha^{\Delta}_{\text{PMS}}=0.3408$, very close to the exactly known value 1/3.

where $c_k=\Gamma(c+k)/\Gamma(c)$ are Pochhammer's symbols. The parameter α is free to choose, and may be used to accommodate any strong-coupling power of $Z(g,\delta)$

$$Z(g,\delta) \xrightarrow{g\to\infty} \kappa(\delta) g^{\alpha}, \tag{17}$$

if this is known. Since in quantum field theory, this is not the case, α will be chosen by the condition of best convergence of the resummed g series $Z_n^{(N)}(g)$ in Eq. (12), as explained in the next section.

C. Optimal choice of strong-coupling power α

An idea of the relevance of α is gained from the study of corresponding models in quantum mechanics, where the strong-coupling behavior can be deduced from scaling arguments. Consider first an $O(M)$-symmetric anharmonic oscillator with $g(x_i^2)^2/4$ interaction, where M is the number of components of the vector $\vec{x}$. Here the functions $Z_0^{(N)}(g)$ in Eq. (12) represent the resummed ground-state energies $E^{(N)}(g)$. In Fig. 1 we have illustrated the convergence of $E^{(N)}$ for the anharmonic oscillator with one x component ($M=1$) at a coupling constant $g/4=0.1$. We have plotted $E^{(N)}$ versus the order of approximation N for various values of the strong-coupling parameter α. At large N, the curves become increasingly independent of α and approach a saturation value which coincides with the ground-state energy.

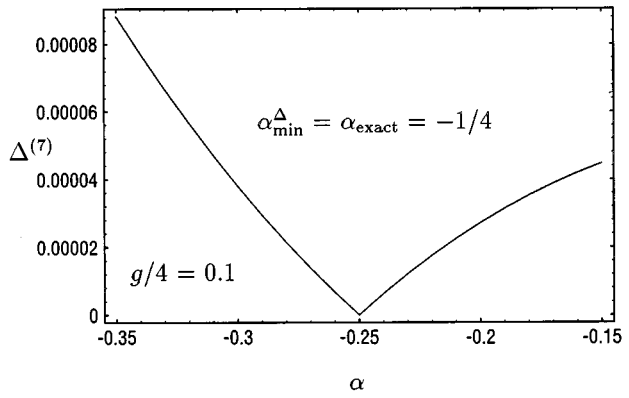

FIG. 2. Judicial function $\Delta^{(7)}(g,\alpha)$ of Eq. (18) to determine the strong-coupling power α for the simple integral $Z(g)=\int dx \exp(-x^2/2-gx^4/4)$, plotted as a function of α for $g/4=0.1$. The optimal parameter $\alpha^{\Delta}_{\rm min}$ lies at the lowest value of $\Delta^{(7)}(g,\alpha)$, and is found to be equal to the exact one. It can be shown that for $\alpha=\alpha_{\rm exact}=-1/4$, already the zeroth order of the resummation algorithm used in this paper reproduces the exact value of $Z(g)$ (Ref. 8). This is the origin of the cusp and $\Delta^{(N_s)}(g,\alpha)=0$ at $\alpha=-1/4$.

This exact result does not depend on α. Therefore, we choose the strong-coupling parameter under the condition that the curvature and the slope of the corresponding curve depend minimally on the variation of α, when approaching the saturation region. This choice complies with the principle of minimal sensitivity (PMS) which has been used with great success to optimize variational perturbation expansions.[16] Using the discretized form of the first and second derivative, the optimal α value, to be denoted by $\alpha^{\Delta}_{\rm PMS}$, is found by calculating the extrema or turning points of the judicial function

$$\Delta^{(N_s)}(g,\alpha)=[(E^{(N_s-1)}-E^{(N_s-2)})^2 + (E^{(N_s)}-2E^{(N_s-1)}+E^{(N_s-2)})^2]^{1/2}/E^{(N_s-2)}, \tag{18}$$

where N_s indicates the beginning of the saturation region. In our special example, $N_s=7$ [see Fig. 1(a)]. In Eq. (18) the value of the coupling constant g is chosen such that the error of the resummation becomes small and the influence of α is isolated. Since the rate of convergence of the resummation decreases for increasing g, we determine the optimal α at a small coupling constant $g/4=0.1$. In Fig. 1(b), we have plotted the α dependence of $\Delta^{N_s}(g,\alpha)$ for $N_s=7$ and $g/4=0.1$. The optimal α value is found to be $\alpha^{\Delta}_{\rm PMS}=0.341$, which is very close to the exactly known strong-coupling parameter $\alpha=1/3$.

We mention that for the zero-dimensional case which corresponds to a simple integral with $gx^4/4$ interaction, the above criterion yields the exact strong-coupling parameter $\alpha=-1/4$ (see Fig. 2).

In most applications of quantum field theory, perturbation series are too short to detect the formation of the saturation plateau with sufficient accuracy. We shall see in the following that for shorter series the criterion given above can be simplified without a significant loss of accuracy by neglecting the curvature term in Eq. (18). Then the slope term is deduced from the two highest known partial sums $E^{(N_{\rm max}-1)}$ and $E^{(N_{\rm max})}$ by forming

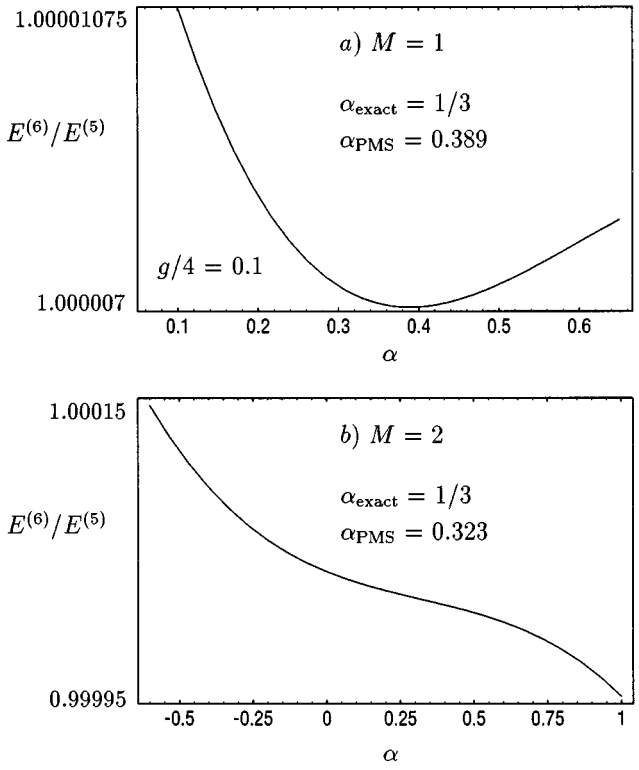

FIG. 3. The α dependence of the ratio $E^{(6)}/E^{(5)}$ in Eq. (19) for the $O(M)$-symmetric anharmonic oscillator at constant coupling strength $g/4=0.1$ and various numbers of field components M. (a) For the simple anharmonic oscillator ($M=1$) the optimal α value $\alpha_{\rm PMS}=0.389$ lies at the minimum. (b) For the two-component oscillator ($M=2$) the optimal value $\alpha_{\rm PMS}=0.323$ lies at the turning point.

$$\Delta^{(N_{\rm max}+1)}(g,\alpha)\approx \frac{E^{(N_{\rm max})}(g,\alpha)}{E^{(N_{\rm max}-1)}(g,\alpha)}-1, \tag{19}$$

where it is important that $E^{(N_{\rm max}-1)}$ and $E^{(N_{\rm max})}$ have gotten over the large-fluctuating initial region. Thus the optimal α value, denoted now by $\alpha_{\rm PMS}$, will be determined from calculating the extrema or turning points of $E^{(N_{\rm max})}(g,\alpha)/E^{(N_{\rm max}-1)}(g,\alpha)$. As before, the coupling constant is chosen to be small $g/4=0.1$ to ensure a sensitive determination of the optimal α.

As the largest available order of approximation for the RG functions in QFT is $N_{\rm max}=6$, we have plotted in Fig. 3 the α dependence of the ratio $E^{(6)}/E^{(5)}$ for the numbers of vector components $M=1$ and $M=2$ at a coupling constant $g/4=0.1$. In the case of $M=1$, a minimum exists at $\alpha_{\rm PMS}=0.389$. For $M=2$, there is no extremum, and the optimal α value lies at the turning point $\alpha_{\rm PMS}=0.323$. In both cases the simplified criterion yields again results for $\alpha_{\rm PMS}$ which are close to the exactly known strong-coupling parameter $\alpha=1/3$.

It is unnecessary to know α to a higher accuracy than that. In order to show this, we have compared the resummed

TABLE I. Convergence of the ground-state energy E_0 of the anharmonic oscillator with $M=1$ for the strong-coupling parameters $\alpha_{\text{exact}}=1/3$ and $\alpha_{\text{PMS}}=0.389$.

E_0	$g/4=0.1$		$g/4=1.0$	
N	α_{exact}	α_{PMS}	α_{exact}	α_{PMS}
1	0.561496	0.56235	0.83055	0.849631
2	0.558592	0.558614	0.78297	0.784942
3	0.559232	0.559254	0.812948	0.816638
4	0.559142	0.559142	0.801761	0.802012
5	0.559143	0.559143	0.802206	0.802487
6	0.559147	0.559147	0.805103	0.805518
7	0.559146	0.559146	0.803901	0.803937
8	0.559146	0.559146	0.803115	0.803072
9	0.559146	0.559146	0.803852	0.803924
exact	0.559146		0.803770	

ground-state energies $E^{(N)}$ of the anharmonic oscillator using various values of α (see Table I). For the coupling constants $g/4=0.1$ and $g/4=1.0$, the rate of convergence to the exact energies depends only very little on α.

Consider now the case that $Z(g,\delta)$ in Eq. (10) represents the ground-state energy $E(g,\delta)$ of the anisotropic quartic oscillator with an interaction

$$V_{\text{int}}=\frac{g}{4}[x^4+2(1-\delta)x^2y^2+y^4], \tag{20}$$

which we have studied in detail in Ref. 15 using the Borel-type resummation algorithm of Refs. 7 and 8. When expressed in terms of the old coupling constants u, v via $g=u+v$ and $\delta=v/(u+v)$, the expression (20) corresponds to the interaction term in the Euclidean action (1) for the number of field components $M=2$. It was found in Ref. 15 that the parameter α is the same for each coefficient $E_n(g)$ in the δ expansion of the ground-state energy $E(g,\delta)=\Sigma_{n=0}^{\infty}E_n(g)\delta^n$, and having at each n the same value $\alpha=1/3$ as in the isotropic case. The approximation $E_n^{(N)}(g)$ which follows from a resummation of the corresponding perturbation series in g up to the order N is becoming less accurate for increasing n since the g expansions have fewer and fewer terms. However, taking into account the smallness of the anisotropy δ, the δ expansion may be truncated at a finite order N. This yields the approximation

$$E^{(N)}(g,\delta)=\sum_{n=0}^{N}E_n^{(N)}(g)\,\delta^n,$$

which was found to be very accurate in a wide region of δ around $\delta=0$. We have compared the result for the ground-state energy resummed at $\alpha=1/3$, which is known from Ref. 15, with the result obtained for $\alpha_{\text{PMS}}=0.323$. Figure 4 shows the δ dependence of the approximated ground-state energy $E^{(6)}(g,\delta)$ for the two different values of the parameter α and various coupling constants $g/4$. For $g/4=0.1$, the two curves for α and α_{PMS} coincide. From Fig. 4 and Table I we can thus conclude that the error which is caused by an inaccurate determination of α is negligible.

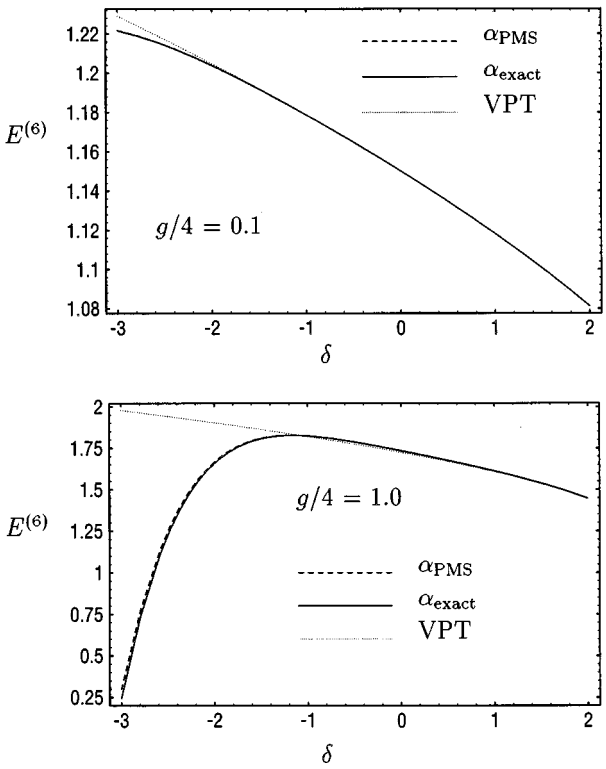

FIG. 4. Ground-state energy E of the anisotropic oscillator with the interacting potential (20), as a function of the anisotropy parameter δ for two coupling constants $g/4=0.1$ and $g/4=1.0$. Comparison is made between the approximation $E^{(6)}$ obtained once for the exact strong-coupling parameter $\alpha_{\text{exact}}=1/3$ and once for $\alpha_{\text{PMS}}=0.323$. In the case $g/4=0.1$, differences are invisible at this graphical resolution. For comparison, results from another resummation scheme, variational perturbation theory (VPT) (Ref. 16), obtained in Ref. 15 are shown as well.

D. Application to quantum field theory

The above analysis of the anisotropic oscillator can now be applied to the corresponding model in quantum field theory. The resummation of the β functions (7) is carried out by combining the formulas (12), (14), and (16), where the function Z stands now for β^u and β^v. The parameters $b_0(n)$ and σ follow from the large-order behavior (9), and are the same for both β functions:

$$b_0(n)=\beta_n+\frac{3}{2}=6+n,$$

$$\sigma=1. \tag{21}$$

The optimal value of the parameter α is chosen to cause minimal sensitivity of the ratios $B_0^{u(6)}/B_0^{u(5)}$ and $B_1^{v(6)}/B_1^{v(5)}$ on α at the small value of coupling constant $g/4=0.1$ [recall Eq. (7)] in accordance with the above observations for the $O(M)$-symmetric anharmonic oscillator.

The optimal values α_{PMS} are found to be

$$\beta^u:\quad \alpha_{\text{PMS}}=1.348,\quad \beta^v:\quad \alpha_{\text{PMS}}=1.225. \tag{22}$$

TABLE II. Numerical result for the isotropic and the cubic fixed point for increasing order of approximation N. For $N \geq 3$, a cubic fixed point is found in the upper half of the coupling constant plane (u,v), i.e., $\delta^*_{\rm cub} > 0$.

N	$g^*_{\rm iso}$	$\delta^*_{\rm iso}$	$g^*_{\rm cub}$	$\delta^*_{\rm cub}$
2	0.560616	0	does not exist	
3	0.440796	0	0.50208	0.291074
4	0.393506	0	0.400199	0.037862
5	0.4012	0	0.411057	0.063068
6	0.389037	0	0.39154	0.015309

For the simultaneous solution of Eqs. (7), we have first determined for each β function all zero-point functions $\delta^{(i)}(g)$ which are implicitly defined by

$$\beta^{u/v}[g, \delta^{(i)}_{u/v}(g)] = 0. \tag{23}$$

From the second equation in Eq. (7), we have then read off a trivial solution $\delta^{(1)}_v(g) \equiv 0$. Restricting attention to the region around the isotropic limit $\delta = 0$, we have found numerically a second nontrivial solution $\delta^{(2)}_v(g)$. For β^u only one solution $\delta_u(g)$ was found. Having obtained $\delta^{(1)}_v$ and δ_u the isotropic fixed point follows from the condition

$$\delta^*_{u\,\rm iso}(g^*_{\rm iso}) = \delta^{*(1)}_{v\,\rm iso}(g^*_{\rm iso}) \equiv 0. \tag{24}$$

The cubic fixed point is similarly obtained by calculating the solution of the equation

$$\delta^*_{u\,\rm cub}(g^*_{\rm cub}) = \delta^{*(2)}_{v\,\rm cub}(g^*_{\rm cub}). \tag{25}$$

III. RESULTS

Table II contains the numerical values of the isotropic and the cubic fixed points for increasing order of approximation N. Starting from the order $N=3$ a cubic fixed point is obtained which lies in the upper half of the plane of coupling constants u and v ($\delta > 0$). Plotting the functions $\delta_u(g)$ and $\delta^{(2)}_v(g)$, yields the cubic fixed point $(g^*_{\rm cub}, \delta^*_{\rm cub})$ via the crossing point. This is shown in Fig. 5 for the orders of approximation $N=2$ and $N=6$. The cubic fixed point is found to lie very close to the isotropic one.

In order to convince ourselves of the stability of the cubic fixed point at the number of field components $M=3$, we calculate the eigenvalues b_1 and b_2 of the matrix

$$B = \left(\begin{array}{cc} \partial_g \beta^g & \partial_\delta \beta^g \\ \partial_g \beta^\delta & \partial_\delta \beta^\delta \end{array} \right) \Bigg|_{g^*, \delta^*} \tag{26}$$

using the resummed β functions (6). The result is contained in Table III. If both eigenvalues are positive, the corresponding fixed point is infrared stable. For $M=3$, this is definitely the case for the cubic fixed point. At the isotropic fixed point,

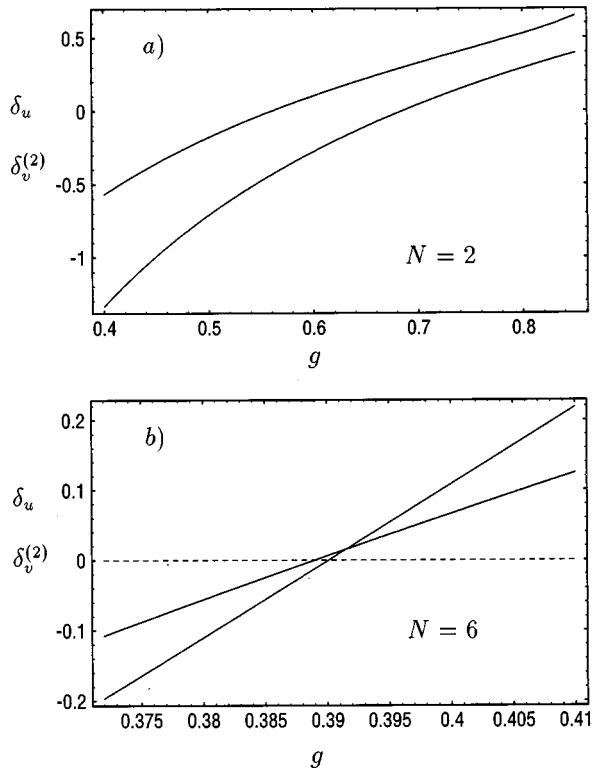

FIG. 5. Determination of the cubic fixed point g^*, δ^*. The nontrivial zeros δ_u and $\delta^{(2)}_v$ of the β functions β^u and β^v are plotted against g [see Eqs. (7) and (23)]. The values of g^* and δ^* are found from the intersection point. (a) Up to the order of approximation $N=2$, no cubic fixed point exists. (b) For the order of approximation $N=6$, the cubic fixed point is given by $g^*=0.39154$, $\delta^*=0.015309$.

on the other hand, one eigenvalue b_2 is negative. As the isotropic and the cubic fixed point interchange their stability at $M=M_c$, the result corroborates the suggestion in Refs. 5 and 6 that the critical value M_c lies below $M=3$.

Thus we conclude that the critical behavior of magnetic phase transitions in anisotropic crystals with cubic symmetry is governed by the cubic, not by the isotropic Heisenberg fixed point. The corresponding critical exponents η and ν follow from the resummed RG functions γ_ϕ and γ_{ϕ^2} via the defining relations

$$\eta = \gamma_\phi(g^*, \delta^*), \quad \nu^{-1} - 2 = -\gamma_{\phi^2}(g^*, \delta^*). \tag{27}$$

TABLE III. Stability of cubic fixed point, as demonstrated by the eigenvalues of the stability matrix B in Eq. (26), calculated from the resummed β functions β_g and β_δ in Eq. (6). The isotropic fixed point is unstable.

N	$b_1^{\rm cub}$	$b_2^{\rm cub}$	$b_1^{\rm iso}$	$b_2^{\rm iso}$
4	0.782796	0.0048920	0.784532	−0.00502046
5	0.764835	0.00851725	0.763966	−0.00886277
6	0.80609	0.00212717	0.80658	−0.00214788

Unfortunately, our result is only of fundamental interest and has no easily measurable experimental consequences. Due to the vicinity of the isotropic fixed point, the difference in the critical exponents is smaller than one percent, so that the cubic universality class is practically indistinguishable from the isotropic class.

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