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CASIMIR EFFECT AT NONZERO TEMPERATURES IN A CLOSED FRIEDMANN UNIVERSE

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For massive and massless scalar fields with an arbitrary coupling to gravity, we investigate finite-size (Casimir) effects on the free energy at nonzero temperatures in a slowly evolving closed Friedmann universe. The renormalized expressions for the free energy and the resulting energy density and pressure are found and their physical properties are discussed. The equation of state turns out to have the form $P = \rho/3$ *for radiation with arbitrary coupling.*

1. Introduction

The global topology of space-time may have effects on the local properties of the universe, such as the energy density and pressure of all fluctuating fields. For example, consider three possible, topologically distinct, Friedmann universes: closed, open flat, and open hyperbolic. The closed universe, in contrast to the open flat and hyperbolic universes, has a finite size and possesses a discrete spectrum of matter and radiation fluctuations. As a result, its partition function contains an additional term, the difference between a spectral sum and an integral, which can be determined by the Euler–MacLaurin formula. In a flat space-time, such differences arise from the energy spectrum of electromagnetic waves between two conducting plates. These give rise to attraction forces known as the Casimir effect [1], which has been discussed extensively in the literature (for reviews, see [2–3]).

An important quantum effect in a closed Friedmann universe is that of particle creation [4–6]. It is a dynamic effect that strongly depends on the evolution velocity. We ignore it here, assuming the evolution to be sufficiently slow, in order to obtain pure finite-size effects (see, e.g., [7, 8]).

The purpose of this note is to calculate the finite-size properties of matter and radiation fluctuations in a closed universe. The difference between the field energies in infinite and finite universes is called the Casimir energy of that field.

Until now, Casimir energies have been investigated mainly at zero temperature [3–6, 9, 10] (the vacuum case). In this paper, we derive it for any temperature. In our previous paper [11], we investigated the Casimir effect at nonzero temperatures in the universe with a 3-torus topology. In that case, space is flat. However, the space topology of the Friedmann universe is a 3-sphere. As a result, the spectra of fluctuating matter and radiation are different for these two cases. Gravity may also affect the spectra through matter coupling. Here we consider massive and massless scalar fields with arbitrary couplings to gravity. If the mass is set equal to zero, the result describes fluctuating radiation. The free energy of the scalar field is obtained by performing functional integration w.r.t. the Fourier components of the free field [12, 13]. In a static universe, this can easily be done since the oscillator frequences are independent of time. The Casimir effect at nonzero temperatures was also investigated in [14], and the main difference between our approach and [14] consists of the consideration of scalar fields with arbitrary coupling to gravity.

Regularization of the infinite sums can be performed using various standard methods described in the literature [15–19], the Abel–Plana summation formula [3–5] being the most useful for our purpose. The expressions for energy density and pressure follow from the free energy by thermodynamic rules.

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It is clear that in systems with entropy $S \gg 1$, as in our universe, the finite-size corrections are negligibly small and no Casimir effect is observable; the Casimir effect gets swamped there. This can be explicitly seen from the high-temperature limit formulas obtained in [11]. The Casimir effect may play an important role in models where the entropy is not too large, e.g., in some inflationary models at the early stages of evolution. The finite-size effects should also be significant in multi-dimensional Kaluza–Klein models [20].

2. Free scalar field in a slowly evolving Robertson–Walker–Friedmann universe

The action of a real scalar field in the background of an arbitrary gravitational field is [5, 6]

$$S = \int d^{D+1}x|g|^{1/2}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right], \tag{2.1}$$

the signature of the $D+1$-dimensional space–time metric $g_{\mu\nu}$ being $-++\cdots+$. For a massive scalar field, the most general harmonic potential $V(\varphi)$ reads as

$$V(\varphi) = \frac{\xi R\varphi^2}{2} + \frac{m^2\varphi^2}{2}, \tag{2.2}$$

where R is the scalar curvature of space–time, m is the mass of scalar field φ, and ξ is the coupling constant. Here, we consider the case of arbitrary ξ. At $\xi = (D-1)/4D$, the massless scalar action becomes conformally invariant.

We choose a Robertson–Walker–Friedmann (RWF) universe as a background and assume that its time evolution is so slow that it can be assumed to be static. Therefore, the metric reads

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = -dt^2 + a^2\, dl^2, \tag{2.3}$$

where a is the scale factor and dl^2 is the metric of a D-dimensional space of constant curvature. We consider the space of dimension $D=3$ with the metric

$$dl^2 = \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta = dr^2 + f^2(r)(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.4}$$

where $f(r)$ can be $\sin r$, r, or $\sinh r$ for different spaces of constant curvature with $\chi = +1, 0, -1$, respectively. The Casimir effect takes place only for $\chi = +1$. (Special types of Ricci flat spaces ($\chi = 0$) and spaces of negative constant curvature ($\chi = -1$) with a finite total volume also exist. It is clear that the Casimir effect should take place for these geometries as well.) The results can be easily generalized to the case of arbitrary D.

The free energy of a system with the temperature $k_BT \equiv 1/\beta$ is

$$F = -k_BT\log Z, \tag{2.5}$$

where the partition function Z is given by the formula

$$Z = \int \mathcal{D}\varphi e^{-S^e}. \tag{2.6}$$

Here S^e is the Euclidean action obtained from the Lorentzian (2.1) by the substitution $t \to -i\tau$. The functional integral is performed over all fields φ that are periodic in the imaginary time τ with a period $\hbar\beta$. For the static metric (2.3) with a spatial part (2.4), the action reads

$$S^e = \frac{1}{2}\int d\tau\, d^3x\, \gamma^{1/2}a^3\left[(\partial_\tau\varphi)^2 + \frac{1}{a^2}\gamma^{\alpha\beta}(\partial_\alpha\varphi)(\partial_\beta\varphi) + M^2\varphi^2\right], \tag{2.7}$$

where we have introduced the effective mass

$$M^2 = \frac{6\chi\xi}{a^2} + m^2. \tag{2.8}$$

To calculate the path integral (2.6), we expand the scalar field φ into the basis of eigenfunctions of the Laplace–Beltrami operator $\Delta_2^{(3)}$. Restricting our attention to the case of positive constant curvature with $\chi = +1$, we obtain the eigenfunctions in the form

$$\Phi_J(\mathbf{x}) \equiv Q^n_{lm}(\mathbf{x}) = \Pi^n_l(r) Y_{lm}(\theta, \phi), \tag{2.9}$$

where $Y_{lm}(\theta, \phi)$ are the scalar spherical harmonics and $\Pi^n_l(r)$ are the "Fock" harmonics [21]. The expansion is

$$\varphi(x) = \frac{1}{2a^{3/2}} \sum_J [\varphi_J(\tau)\Phi_J(\mathbf{x}) + \text{c.c.}]. \tag{2.10}$$

The coefficient functions that satisfy periodic boundary conditions are

$$\varphi_J(\tau = 0) = \varphi_J(\tau = \hbar\beta) = 0. \tag{2.11}$$

Substituting (2.10) into (2.7) and using the orthonormality relations for the spherical harmonics [5, 6, 21, 22], we find the Euclidean action

$$S^{\mathrm{e}} = \sum_J \frac{1}{2} \int d\tau \left[|\dot{\varphi}_J|^2 + \omega_n^2 |\varphi_J|^2 \right], \tag{2.12}$$

where the dot denotes differentiation with respect to τ and $\omega_n^2 = M^2 + (n^2 - 1)/a^2$, $n = 1, 2, 3\ldots$, are the eigenfrequencies. Decomposition reduces the functional integral for the partition function (2.6) to a product of simple path integrals of harmonic oscillators. Then it is easy to calculate the total free energy [12, 13],

$$F = k_B T \sum_n n^2 \log\left(1 - e^{-\frac{\hbar\omega_n}{k_B T}}\right) + \sum_n n^2 \frac{\hbar\omega_n}{2}, \tag{2.13}$$

for the frequencies

$$\omega_n^2 = \frac{6\chi\xi}{a^2} + m^2 + \frac{n^2 - \chi}{a^2} = m^2 + \frac{n^2 + (6\xi - 1)\chi}{a^2}. \tag{2.14}$$

The factor n^2 in the summations is due to the degeneracy of the eigenvalues in the isotropic spaces. The second term in (2.13) is the divergent zero-point fluctuation energy. This expression may be regularized by some standard method, which we do in the next section. In the case of an open universe with $\chi = 0$ or 1, the summations in (2.13) are replaced by integrals [4, 5] and the standard regularization amounts to dropping the last integral ([12]).

3. The Casimir effect

To perform the summation in the free energy expression (2.15), we rewrite it as follows:

$$F = k_B T \sum_n n^2 \log\left[2\sinh\frac{\hbar\omega_n}{2k_B T}\right], \tag{3.1}$$

where

$$\omega_n^2 = m^2 + \frac{n^2 - n_c^2}{a^2} \tag{3.2}$$

and

$$n_c^2 = (1 - 6\xi)\,\chi, \qquad \chi = 1. \tag{3.3}$$

The parameter ξ is usually assumed to lie in the interval $0 \le \xi \le \frac{1}{6}$ [6]. Thus, $0 \le n_c \le 1$, while $n_c = 0$ for the conformal coupling ($\xi = \frac{1}{6}$) and $n_c = 1$ for the minimal coupling ($\xi = 0$).

Due to its simplicity, we first consider a massive scalar field with the conformal coupling ($n_c = 0$). Then,

$$\omega_n^2 = m^2 + \frac{n^2}{a^2}. \tag{3.4}$$

Using the dimensionless frequencies

$$\tilde{\omega}_n^2 = m^2 a^2 + n^2, \tag{3.5}$$

we rewrite (3.1) as

$$F = k_B T \sum_{n=1}^{\infty} n^2 \log\left[2\sinh\frac{\hbar\tilde{\omega}_n}{2\Theta}\right] \tag{3.6}$$

with the reduced temperature parameter

$$\Theta \equiv k_B T \cdot a \tag{3.7}$$

measuring the temperature in units of $\frac{\hbar}{a k_B}$ (for $c \equiv 1$). To isolate the finite-sized effects, we add and subtract F_∞, the free energy of the infinite universe ($a \to \infty$):

$$F = (F - F_\infty) + F_\infty \equiv F_{\text{fs}} + F_\infty. \tag{3.8}$$

The expression for F_∞ is divergent,

$$F_\infty = k_B T \int_0^\infty dn\, n^2 \log\left[2\sinh\frac{\hbar\tilde{\omega}}{2\Theta}\right]. \tag{3.9}$$

After a standard zeta-function regularization (which makes $\int_0^\infty dn n^2 \tilde{\omega}_n = 0$), we obtain

$$\begin{aligned} F_{\infty,\text{ren}} &= k_B T \int_0^\infty dn\, n^2 \log\left[1 - \exp\left(-\sqrt{\left(\frac{m\hbar}{k_B T}\right)^2 + \left(\frac{\hbar n}{\Theta}\right)^2}\right)\right] = \\ &= -\frac{a^3}{\hbar^3}\frac{(k_B T)^4}{3} \int_0^\infty dx\, x^4 \frac{1}{\sqrt{\left(\frac{m\hbar}{k_B T}\right)^2 + x^2}\left[\exp\left(\sqrt{\left(\frac{m\hbar}{k_B T}\right)^2 + x^2}\right) - 1\right]}. \end{aligned} \tag{3.10}$$

The effects due to the finite size of the universe (the Casimir effect) follow from the difference between a discrete summation and an integral expression,

$$F_{\text{fs}} = k_B T\left[\sum_{n=1}^{\infty} n^2 \log\left(2\sinh\frac{\hbar\tilde{\omega}_n}{2\Theta}\right) - \int_0^\infty dn\, n^2 \log\left(2\sinh\frac{\hbar\tilde{\omega}_n}{2\Theta}\right)\right]. \tag{3.11}$$

Here it is convenient to use the Abel–Plana summation formula [3-5, 23]

$$\left[\sum_{n=1} f(n) - \int_0^\infty f(n)\, dn\right] = -\frac{1}{2} f(0) + i\int_0^\infty \frac{f(i\nu) - f(-i\nu)}{[\exp(2\pi\nu) - 1]}\, d\nu, \tag{3.12}$$

which is correct if $f(\nu)$ is regular for $\mathrm{Re}\,\nu \geq 0$ (on the imaginary axis, $f(\nu)$ may have poles and branch points, which are passed on the right during integration, i.e., with $\mathrm{Re}\nu > 0$). Then

$$F_{\mathrm{fs}} = k_B T i \int_0^\infty \frac{f(i\nu) - f(-i\nu)}{[\exp(2\pi\nu) - 1]}\, d\nu, \tag{3.13}$$

with

$$\begin{aligned} f(n) &= n^2 \log\left(2\sinh\frac{\hbar}{2\Theta}\sqrt{m^2a^2+n^2}\right) = \\ &= n^2 \log\left\{\frac{\hbar}{\Theta}\sqrt{m^2a^2+n^2}\prod_{p=1}^{\infty}\left[1+\left(\frac{\hbar}{2\pi p\Theta}\right)^2(m^2a^2+n^2)\right]\right\}. \end{aligned} \tag{3.14}$$

The function $f(n)$ has logarithmic branch points on the imaginary axis with constant discontinuities starting at $n = \left[m^2a^2 + \left(\frac{2\pi p\Theta}{\hbar}\right)^2\right]^{1/2}$ for $p = 0, 1, 2 \ldots$. Therefore, integral (3.13) becomes a summation,

$$\begin{aligned} F_{\mathrm{fs}} &= k_B T 2\pi \int_0^\infty dn\, n^2 {\sum_{p=0}^{\infty}}' \theta\left[n^2 - m^2a^2 - \left(\frac{2\pi p\Theta}{\hbar}\right)^2\right]\frac{1}{\exp(2\pi n)-1} = \\ &= k_B T 2\pi {\sum_{p=0}^{\infty}}' \int_{n_p}^\infty dn\, n^2 \frac{1}{\exp(2\pi n)-1}, \end{aligned} \tag{3.15}$$

where the integrals start at

$$n_p \equiv \left[m^2a^2 + \left(\frac{2\pi p\Theta}{\hbar}\right)^2\right]^{1/2} \tag{3.16}$$

and the prime on the summation indicates that the term with $p = 0$ should be counted with the weight $\frac{1}{2}$. Going back from the momentum quantum number n to the "physical" wave vectors $k = \frac{n}{a}$, we see that (3.15) corresponds to a Planck distribution form with the effective temperature $T_{\mathrm{eff}} = \frac{\hbar}{ak_B}$. The appearance of such an effective temperature is typical for the Casimir effect [10, 3].

The total renormalized free energy is

$$\begin{aligned} F_{\mathrm{ren}} = F_{\mathrm{fs}} + F_{\infty,\mathrm{ren}} = &\frac{k_B T}{4\pi^2}{\sum_{p=0}^{\infty}}' \int_{2\pi n_p}^\infty \frac{x^2\,dx}{e^x - 1} - \\ &- \frac{a^3}{\hbar^3}\frac{(k_BT)^4}{3}\int_0^\infty \frac{x^4\,dx}{\sqrt{\left(\frac{m\hbar}{k_BT}\right)^2 + x^2}\left[\exp\left(\sqrt{\left(\frac{m\hbar}{k_BT}\right)^2+x^2}\right)-1\right]}. \end{aligned} \tag{3.17}$$

The density of the internal energy fluctuations follows from standard thermodynamic formulas,

$$\rho = \frac{1}{V}U = \frac{1}{2\pi^2a^3}\frac{\partial(\beta F_{\mathrm{ren}})}{\partial\beta} = \rho_{\mathrm{fs}} + \rho_{\infty,\mathrm{ren}}\,, \tag{3.18}$$

where $V = 2\pi^2 a^3$ is the total volume of the closed RWF universe. Explicitly,

$$\rho_{\rm fs} = 8\pi^2 \frac{(k_B T)^4}{\hbar^3} \sum_{p=1}^{\infty} \frac{p^2 \left(\left(\frac{m\hbar}{2\pi k_B T} \right)^2 + p^2 \right)^{1/2}}{\exp\left(\frac{4\pi^2 \Theta}{\hbar} \cdot \sqrt{\left(\frac{m\hbar}{2\pi k_B T} \right)^2 + p^2} \right) - 1}, \tag{3.19}$$

$$\rho_{\infty,{\rm ren}} = \frac{1}{2\pi^2} \frac{(k_B T)^4}{\hbar^3} \int_0^{\infty} \frac{x^2 \cdot \sqrt{\left(\frac{m\hbar}{k_B T} \right)^2 + x^2 dx}}{\exp\left(\sqrt{\left(\frac{m\hbar}{k_B T} \right)^2 + x^2} \right) - 1}. \tag{3.20}$$

It is convenient to deduce the energy density from (3.18)–(3.20) in the high-temperature limit $k_B T \gg \hbar/a$ ($\Theta/\hbar \gg 1$). It is clear that the main contribution comes from the term with $p = 1$.

To find the pressure of the fluctuation, we use the formula

$$P = -\frac{\partial F_{\rm ren}}{\partial V} = -\frac{1}{6\pi^2 a^2} \frac{\partial F_{\rm ren}}{\partial a} = P_{\rm fs} + P_{\infty,{\rm ren}}, \tag{3.21}$$

where

$$P_{\rm fs} = \frac{1}{3}\rho_{\rm fs} + \frac{2}{3} \frac{(k_B T)^4}{\hbar^3} \left(\frac{m\hbar}{k_B T} \right)^2 \sum_{p=0}^{\infty}{}' \frac{\left(\left(\frac{m\hbar}{2\pi k_B T} \right)^2 + p^2 \right)^{1/2}}{\exp\left(\frac{4\pi^2 \Theta}{\hbar} \cdot \sqrt{\left(\frac{m\hbar}{2\pi k_B T} \right)^2 + p^2} \right) - 1}, \tag{3.22}$$

$$P_{\infty,{\rm ren}} = \frac{1}{6\pi^2} \frac{(k_B T)^4}{\hbar^3} \int_0^{\infty} \frac{x^4 dx}{\sqrt{\left(\frac{m\hbar}{k_B T} \right)^2 + x^2} \left[\exp\left(\sqrt{\left(\frac{m\hbar}{k_B T} \right)^2 + x^2} \right) - 1 \right]}. \tag{3.23}$$

In the high-temperature limit $\frac{\Theta}{\hbar} \gg 1$, only the terms with $p = 0$ and $p = 1$ should be kept in (3.22) as they give the main contribution to P.

An alternative form. It is not convenient to use Eq. (3.17) in the low-temperature limit $\Theta/\hbar \ll 1$. For this, it is useful to use the Poisson summation formula [15–17]

$$\sum_{p=-\infty}^{\infty} \sigma(p) = 2\pi \sum_{p=-\infty}^{\infty} c(2\pi p), \tag{3.24}$$

where $\sigma(p)$ and $c(p)$ are connected by the Fourier transform

$$c(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sigma(x) e^{-i\alpha x}\, dx. \tag{3.25}$$

In our case,

$$\sigma(p) = \int_{2\pi n_p}^{\infty} \frac{x^2\, dx}{e^x - 1}, \tag{3.26}$$

so we can rewrite (3.15) as

$$F_{\rm fs} = \frac{k_B T}{4\pi^2} \sum_{p=0}^{\infty}{}' \sigma(p) = \frac{k_B T}{2\pi} \sum_{p=0}^{\infty}{}' c(2\pi p). \tag{3.27}$$

The expression for $c(0)$ is easily obtained,

$$c(0) = \frac{\hbar}{4\pi^3\Theta}\int_{2\pi ma}^{\infty}\frac{\sqrt{x^2-(2\pi ma)^2}\,x^2\,dx}{e^x-1}, \tag{3.28}$$

and (3.28) is proportional to the free energy of the vacuum fluctuations F_{vf},

$$F_{\text{vf}} = \sum_{n=1}^{\infty} n^2\frac{\hbar\omega_n}{2} - \int_0^{\infty} n^2\frac{\hbar\omega_n}{2}dn = \frac{\hbar}{a}\frac{1}{(2\pi)^4}\int_{2\pi ma}^{\infty}\frac{\sqrt{x^2-(2\pi ma)^2}x^2\,dx}{e^x-1}, \tag{3.29}$$

where we use the Abel–Plana summation formula (3.12) for the regularization F_{vf}. Equation (3.27) takes the form

$$F_{\text{fs}} = F_{\text{vf}} + \frac{k_BT}{2\pi}\sum_{p=1}^{\infty} c(2\pi p), \tag{3.30}$$

where, for $p \neq 0$,

$$c(\alpha) = \frac{1}{\pi\alpha}\left[-\frac{d^2}{dz^2}+(2\pi ma)^2\right]\int_{2\pi ma}^{\infty}\frac{\sin\left(z\sqrt{x^2-(2\pi ma)^2}\right)}{e^x-1}dx|_{z=\frac{\alpha\hbar}{4\pi^2\Theta}}. \tag{3.31}$$

Thus, the alternative expression for the total renormalized free energy is $F_{\text{ren}} = F_{\text{fs}} + F_{\infty,\text{ren}}$ with F_{fs} being determined by (3.30), (3.31).

This is the most convenient expression for dealing with the massless case, as we demonstrate below. For the low-temperature limit in massive scalar field theory, it is preferable to find yet another representation of the same expression.

The low-temperature limit for nonzero mass. For $m \neq 0$, integral (3.31) cannot be calculated exactly. Then, the low-temperature limit for F_{fs} can be obtained directly from Eq. (3.11), which can be written in the form

$$F_{\text{fs}} = k_BT\sum_{1}^{\infty} n^2\log\left[1-\exp\left(-\frac{\hbar\tilde{\omega}_n}{\Theta}\right)\right] + F_{\text{vf}} - F_{\infty,\text{ren}}\,. \tag{3.32}$$

Adding $F_{\infty,\text{ren}}$ to this expression, in the low-temperature limit, we have

$$F_{\text{ren}} = k_BT\sum_{1}^{\infty} n^2\log\left[1-\exp\left(-\frac{\hbar\tilde{\omega}_n}{\Theta}\right)\right] + F_{\text{vf}} \approx -k_BT\exp\left(-\frac{\hbar\tilde{\omega}_1}{\Theta}\right) + F_{\text{vf}}. \tag{3.33}$$

Substituting (3.33) into (3.18), we obtain

$$\rho = \frac{\hbar}{2\pi^2a^4}\sum_{1}^{\infty}\frac{n^2\tilde{\omega}_n}{\exp\left(\frac{\hbar\tilde{\omega}_n}{\Theta}\right)-1} + \rho_{\text{vf}} \approx \frac{\hbar\tilde{\omega}_1}{2\pi^2a^4}\exp\left(-\frac{\hbar\tilde{\omega}_1}{\Theta}\right) + \rho_{\text{vf}}, \tag{3.34}$$

where ρ_{vf} is the well-known expression for the energy density of the vacuum fluctuations of a massive scalar field [3–6, 9, 10],

$$\rho_{\text{vf}} = \frac{\hbar}{a^4\pi(2\pi)^5}\int_{2\pi ma}^{\infty}\frac{\sqrt{x^2-(2\pi ma)^2}\,x^2\,dx}{e^x-1}. \tag{3.35}$$

From (3.21) and (3.33), the pressure is

$$P = \frac{\hbar}{6\pi^2\alpha^4}\sum_{1}^{\infty}\frac{n^4}{\tilde{\omega}_n\left[\exp\left(\frac{\hbar\tilde{\omega}_n}{\Theta}\right)-1\right]} + P_{\text{vf}} \approx \frac{\hbar}{6\pi^2a^4\tilde{\omega}_1}\exp\left(-\frac{\hbar\tilde{\omega}_1}{\Theta}\right) + P_{\text{vf}}, \tag{3.36}$$

where

$$P_{\text{vf}} = \frac{1}{3}\rho_{\text{vf}} + \frac{(ma)^2\hbar}{24\pi^2a^4}\int_{2\pi ma}^{\infty}\frac{x^2\,dx}{\sqrt{x^2-(2\pi ma)^2}(e^x-1)}. \tag{3.37}$$

The integral in (3.37) converges.

The massless case. The formulas obtained above are considerably simplified in case of radiation $(m = 0)$. Equation (3.17) for F_{ren} reads as

$$F_{\text{ren}} = \frac{k_B T}{4\pi^2} \sum_{p=0}^{\infty}{}' \int_{\frac{4\pi^2\Theta}{\hbar}p}^{\infty} \frac{x^2 dx}{e^x - 1} - \frac{a^3}{\hbar^3}\frac{\pi^4}{45}(k_B T)^4. \tag{3.38}$$

From this, the energy density of the fluctuations of a massless scalar field follows:

$$\rho = 8\pi^2 \frac{(k_B T)^4}{\hbar^3} \sum_{p=1}^{\infty} \frac{p^3}{\exp\left(\frac{4\pi^2 p\Theta}{\hbar}\right) - 1} + \frac{\pi^2}{30}\frac{(k_B T)^4}{\hbar^3}. \tag{3.39}$$

Here, the second term is the usual black-body energy density. The first term corresponds to the finite-sized effects. In principle, this formula provides us with the possibility of gaining information on the global topology of our universe by measuring the microwave radiation. Unfortunately, as we show later, the corrections to the black-body energy density are exponentially small, at the present moment, for the standard model of the hot Friedmann universe. If we stipulate the preservation of the total fluctuation energy during the evolution of the universe, i.e., $\rho \cdot a^4 = \text{const}$ [24], then we have, from (3.39),

$$k_B T \cdot a = \Theta = \text{const}. \tag{3.40}$$

This is the usual relation between the temperature of radiation and the scale factor.

It is convenient to deduce the energy density in the high-temperature limit $k_B T \gg \frac{\hbar}{a}$ from (3.39), which, in the present units, amounts to $\frac{\Theta}{\hbar} \gg 1$:

$$\rho \approx 8\pi^2 \frac{\hbar}{a^4} \left(\frac{\Theta}{\hbar}\right)^4 \exp\left(-4\pi^2 \frac{\Theta}{\hbar}\right) + \frac{\pi^2}{30}\frac{\hbar}{a^4}\left(\frac{\Theta}{\hbar}\right)^4. \tag{3.41}$$

To find the pressure of the fluctuations, we use Eq. (3.21) and obtain

$$P = \frac{\rho}{3}, \tag{3.42}$$

where ρ is defined by (3.39). This is the usual equation of state for radiation.

As was stressed above, formulas of type (3.39) are not yet useful at the low-temperature limit $\frac{\Theta}{\hbar} \ll 1$. For this purpose, we use Eq. (3.30) and, in the massless case, we have

$$F_{\text{vf}} = \frac{1}{240}\frac{\hbar}{a} \tag{3.43}$$

and from [17],

$$c(\alpha) = -\frac{1}{\pi\alpha}\left\{4\pi^3 \frac{\left[1 + \exp\left(-\frac{\alpha\hbar}{2\pi\Theta}\right)\right]\exp\left(-\frac{\alpha\hbar}{2\pi\Theta}\right)}{\left[1 - \exp\left(-\frac{\alpha\hbar}{2\pi\Theta}\right)\right]^3} - \frac{(4\pi^2\Theta)^3}{\alpha^3\hbar^3}\right\}. \tag{3.44}$$

Expression (3.43) is the well-known free energy of vacuum fluctuations of a massless scalar field in the closed Friedmann universe [3–6, 9, 10]. As a result, we have an alternative formula for the free energy,

$$F_{\text{ren}} = \frac{1}{240}\frac{\hbar}{a} - k_B T \sum_{n=1}^{\infty} \frac{1}{n} \frac{\left[1 + \exp\left(-\frac{n\hbar}{\Theta}\right)\right]\exp\left(-\frac{n\hbar}{\Theta}\right)}{\left[1 - \exp\left(-\frac{n\hbar}{\Theta}\right)\right]^3}. \tag{3.45}$$

Then the low-temperature limit $\frac{\Theta}{\hbar} \ll 1$ reads

$$F_{\text{ren}} \approx \frac{1}{240}\frac{\hbar}{a} - k_B T \exp\left(-\frac{\hbar}{\Theta}\right). \tag{3.46}$$

This estimate can be obtained directly from (3.32) and (3.33). Substituting (3.45) into (3.18), we obtain an alternative form for the energy density,

$$\rho = \frac{\hbar}{480\pi^2}\frac{1}{a^4} + \frac{\hbar}{2\pi^2}\frac{1}{a^4}\sum_{n=1}^{\infty}\frac{\left[1+4\exp\left(-\frac{n\hbar}{\Theta}\right)+\exp\left(-\frac{2n\hbar}{\Theta}\right)\right]\exp\left(-\frac{n\hbar}{\Theta}\right)}{\left[1-\exp\left(-\frac{n\hbar}{\Theta}\right)\right]^4}, \tag{3.47}$$

which converges rapidly at low temperatures. It is easy to see from this formula that the requirement of a constant total energy gives the condition (3.40) $k_BT \cdot a = \text{const}$. Also in the low-temperature limit $k_BT \ll \frac{\hbar}{a}$, we obtain the approximation

$$\rho \approx \frac{\hbar}{480\pi^2}\frac{1}{a^4} + \frac{\hbar}{2\pi^2}\frac{1}{a^4}\exp\left(-\frac{\hbar}{\Theta}\right). \tag{3.48}$$

For the pressure of the fluctuations, we can find the equation of state (3.42) using of (3.21), where ρ is now defined by Eq. (3.47).

Nonconformal coupling. Now let us consider a scalar field with an arbitrary coupling $0 \le \xi \le \frac{1}{6}$. For simplicity, we study only the massless case. Generalization of the formulas obtained in the case $m \neq 0$ is straightforward.

In this case, the frequency (3.2) is $\omega_n^2 = \frac{(n^2-n_c^2)}{a^2}$, and, after rotation in the complex n_c-plane $n_c \to i\nu_c$, it is reduced to

$$\omega_n^2 = \frac{\nu_c^2}{a^2} + \frac{n^2}{a^2} \equiv m^2(a) + \frac{n^2}{a^2}, \tag{3.49}$$

or

$$\tilde{\omega}_n^2 = a^2 \cdot \omega_n^2 = \nu_c^2 + n^2, \tag{3.50}$$

where $m(a) = \frac{\nu_c}{a}$ plays the role of the scalar field mass that depends on the scale factor. Now we can use the formulas derived for the massive scalar field with conformal coupling. At the end, we rotate ν_c back to its original imaginary value $\nu_c \to -in_c$.

The expression for the nonregularized free energy takes the form

$$F = k_BT\sum_{n=1}^{\infty} n^2 \log\left[2\sinh\frac{\hbar\sqrt{\nu_c^2+n^2}}{2\Theta}\right]. \tag{3.51}$$

Eventually, performing a regularization similar to (3.6)–(3.15), we have

$$\begin{aligned} F_{\text{fs}} &= k_BT2\pi\int_0^{\infty} dn\, n^2 {\sum_{p=0}^{\infty}}'\, \theta\left[n^2+n_c^2-\left(\frac{2\pi p\Theta}{\hbar}\right)^2\right]\frac{1}{\exp(2\pi n)-1} = \\ &= k_BT\pi(1+2p_c)\int_0^{\infty}\frac{n^2\,dn}{\exp(2\pi n)-1} + k_BT2\pi\sum_{p=p_c+1}^{\infty}\int_{\sqrt{\left(\frac{2\pi p\Theta}{\hbar}\right)^2-n_c^2}}^{\infty} dn\, n^2\left[\exp(2\pi n)-1\right]^{-1}, \end{aligned} \tag{3.52}$$

where

$$p_c = \left[\frac{\hbar n_c}{2\pi\Theta}\right] \tag{3.53}$$

is the largest integer $\le \frac{\hbar n_c}{2\pi\Theta}$. It is clear from Eqs. (3.52) and (3.53) that p_c can be treated as the infrared cut-off.

The renormalized $a \to \infty$ limit of (3.51) is

$$F_{\infty,\text{ren}} = -\frac{a^3}{3\hbar^3}(k_BT)^4\int_0^{\infty}\frac{x^4\,dx}{\sqrt{\left(\frac{\hbar\nu_c}{\Theta}\right)^2+x^2}\left[\exp\left(\sqrt{\left(\frac{\hbar\nu_c}{\Theta}\right)^2+x^2}\right)-1\right]} \underset{a\to\infty}{\longrightarrow} -\frac{\pi^4}{45}\frac{a^3}{\hbar^3}(k_BT)^4. \tag{3.54}$$

This is the usual black-body free energy, a predictable result, since the mass term $m(a)$ tends to zero as $a \to \infty$. Thus, $F_{\infty,\text{ren}}$ has the same form for any coupling ξ.

The renormalized expression for the free energy can be found as the sum

$$F_{\text{ren}} = F_{\text{fs}} + F_{\infty,\text{ren}}\,. \tag{3.55}$$

Then, using Eqs. (3.18) and (3.21), we obtain, for the energy density ρ,

$$\rho = 8\pi^2 \frac{(k_B T)^4}{\hbar^3} \sum_{p=p_c+1}^{\infty} \frac{p^2 \left(p^2 - \left(\frac{n_c \hbar}{2\pi\Theta}\right)^2\right)^{1/2}}{\exp\left(\frac{4\pi^2\Theta}{\hbar} \cdot \sqrt{p^2 - \left(\frac{n_c\hbar}{2\pi\Theta}\right)^2}\right) - 1} + \frac{\pi^2}{30} \frac{(k_B T)^4}{\hbar^3}. \tag{3.56}$$

For the pressure, we obtain the usual equation of state for radiation: $P = \frac{1}{3}\rho$. The requirement of preserving the total energy during the evolution of the universe gives the condition (3.40) once again.

Formulas (3.52) and (3.56) are convenient for making estimations in the high-temperature limit $\frac{\Theta}{\hbar} \gg 1$. The leading contribution in the summations is given by the term with $p = p_c + 1$.

The presence of "mass" in this model does not permit us to calculate the coefficients $c(\alpha)$ in (3.31) exactly. To obtain an estimate in the low-temperature limit $\frac{\Theta}{\hbar} \ll 1$, we write, in analogy with (3.33), the expression for F_{ren} in the form

$$F_{\text{ren}} = k_B T \sum_{n=n^*+1}^{\infty} n^2 \log\left[1 - \exp\left(-\frac{\hbar\tilde{\omega}_n}{\Theta}\right)\right] + F_{\text{vf}}, \tag{3.57}$$

where we introduced an infrared cut-off similar to (3.53): $n^* = [n_c]$ and

$$n^* = \begin{cases} 0, & n_c < 1, \\ 1, & n_c = 1. \end{cases} \tag{3.58}$$

The energy F_{vf} is the free energy of the vacuum fluctuations

$$F_{\text{vf}} = \frac{\hbar}{a} \frac{1}{(2\pi)^4} \int_0^\infty \frac{\sqrt{x^2 + (2\pi n_c)^2}\, x^2\, dx}{e^x - 1}. \tag{3.59}$$

The corresponding energy density is

$$\rho_{\text{vf}} = \frac{\hbar}{a^4 \pi (2\pi)^5} \int_0^\infty \frac{\sqrt{x^2 + (2\pi n_c)^2}\, x^2\, dx}{e^x - 1}, \tag{3.60}$$

with the pressure satisfying the state equation $P_{\text{vf}} = \frac{1}{3}\rho_{\text{vf}}$. In the low-temperature limit $\frac{\Theta}{\hbar} \ll 1$, we obtain

$$\rho = \frac{1}{2\pi^2 a^3} \frac{\partial}{\partial\beta}(\beta F_{\text{ren}}) \approx \frac{\hbar (n^* + 1)^2 \tilde{\omega}^*_{n+1}}{2\pi^2 a^4} \exp\left(-\frac{\hbar\tilde{\omega}^*_{n+1}}{\Theta}\right) + \rho_{\text{vf}}, \tag{3.61}$$

and the pressure, again, satisfies the equation of state $P = \frac{\rho}{3}$.

The requirement of a constant total energy $\rho \cdot a^4 = \text{const}$ leads to condition (3.40) once more. It is remarkable that the finite-sized effects do not alter this formula.

The role of Casimir effects in the modern universe. It is interesting to estimate the value of the reduced temperature Θ for our universe. If we adopt the standard model of a hot universe [24], which most cosmologists side with when describing the evolution of the now-observable universe, we obtain $a \sim 12.48 \cdot 10^{27}$cm for the present value of the scale factor. For the temperature of the microwave radiation $T \sim 2.7\,$K, we find $\frac{\Theta}{\hbar} \sim 1.5 \cdot 10^{29}$. Approximately the same value of Θ is obtained for the relict neutrinos and gravitons [24]. Thus, at the present state of evolution of these radiation processes, our universe is in the high-temperature limit and finite-sized quantum effects are certainly unobservable. This estimate of the reduced temperature shows us that, in the standard model of a hot universe, it is impossible to use formula (3.39) to answer the question about the global topology of our universe, at least on the basis of the observed microwave radiation.

The method developed here for calculating the temperature and finite-sized effects should be useful in applications to models of a cold universe or to other models with nontrivial topology, where the dimensionless parameter Θ is not so dramatically large as in the case of a hot universe.

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