27 October 1997

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 235 (1997) 105-112

Supersymmetry in stochastic processes with higher-order time derivatives

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Received 27 May 1997; revised manuscript received 11 August 1997; accepted for publication 12 August 1997
Communicated by P.R. Holland

Abstract

A supersymmetric path-integral representation is developed for stochastic processes whose Langevin equation contains any number N of time derivatives, thus generalizing the presently available treatment of first-order Langevin equations by Parisi and Sourlas [Phys. Rev. Lett. 43 (1979) 744; Nucl. Phys. B 206 (1982) 321] to systems with inertia (Kramers' process) and beyond. The supersymmetric action contains N fermion fields with first-order time derivatives whose path integral is evaluated for fermionless asymptotic states. © 1997 Published by Elsevier Science B.V.

1. For a stochastic time-dependent variable x_t obeying a first-order Langevin equation

$$L_t[x] \equiv \dot{x}_t + F(x_t) = \eta_t, \tag{1}$$

driven by a white noise η_t with $\langle \eta_t \rangle = 0$, $\langle \eta_t \eta_{t'} \rangle = \delta_{tt'}$, the correlation functions $\langle x_{t_1} \dots x_{t_n} \rangle$ can be derived from a generating functional

$$Z[J] = \langle \mathrm{e}^{\mathrm{i}\int \mathrm{d}t\, Jx} \rangle = \int \mathcal{D}x\, \Delta\, \mathrm{e}^{-S_\mathrm{b} + \mathrm{i}\int \mathrm{d}t\, Jx}, \tag{2}$$

with an action $S_\mathrm{b} = \frac{1}{2}\int \mathrm{d}t\ L_t^2$, and a Jacobian $\Delta = \det\ \delta_{x_{t'}} L_t$. We denote by $\delta_{x_{t'}} L_t$ the functional derivative $\delta L_t[x]/\delta x_{t'}$. Explicitly, $\delta_{x_{t'}} L_t = [\partial_t + F'(x_t)]\delta_{tt'}$. The time variable is written as a subscript to have room for functional arguments after a symbol. It was pointed out by Parisi and Sourlas [1] that by expressing the Jacobian Δ as a path integral over Grassmann variables,

$$\Delta = \int \mathcal{D}\bar{c}\, \mathcal{D}c\, \mathrm{e}^{-S_\mathrm{f}}, \tag{3}$$

with a fermionic action

$$S_\mathrm{f} = \int \mathrm{d}t\, \mathrm{d}t'\ \bar{c}_t\, \delta_{x_{t'}} L_t\, c_{t'} = \int \mathrm{d}t\ \bar{c}_t(\partial_t + F')c_t, \tag{4}$$

the combined action $S \equiv S_\mathrm{b} + S_\mathrm{f}$ becomes invariant under supersymmetry transformations generated by the nilpotent ($Q^2 = 0$) operator

$$Q = \int \mathrm{d}t\ (\mathrm{i}c\delta_x - \mathrm{i}L_t\delta_{\bar{c}}). \tag{5}$$

The supersymmetry implies $QS = 0$.

The determinant (3) should not be confused with the partition function formed with the Hamiltonian corresponding to the fermionic action (4). The partition function is equal to the trace over external states,

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PII S0375-9601(97)00660-9

the determinant to the vacuum-to-vacuum transition amplitude for the imaginary-time interval. When calculating the determinant (3), one must specify the boundary conditions which are not just antiperiodic as for ordinary Fermi fields, but causal. Within the coherent-state representation [2], the fermionic path integral (3) requires setting $\bar{c}_t$ and c_t equal to zero at the initial and final times, respectively [3].

The generating function (2) can also be rewritten in a canonical Hamiltonian form by introducing an auxiliary Gaussian integral over momentum variables p_t, and replacing S_b by $S_b^H = \int dt\ (p_t^2/2 - ip_tL_t)$. The generator of supersymmetry for the canonical action is $Q^H = \int dt\ (ic_t\delta_{x_t} - p_t\delta_{\bar{c}_t})$. This form has the important advantage, to be used later, that it does not depend explicitly on D_t, so that the above analysis remains valid also for more general colored noises with an arbitrary correlation function $\langle \eta_{at}\eta_{bt'}\rangle = (D_{ab})_{tt'} \neq \delta_{tt'}$.

Inserting (3) into (2), the generating functional becomes

$$Z[J] = \langle e^{i\int dt\, Jx}\rangle = \int \mathcal{D}p\ \mathcal{D}x\ \mathcal{D}\bar{c}\ \mathcal{D}c\ e^{-S^H - S_f + i\int dt\, Jx}. \quad (6)$$

This representation makes the stochastic process (1) equivalent to a supersymmetric quantum mechanical system in imaginary time. The supersymmetry gives rise to an infinity of Ward identities between the correlation functions which can be collected in the functional relation

$$\int \mathcal{D}p\ \mathcal{D}x\ \mathcal{D}\bar{c}\ \mathcal{D}c\ e^{-S^H} Q^H\, \Phi = 0, \quad (7)$$

valid for an arbitrary functional $\Phi \equiv \Phi[p,x,\bar{c},c]$. The Ward identities simplify a perturbative computation of the correlation functions.

A proof of the equivalence between (1) and (2) requires a regularization of the path integral, most simply by time slicing. This is not unique, since there are many ways of discretizing the Langevin equation (1). If one sets $t_k = k\epsilon$, for $k = 0,1,2,\dots,M$, $x_k = x_{t_k}$, and $F_k = F(x_k)$, then the velocity $\dot{x}$ may be approximated by $(x_k - x_{k-1})/\epsilon$. On the sliced time axis, the force $F(x_t)$ may act at any time within the slice (t_k, t_{k-1}), which is accounted for by a parameter a and a discretization $F \to aF_k + (1-a)F_{k-1}$. Note that the discretized Langevin equation is assumed to be causal, meaning that given the initial value of the stochastic variable x_0 and the noise configurations $\eta_0, \eta_1, \dots, \eta_{M-1}$, the Langevin equation uniquely determines the configurations of the stochastic variable at later times, $x_1, x_2, \dots, x_M$. The simplest choice of the right-hand side of the Langevin equation compatible with the causality is to set it equal to η_{k-1}. In general, one can replace η_{k-1} by $\sum_1^M A_{k-1,j-1}\eta_{j-1}$, with A being an orthogonal matrix ($A^TA = 1$). The latter is just evidence for the symmetry of the white-noise stochastic process with respect to orthogonal transformations $\eta_t \to (A\eta)_t$.

Some specific values of the interpretation parameter a have been favored in the literature, such as $a = 0$ or $1/2$ corresponding to the so-called Itô- or Stratonovich-related interpretation of the stochastic process (1), respectively [2,4]. In the time-sliced path integral, these values correspond to a prepoint or midpoint sliced action [2,5]. Emphasizing the a-dependence of the sliced action, we shall denote it by S_a^H. This action is supersymmetric for any a: $Q^H S_a^H = 0$. The sliced generator $Q^H = \sum_k (ic_k\partial_{x_k} - p_k\partial_{\bar{c}_k})$ is independent of both the interpretation parameter a and the width ϵ of time slicing [2]. A shift of a changes the action by the Q-exact term

$$S_{a+\delta a}^H = S_a^H + \delta a\, Q^H\, G, \quad (8)$$

in which G is a function of a and a functional of $p, x, \bar{c}, c$. This makes the Ward identities independent of a, i.e. on the interpretation of the Langevin equation. Indeed, setting $\delta a = -a$ we find $e^{-S_a^H} = e^{-S_0^H} e^{aQ^H\Phi} \equiv e^{-S_0^H}(1 + Q^H\Phi_a')$. Substituting this relation into (7) we observe that the a-dependence drops out from the Ward identities, because of the supersymmetry $Q^H S_0^H = 0$ and the nilpotency $(Q^H)^2 = 0$.

The simplest situation arises for the Itô choice, $a = 0$. Then the sliced fermion determinant Δ becomes a trivial constant independent of x. In the continuum limit of the path integral, however, this choice is inconvenient since the associated limiting action S_0 *cannot* be treated as an ordinary time integral over the con-

[3] The conventional coherent-state path integral can also be used to represent the determinant (refdet) [3], but it has the disadvantage that quantum corrections (terms proportional to $\hbar$) must be added by hand to the classical bosonic action explicitly.

tinuum Lagrangian. Instead, $S_{a=0}$ goes over into a so-called Itô stochastic integral [2]. The Itô integral calculus [4] differs in several respects form the ordinary one, most prominently by the property $\int dx \neq \int dt\, \dot{x}$. This difficulty is avoided by taking the Stratonovich value $a = 1/2$, for which the continuum limit of $S_{1/2}$ *is* an ordinary integral [2] [4]. Splitting (8) as $S_a = S_{1/2} + (a - 1/2)Q\, G$, the non-Stratonovich part vanishes in the continuum limit because Q does not depend on the slicing parameter ϵ, whereas G is proportional to $\epsilon \to 0$ [2]. For $a = 1/2$, formula (6) has a conventional continuous interpretation as a sum over paths, and can be treated by standard rules of continuum path integration based on the perturbation expansion around Gaussian measures. The price to pay is the additional fermion interaction which possesses, as a compensation, the additional supersymmetry.

The aim of our work is to extend this supersymmetric path-integral representation to stochastic processes with higher time derivatives

$$L_t = \gamma(\partial_t)\dot{x}_t + F(x_t) = \eta_t, \tag{9}$$

where γ is a polynom of any order $N - 1$, thus producing N time derivatives on x_t. This Langevin equation may account for inertia via a term $m\,\partial_t$ in $\gamma(\partial_t)$, and/or an arbitrary non-local friction $\int d\tau\, \gamma_\tau \dot{x}_{t-\tau} \approx \sum_{n=0}^{N-1} \gamma_n \partial_t^{n+1} x_t$ where $\gamma_n = \int d\tau\, \gamma_\tau(-\tau)^n/n!$. The main problem is to find a proper representation of the more complicated determinant $\Delta = \det \delta_{x_{t'}} L_t$ in terms of Grassmann variables. What we need is an appropriate generalization of the causal boundary condition discussed after (5). The way to resolve the boundary condition problem is to go to the corresponding operator formalism [2]. However, quantum theories with higher-order time derivatives have many unphysical features, in particular states with negative norms [6,7]. Thus, it is a priori unclear how to impose the causal boundary conditions for the associated fermionic path integral. In gauge theories, the Faddeev–Popov ghosts give an example of a fermionic theory with higher (second) order derivatives. There, unphysical consequences of the negative norms of the ghost states are avoided by imposing the so-called BRST invariant boundary conditions upon the path integral. For the above stochastic determinant with higher-order derivatives, the correct boundary condition are unknown.

2. The solution proposed by us in this work is best illustrated by first treating Eq. (9) for the case of one more time derivative

$$L_t = \ddot{x}_t + \gamma\dot{x}_t + F(x_t) = \eta_t, \tag{10}$$

accounting for particle inertia (Kramers' process for a unit mass $m \equiv 1$). Omitting the time subscript of the stochastic variables, for brevity, we replace the stochastic differential equation (10) by two coupled first-order equations

$$L_v \equiv \dot{v} + \gamma v + F(x) = \nu_v, \tag{11}$$

$$L_x \equiv \dot{x} - v = \nu_x. \tag{12}$$

There are now two independent noise variables, which fluctuate according to the path integral

$$\langle F[x,v]\rangle = \int \mathcal{D}\nu_x\, \mathcal{D}\nu_v\, F[x,v] \times \exp\left[-\frac{1}{2}\int dt \left(\frac{\nu_v^2}{2(1-\sigma)} + \frac{(\dot{\nu}_x + \gamma\nu_x)^2}{2\sigma}\right)\right]. \tag{13}$$

A parameter σ regulates the average size of deviations of $\dot{x}$ from v in Eq. (12). If we regard the basic noise correlation functions as functional matrices $(D_n)_{tt'} = \langle \nu_{nt}\nu_{nt'}\rangle$ for $n = x, v$, which act on functions of time as linear operators $D_n f_t = \int dt'\, (D_n)_{tt'} f_{t'}$, the noise correlation functions associated with (13) are

$$D_v = 1 - \sigma,$$
$$D_x = \sigma e^{-\gamma t}\left(-\partial_t^{-1} e^{2\gamma t} \partial_t^{-1}\right) e^{-\gamma t}. \tag{14}$$

Substituting (12) into (11) we find Eq. (10) now driven by the combined noise

$$\eta \to \eta_\sigma = \nu_v + \dot{\nu}_x + \gamma\nu_x. \tag{15}$$

This noise is white for *any* choice of σ,

$$\langle \eta_{\sigma t}\rangle = 0, \quad \langle \eta_{\sigma t}\eta_{\sigma t'}\rangle = \delta_{tt'}. \tag{16}$$

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[4] In Section 10.5 of Ref. [5] it is shown that the correct time slicing of an interaction $\int dt\, \dot{q}F(q)$ in a path integral is of the midpoint type, corresponding to $a = 1/2$. Sometimes this is referred to as the *midpoint prescription* for defining the sliced action, but it can actually be *derived* from the short-time action along a classical orbit.

Let $x_t[\eta]$ be a solution of the original Langevin equation (10), and $x_t[\eta_\sigma]$ a solution of the system (12), (11). The property (16) implies that $x_t[\eta_\sigma]$ has the same correlation functions as $x_t[\eta]$ for *any* σ, thus describing a stochastically equivalent process.

Once we have transformed Kramers' process into a system of coupled first-order Langevin equations (11) and (12), which is a trivial extension of the first-order Eq. (1) to a matrix form, there obviously exists a path integral representation analogous to (6). This was precisely the purpose of introducing two noise variables and a fluctuating relation between $\dot{x}$ and v in Eq. (12). There is a complication though, that the noise ν_x is no longer white since D_x is non-local in time. However, as observed above, this does not affect the supersymmetry in the canonical form (6) of the path integral since the supersymmetry generator Q^{H} does not depend on D_x (in contrast to Q).

Thus, having established the supersymmetric path integral representation of the equivalent system of first-order stochastic equations, our strategy is the following. We shall integrate out the auxiliary bosonic variable v. Since Eqs. (11) and (12) are linear in v, the corresponding determinant does not depend on it. Therefore the integration will not affect the fermionic integral whose proper boundary conditions will be guaranteed by the coherent-state path-integral representation of the determinant (3) associated with the system (11), (12) which, by construction, is equivalent to the original second-order stochastic process. From the supersymmetry generator in the extended space we shall obtain the corresponding generator in the original bosonic configuration space by substituting solutions of the equations of motion for the auxiliary variable. The so-obtained effective action and the supersymmetry generator are shown to be regular in the vicinity of $\sigma = 0$. This allows us to take the limit $\sigma \to 0$, in which the effective action and the supersymmetry generator are both local in time.

To prepare the notation for the later generalization to a stochastic differential equation with N derivatives, we rename the variables x and v as x_n, with $\alpha = 1, \dots, N$, where $N = 2$, for the moment. Only the equation for x_N contains the force $F = F(x_1)$. The other equation just establishes a fluctuating equality between $\dot{x}_1$ and x_2, the original process being described by $x \equiv x_1$. Inserting the stochastic equations (11) and (12) into the exponent of (13), we apply the previously discussed time-slicing procedure to change the integration variables $\nu_n = \nu_n[x]$. Choosing mid-point slicing with $a = 1/2$ à la Stratonovich, we obtain the path-integral representation of the generating functional

$$Z[J] = \int \mathcal{D}p\, \mathcal{D}x\, \mathcal{D}\bar{c}\, \mathcal{D}c\, \mathrm{e}^{-S^{\mathrm{H}} + \mathrm{i}\int \mathrm{d}t\, Jx}, \tag{17}$$

$$S^{\mathrm{H}} = \sum_{n=1}^{2} \int \mathrm{d}t\, \left(\tfrac{1}{2} p_n D_n p_n - \mathrm{i} p_n L_n + \bar{c}_n \delta_{x_m} L_n c_m\right).$$

The generator of supersymmetry is

$$Q^{\mathrm{H}} = \sum_{n=1}^{2} \int \mathrm{d}t\, (\mathrm{i} c_n \delta_{x_n} - p_n \delta_{\bar{c}_n}). \tag{18}$$

It is readily verified that $Q^{\mathrm{H}} S^{\mathrm{H}} = 0$, using the fact that $\sum_{kmn} \bar{c}_m c_k (\delta_{x_k} \delta_{x_n} L_m) c_n \sim \sum_n c_n^2 = 0$ due to the Grassmann nature of c_n. Explicitly, the Fermi part of the action S^{H} reads

$$S_{\mathrm{f}} = \int \mathrm{d}t\, [\bar{c}_x \dot{c}_x + \bar{c}_v \dot{c}_v - \bar{c}_x c_v + \bar{c}_v c_x F'(x) + \gamma \bar{c}_v c_v]. \tag{19}$$

The Gaussian path integral over momenta in (17) has a meaning without time slicing, and can be performed to recover the Lagrangian version of the supersymmetric action

$$S_\sigma = \sum_{n=1}^{2} \int \mathrm{d}t\, \tfrac{1}{2} L_n D_n^{-1} L_n + S_{\mathrm{f}}. \tag{20}$$

The associated generator of supersymmetry is obtained from (18) by substituting into Q^{H} the solutions of the Hamilton equations of motion $p_n = \mathrm{i} D_n^{-1} L_n$ which extremize S^{H} ($\delta_{p_n} S^{\mathrm{H}} = 0$), leading to

$$Q_\sigma = \sum_{n=1}^{2} \int \mathrm{d}t\, (\mathrm{i} c_n \delta_{x_n} - \mathrm{i} D_n^{-1} L_n \delta_{\bar{c}_n}). \tag{21}$$

The final step consists in integrating out the auxiliary variable $x_2 = v$ which only appears quadratically in the bosonic part of the action. Making use of the explicit form of D_n given in (14), we obtain the Lagrangian form of the supersymmetric action

$$S_\sigma = \int \mathrm{d}t\, \tfrac{1}{2} L_t \left(1 + \frac{1}{1-\sigma} \partial_t D_x \partial_t\right) L_t + S_{\mathrm{f}}, \tag{22}$$

where L_t is now given by (10). At this stage, the effective action is non-local in time. Now we take advantage of the freedom in choosing the parameter σ. We go to the limit $\sigma \to 0$, in which case $D_x \sim \sigma$ vanishes, reducing the action to the local form

$$S = S_0 = \int \mathrm{d}t\ \tfrac{1}{2}L_t^2 + S_\mathrm{f}. \tag{23}$$

To find the generator of supersymmetry in this representation, we omit $\delta_{x_N} \equiv \delta_v$ in (21), and replace $x_N \equiv v$ by the solution of the equation of motion

$$\delta_v S_\sigma = -D_x^{-1} L_x + \frac{1}{1-\sigma}(-\partial_t + \gamma) L_v = 0. \tag{24}$$

According to (12), the quantity L_x fluctuates with the width proportional to σ. In the limit $\sigma \to 0$, we recover the non-fluctuating relation $L_x = v - \dot{x} = 0$. Although the operator $D_x^{-1} \sim \sigma^{-1}$ diverges in this limit, so that $D_x^{-1} L_x$ is uncertain, the equation of motion for the velocity v holds for *all* values of σ. In fact, in the limit $\sigma \to 0$, the value of $D_x^{-1} L_x$ is equal to $(\partial_t - \gamma) L_v$ when v satisfies the classical equation of motion (24). Since by construction the operator (21) is evaluated by inserting the solution of the classical equation of motion for the auxiliary variable v, the relation (24) also determines the operator (21) at $\sigma = 0$. The supersymmetry generator assumes the final form

$$Q = Q_0 = \int \mathrm{d}t\ \left[\mathrm{i}c_x\delta_x - \mathrm{i}(-\partial_t + \gamma)L_t\delta_{\bar{c}_x} - \mathrm{i}L_t\delta_{\bar{c}_v}\right]. \tag{25}$$

The action (23) provides us with the desired supersymmetric description of Kramers' process (10).

An important feature of the supersymmetry generated by Q is that the supermultiplet contains one boson field and two fermion fields. The reason for this is, of course, that a quantized boson field with N time derivatives in the action carries N particles, each of which must have a supersymmetric fermionic partner. In our matrix formulation (11), (12) of Kramers' process, the fermion degrees of freedom have the conventional first-order action, which permits us to impose the vacuum-to-vacuum boundary conditions within the coherent-state representation of fermionic path integrals [2]. So, the problem of the boundary condition for the determinant of the second-order operator has been circumvented by enlarging the number of Fermi fields, thereby reducing the problem to the known one for the determinant of the single-derivative operator. The boundary conditions ($x_{t=0} = x_0$ and $\dot{x}_{t=0} = v_0 = \dot{x}_0$) for the bosonic path integral pose no problem.

What happens if we integrate out the auxiliary Grassmann variables $\bar{c}_v$, c_v? In these variables, the action (19) is harmonic, driven by external forces $\bar{c}_x$ and $c_x F'(x)$. After a quadratic completion the integration with the vacuum-to-vacuum boundary condition yields $\det(\partial_t + \gamma)$. The effective action for the other fermion pair becomes non-local,

$$S_\mathrm{f} = \int \mathrm{d}t\ \left[\bar{c}_x\dot{c}_x + \bar{c}_x(\partial_t + \gamma)^{-1}(F'(x)c_x)\right]. \tag{26}$$

The total effective action $S = S_\mathrm{b} + S_\mathrm{f}$ still exhibits the supersymmetry now generated by the operator (25), if the last term in Q is dropped. The action (26) is a first-order action. So, with the vacuum-to-vacuum boundary condition, the integral over $\bar{c}_x, c_x$ would also give a determinant. Thus we obtain the product

$$\Delta = \det[\partial_t + \gamma]\,\det[\partial_t + (\partial + \gamma)^{-1}F'(x)]. \tag{27}$$

Invoking the formula for the determinant of a block matrix, the non-locality in the second determinant can be removed, while maintaining the linearity in the time derivative

$$\Delta = \det\begin{pmatrix} \partial_t + \gamma & F' \\ -1 & \partial_t \end{pmatrix}. \tag{28}$$

This is exactly the determinant arising from the two-noise process (11), (12). In this way we have represented the determinant of the second-order operator as a determinant of a first-order operator acting upon a higher-dimensional space for which the boundary conditions are known.

Thus, with the help of two coupled equations driven by auxiliary noises we have succeeded in giving a unique meaning to the path-integral representation of the Kramers process. The final path integral can be time-sliced in any desired way (prepoint, postpoint, midpoint, or any mixture of these). As long as the slicing is done equally in the bosonic and the fermionic actions, the fermions will compensate all slicing ambiguities in the bosonic variables. In Section 4 the procedure will be generalized to the case of a friction coefficient γ being a function of x. Now we proceed to generalize our construction to stochastic processes of an arbitrary order N. As a result we shall arrive at a supersymmetric extension of general higher-order Lagrangian systems with a supermultiplet of N fermion fields which all possess a good quantum theory due to their first-order dynamics.

3. Consider a system of coupled stochastic processes

$$L_N = \dot{x}_N + \sum_{n=1}^{N} \gamma_{n-1} x_n + F(x_1) = \nu_N, \tag{29}$$

$$L_n = \dot{x}_n - x_{n+1} = \nu_n, \quad n = N-1, N-2, \ldots, 1, \tag{30}$$

where $x_1 \equiv x$. This stochastic process is equivalent to the original one if we assume the noise average as being taken with the weight e^{-S_ν}, generalizing that in (13) to

$$S_\nu = \tfrac{1}{2} \int dt \left(\frac{1}{1-\sigma} \nu_N^2 + \sum_{n=1}^{N-1} \frac{1}{\sigma_n} (\Lambda_{N-n} \nu_n)^2 \right), \tag{31}$$

where $\sigma = \sum_{n=1}^{N-1} \sigma_n$ and $\Lambda_n = \sum_{m=0}^{n} \gamma_{N-m} \partial_t^{n-m}$, $\gamma_N \equiv 1$. As for $N = 2$, Eqs. (29) and (30) can be combined into a single equation $L_t = \nu_N + \sum_{n=1}^{N-1} \Lambda_{N-n} \nu_n \equiv \eta_\sigma$. From (31) follows that $\langle \eta_{\sigma t} \rangle = 0$ and $\langle \eta_{\sigma t} \eta_{\sigma t'} \rangle = \delta_{tt'}$. Thus the correlation functions of the system (29) are the same as those of the original one. As before, the correlation functions for the combined noise η_σ do not depend on the parameters σ_n. At the end, all of these will be chosen to be zero, to obtain a formulation local in time.

The Hamiltonian path integral for the stochastic system (29) and (30) has the form (17), where the label n runs now from 1 to N. With the same extension of the index sum, the operator Q^{H} in (18) generates supersymmetry. The noise correlation functions (14) are generalized to $D_N = 1 - \sum_{n=1}^{N-1} \sigma_n$ and $D_n = \sigma_n (\Lambda_{N-n}^\dagger \Lambda_{N-n})^{-1}$. After integrating out the momenta p_n, we arrive at the action (20) with the extended sum, and the generator of supersymmetry assumes the form (21) with the extended sum.

Integrating out the auxiliary variables x_n, $n > 1$, is now technically more involved, but the integral is still Gaussian. A successive integration is possible by observing that the fermion action does not depend on the variables x_n for $n > 1$, the stochastic process being non-linear only in the physical variable $x_1 \equiv x$. The classical equations of motion $\delta_{x_n} S_\sigma = 0$ can be written in the form

$$-D_{n-1}^{-1} L_{n-1} - \partial_t D_n^{-1} L_n + \gamma_{n-1} D_N^{-1} L_N = 0, \tag{32}$$

for $n = 2, 3, \ldots, N$. Combining the equations for $n = N$ and $n = N-1$, and the result with the equation for $n = N-2$, and so on, we derive the relation

$$D_n^{-1} L_n = \frac{1}{1-\sigma} \left(\sum_{k=0}^{N-n} (-1)^k \gamma_{n+k} \partial_t^k \right) L_N, \tag{33}$$

having inserted $D_N = 1 - \sigma$ and with $n = 2, 3, \ldots, N$. These expressions may be substituted into the action (20), and the generator (21). As in the case $N = 2$, the supersymmetric Lagrangian action and the operator Q turn out to have a smooth limit $\sigma_n \to 0$. Relation (33) specifies the value of $D_n^{-1} L_n$ in the limit $\sigma_n \to 0$. The generator of supersymmetry becomes

$$Q = i \int dt \left[c_1 \delta_x - \sum_{n=1}^{N} \left(\sum_{k=0}^{N-n} (-1)^k \gamma_{n+k} \partial_t^k L_t \right) \delta_{\bar{c}_n} \right]. \tag{34}$$

For convenience, we give the fermion action explicitly,

$$S_f = \int dt \left[\sum_{n=1}^{N} \bar{c}_n \dot{c}_n - \bar{c}_n c_{n+1} + c_N \left(\sum_{n=0}^{N} \gamma_{n-1} c_n + F'(x) c_1 \right) \right]. \tag{35}$$

The operator (34) transforms the original stochastic variable $x = x_1$ into the Grassmann variable c_1, $Qx = ic_1$, whereas all the fermionic variables are transformed into some functions of the only bosonic variable x. The fermionic action (35) is constructed in such a way that QS_f depends only on c_1. The terms containing the other Grassmann variables are cancelled amongst each other. The c_1 term is cancelled against the term resulting from QS_b, i.e. $Q(S_b + S_f) = 0$. It is important to realize that the fermions are coupled with each other, and thus belong to an irreducible supermultiplet. The number of fermions is equal to the highest order of the time derivative entering the bosonic action, as observed before for $N = 2$.

4. The idea of splitting the higher-order Langevin equation into a system of coupled first-order stochastic processes with a combined noise can also be applied to construct a supersymmetric quantum theory associated with the higher-order stochastic process where the coefficients γ_n are functions of x_t. We illustrate this with the example of Kramers' process in which the friction coefficient being a function of the stochastic variable x_t.

A straightforward replacement of γ by $\gamma(x)$ in (11) would create a problem because the combined noise η_σ appears to be a function of x_t, making the system (11), (12) inequivalent to the original stochastic process (if the Gaussian distributions for the auxiliary noises are assumed). To resolve this problem, we take two coupled non-linear first-order processes

$$L_v = \dot{v} + v + \lambda_v(x) = \nu_v, \tag{36}$$

$$L_x = \dot{x} - v + \lambda_x(x) = \nu_x. \tag{37}$$

The functions $\lambda_{x,v}$ are subject to the condition

$$\lambda_x' = \gamma - 1, \quad \lambda_v = F - \lambda_x. \tag{38}$$

With the noise average defined by (13), where the constant γ is set equal to one, and the condition (38), the stochastic system (36), (37) is equivalent to the original system $L_t = \ddot{x} + \gamma(x)\dot{x} + F(x) = \eta$.

The difference between (37) and (12) is just the extra force λ_x, which does not affect the derivation of the associated supersymmetric action. Repeating calculations of Section 2, we arrive at the supersymmetric action $S = S_{\mathrm{b}} + S_{\mathrm{f}}$, where

$$S_{\mathrm{b}} = \frac{1}{2}\int \mathrm{d}t\ \left[\ddot{x} + \gamma(x)\dot{x} + F(x)\right]^2, \tag{39}$$

$$S_{\mathrm{f}} = \int \mathrm{d}t\ \{\bar{c}_x\dot{c}_x + \bar{c}_v\dot{c}_v + \bar{c}_x c_x[\gamma(x) - 1] + \bar{c}_v c_v - \bar{c}_v c_x[F'(x) - \gamma(x) + 1] - \bar{c}_x c_v\}. \tag{40}$$

The supersymmetry generator has the form

$$Q = \int \mathrm{d}t\ [\mathrm{i}c_x\delta_x - \mathrm{i}L_t\delta_{\bar{c}_v} - \mathrm{i}(-\partial_t + 1)L_t\delta_{\bar{c}_x}]. \tag{41}$$

It is not hard to verify that $QS = 0$.

If we set γ to be independent of x in (40), the fermionic action does not turn into (19). In fact, when $\gamma(x) = \gamma = \text{const.}$, the system (36), (37) does not coincide with the system (11), (12), and the correlation function D_x of the auxiliary noise is different in both systems. Nonetheless, the integral (3) for the actions (40) and (19) is equal to the very same determinant (28). The reason is that the fermionic path integral exhibits a large symmetry associated with general canonical transformations on the Grassmann phase space spanned by $\bar{c}$ and c. Recall that under canonical transformations the canonical one-form $\sum_n \bar{c}_n\, \mathrm{d}c_n$ is invariant up to a total differential $dF(\bar{c}, c)$. Also the measure $\prod_n \mathrm{d}\bar{c}_n\, \mathrm{d}c_n$ remains unchanged. Thus there exists infinitely many equivalent supersymmetric representations of the same stochastic process. The situation is similar to the BRST symmetry [7] in gauge theories where the BRST charge is defined up to a general canonical transformation. This freedom can be used to simplify the fermionic action or the Fermi part of the supersymmetry generator.

This formal invariance of the continuum phase-space path-integral measure with respect to canonical transformations has been studied thoroughly [8] for bosonic phase spaces. A regularization of the continuum phase-space path-integral measure with respect to general canonical transformations in a phase space which is a Grassmann manifold is still an open problem.

The authors are grateful to Drs. Glenn Barnich and Axel Pelster for many useful discussions, and to Professor John Klauder for comments.

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