General Relativity and Gravitation, Vol. 32, No. 5, 2000

Nonholonomic Mapping Principle for Classical and Quantum Mechanics in Spaces with Curvature and Torsion

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Received September 5, 1997. Rev. version August 16, 1999

I explain the geometric basis for the recently-discovered nonholonomic mapping principle which permits deriving laws of nature in spacetimes with curvature and torsion from those in flat spacetime, thus replacing and extending Einstein's equivalence principle. As an important consequence, it yields a new action principle for determining the equation of motion of a free spinless point particle in such spacetimes. Surprisingly, this equation contains a torsion force, although the action involves only the metric. This force makes trajectories autoparallel rather than geodesic, as a manifestation of inertia. A generalization of the mapping principle transforms path integrals from flat spacetimes to those with curvature and torsion, thus playing the role of a *quantum equivalence principle*. This generalization yields consistent results only for completely antisymmetric or for gradient torsion.

KEY WORDS : Autoparallel versus geodetic motion

1. INTRODUCTION

Present generalizations of Einstein's theory of gravity to spacetimes with torsion proceed by setting up model actions in which gravity is coupled minimally to matter, and deriving field equations from extrema of these actions [1–6]. So far, there is no way of verifying experimentally the correctness of such theories due to the smallness of torsion effects upon gravi-

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tating matter. The presently popular field equations are a straightforward extension of Einstein's equation which postulate the proportionality of the Einstein–Cartan tensor with the energy-momentum tensor of matter. When forming a spacetime derivative of these equation, the purely geometric Bianchi identity for the Einstein–Cartan tensor which expresses the single-valuedness of the connection is balanced by the conservation law for the energy-momentum tensor. For spinless point particles, this law yields directly the trajectories of such particles, which turn out to be geodesics [7], the shortest paths in spacetime. The appeal of the mathematics and the success of the original Einstein equation left little doubt as to the physical correctness of this result.

In this paper I shall try to convince the reader that the result is nevertheless physically incorrect, and that spinless particles move on autoparallels after all, thus calling for a revision of the field equations. My conclusions are derived from a study of point mechanics in a given spacetime with curvature and torsion, leaving the origin of the geometry open. The equations of motion imply a simplified covariant conservation law for the energy momentum tensor, which is no longer completely analogous to the Bianchi identity, thus preventing me from writing down a field equation as usual, a problem which is left to the future.

My conclusions are based on a careful reanalysis of the action principle in spacetimes with torsion. Due to the fact that in the presence of torsion, parallelograms are in general not closed but exhibit a closure failure proportional to the torsion, the standard variational procedure for finding the extrema of the action must be modified. Whereas usually, paths are varied keeping the endpoints fixed, such that variations form closed paths, the closure failure makes the variation at the final point nonzero, and this gives rise to a torsion force.

In quantum mechanics, the nonholonomic mapping principle was essential for solving the path integral of the hydrogen atom. Its time-sliced version has existence problems, but a nonholonomic coordinate transformation to a space with torsion makes it harmonic and solvable. In the absence of truly gravitating systems with torsion, the hydrogen atom in that description may serve as a testing ground for theories with torsion.

2. NEW EQUIVALENCE PRINCIPLE

Some time ago it was pointed out [8–11] that Einstein's rules for finding correct equations of motion in spacetimes with curvature can be replaced by a more efficient *nonholonomic mapping principle*, which has additional predictive power by being applicable also in the presence of torsion. This new principle was originally discovered for the purpose of transforming nonrelativistic path integrals correctly from flat spacetime to spacetimes with torsion [10]. In that context it appeared as a *quantum equivalence principle*. Evidence for its correctness was derived from its essential role in solving the path integral of the hydrogen atom via a nonholonomic Kustaanheimo–Stiefel transformation [10].

Recall that Einstein found the laws of nature in curved space via the following two steps. First, he went from rectilinear coordinates x^a ($a = 0, 1, 2, 3$) to arbitrary curvilinear ones q^λ ($\lambda = 0, 1, 2, 3$) by a coordinate transformation

$$x^a = x^a(q). \tag{1}$$

This brought the flat Minkowski metric

$$\eta_{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{ab} \tag{2}$$

to the induced metric

$$g_{\lambda\mu}(q) = e^a{}_\lambda(q) e^b{}_\mu(q) \eta_{ab}\,, \qquad e^a{}_\lambda(q) \equiv \partial x^a(q)/\partial q^\lambda, \tag{3}$$

with the same flat geometry as before, only parametrized in an arbitrary way. Here Einstein *postulated* that when written in such generalized coordinates, the flat-spacetime laws of nature remain valid in spacetimes with curvature.

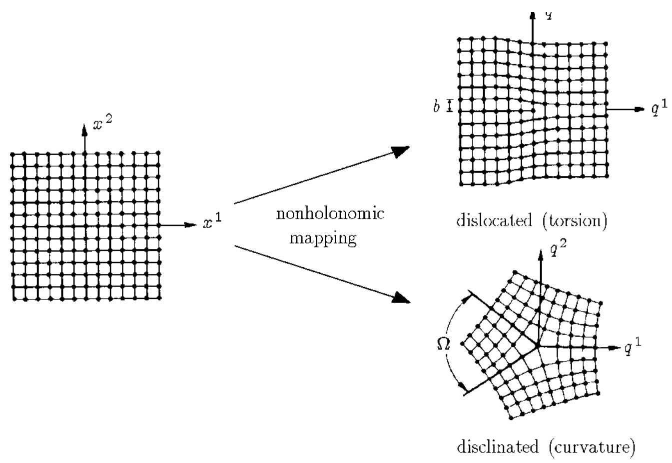

Figure 1. Crystal with dislocation and disclination generated by nonholonomic coordinate transformations from an ideal crystal. Geometrically, the former transformation introduces torsion and no curvature, the latter curvature and no torsion.

The new formulation and extension of this procedure [10] was inspired by a standard technique in describing line-like topological defects in crystals [6,12–14]. In that context it was recognized, that crystalline defects may be generated via a thought experiment, a so-called *Volterra process*, in which layers or sections of matter are cut from a crystal, with a subsequent smooth rejoining of the cutting surfaces (see Fig. 1).

Mathematically, this cutting and joining may be described by *active nonholonomic mappings* of the next-neighbor atomic distance vectors. Since there are missing or excess atoms in the image space, the mapping is not integrable to a global coordinate transformation (1). Instead, it is described by a local transformation

$$dx^a = e^a{}_\lambda(q)\, dq^\lambda, \tag{4}$$

whose coefficients $e^a{}_\lambda(q)$ have a nonvanishing curl

$$\partial_\mu\, e^a{}_\lambda(q) - \partial_\lambda\, e^a{}_\mu(q) \neq 0, \tag{5}$$

implying that any candidate for a coordinate transformation $x^a(q)$ corresponding to (4) must disobey the integrability conditions of Schwarz, i.e., its second derivatives do not commute:

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)x^a(q) \neq 0. \tag{6}$$

The functions $x^a(q)$ must therefore be *multivalued*, thus being no proper functions of mathematical textbooks, which require them to be single-valued. We shall see that such functions are the ideal tools for constructing the *nonholonomic* coordinate transformations which carry theories in flat space to spaces with curvature and torsion. It is therefore important to learn how to handle such functions.

As a matter of fact, the multivaluedness of the coordinate transformations $x^a(q)$ implied by (6) is not enough to describe all topological defects in a crystal. Also the coefficient functions $e^a{}_\lambda(q)$ themselves will have to violate the Schwarz criterion by having noncommuting derivatives [6]:

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)e^a{}_\lambda(q) \neq 0\,. \tag{7}$$

They are called *multivalued basis tetrads*.[2] The multivaluedness distinguishes them in an essential way from the well-known similar-looking objects *tetrad* or *vierbein* formalism used in the standard literature on gravity

[2] In [6], these were called *basis tetrads*. By adding the adjective *multivalued* we emphasizes the difference with respect to the usual single-valued tetrads or vierbeins of gravitational physics.

to be found in all major textbooks (for instance Ref. 15). In contrast to our basis tetrads, those are single-valued. The difference will be explained below and in more detail in Section 13.

As in the standard tetrad formalism, the induced metric (3) can be used to introduce *reciprocal multivalued tetrads*

$$e_a{}^\mu(q) \equiv \eta_{ab} g^{\mu\nu}(q) e^b{}_\nu(q). \tag{8}$$

They satisfy the orthogonality and completeness relations

$$e_a{}^\lambda(q)\, e^a{}_\mu(q) = \delta^\lambda{}_\mu\,, \qquad e_a{}^\lambda(q)\, e^b{}_\mu(q) = \delta_a{}^b\,. \tag{9}$$

Parallel transport of a vector field is defined by a vanishing covariant derivative

$$\begin{aligned} D_\mu v_\nu(q) &= \partial_\mu v_\nu(q) - \Gamma_{\mu\nu}{}^\lambda(q) v_\lambda(q), \\ D_\mu v^\lambda(q) &= \partial_\mu v^\lambda(q) + \Gamma_{\mu\nu}{}^\lambda(q) v^\nu(q), \end{aligned} \tag{10}$$

where $\Gamma_{\mu\nu}{}^\lambda(q)$ is the affine connection

$$\Gamma_{\mu\nu}{}^\lambda(q) \equiv e_a{}^\lambda(q) \partial_\mu\, e^a{}_\nu(q) = -e^a{}_\nu(q)\, \partial_\mu\, e_a{}^\lambda(q). \tag{11}$$

Note that by definition, the multivalued tetrads themselves form a parallel field:

$$D_\mu e_a{}^\lambda(q) = 0, \qquad D_\mu e^a{}_\nu(q) = 0, \tag{12}$$

implying that the induced metric is a parallel tensor field (*metricity condition*):

$$D_\lambda g_{\mu\nu}(q) = 0\,. \tag{13}$$

The antisymmetric part of the affine connection $\Gamma_{\mu\nu}{}^\lambda(q)$ is defined as the torsion tensor

$$S_{\mu\nu}{}^\lambda(q) \equiv \tfrac{1}{2}[\Gamma_{\mu\nu}{}^\lambda(q) - \Gamma_{\nu\mu}{}^\lambda(q)]. \tag{14}$$

By expressing the right-hand side in terms of the multivalued tetrads according to (11),

$$S_{\mu\nu}{}^\lambda(q) = \tfrac{1}{2} e_a{}^\lambda(q)\, [\partial_\mu\, e^a{}_\nu(q) - \partial_\nu\, e^a{}_\mu(q)], \tag{15}$$

we see that it measures directly the violation of the integrability condition as in (5), and thus the noncommutativity (6) of the derivatives in front of $x^a(q)$.

While torsion measures the degree of violation of the Schwarz integrability condition of the nonholonomic coordinate transformations in (6), the violation of the condition in (7) defines curvature tensor:

$$R_{\mu\nu\lambda}{}^{\kappa}(q) = e_a{}^{\kappa}(q)\,(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)e^a{}_\lambda(q). \tag{16}$$

Indeed, using (11), we find for $R_{\mu\nu\lambda}{}^{\kappa}(q)$ the covariant curl of the connection

$$R_{\mu\nu\lambda}{}^{\kappa} = \partial_\mu\Gamma_{\nu\lambda}{}^{\kappa} - \partial_\nu\Gamma_{\mu\lambda}{}^{\kappa} - \Gamma_{\mu\lambda}{}^{\sigma}\Gamma_{\nu\sigma}{}^{\kappa} + \Gamma_{\nu\lambda}{}^{\sigma}\Gamma_{\mu\sigma}{}^{\kappa}, \tag{17}$$

which is the defining equation for the Riemann–Cartan curvature tensor. By constructing, the curvature tensor is antisymmetric in the first index pair.

In spite of the multivaluedness of the tetrads $e^a{}_\mu(q)$, the metric and connection must be single-valued so that their second derivatives commute:

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Gamma_{\sigma\tau}{}^{\lambda}(q) = 0, \qquad (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)g_{\sigma\tau}(q) = 0\,. \tag{18}$$

In fact, these properties are the origin of the first and second Bianchi identities of general relativity, respectively.

From the integrability condition for the metric in (18) we derive the antisymmetry of $R_{\mu\nu\lambda\kappa}$ with respect to the second index pair, namely

$$R_{\mu\nu\lambda\kappa} = -R_{\mu\nu\kappa\lambda} \tag{19}$$

where $R_{\mu\nu\lambda\kappa} \equiv R_{\mu\nu\lambda}{}^{\sigma}g_{\kappa\sigma}$: from the definition (44) we calculate directly

$$\begin{aligned} R_{\mu\nu\lambda\kappa} + R_{\mu\nu\kappa\lambda} &= e_{a\kappa}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)e^a{}_\lambda + e_{a\lambda}(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)e^a{}_\kappa \\ &= \partial_\mu\partial_\nu(e_{a\kappa}e^a{}_\lambda) - \partial_\nu\partial_\mu(e_{a\kappa}e^a{}_\lambda) \\ &= (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)g_{\lambda\kappa}\,. \end{aligned} \tag{20}$$

The second Bianchi identity follows from the integrability condition for the affine connection in (18) as follows. First we simplify the algebra by using a vector notation $\mathbf{e}_\mu$ for the basis tetrads $e^a{}_\mu$, and defining a corresponding quantity

$$\mathbf{R}_{\sigma\nu\mu} \equiv (\partial_\sigma\partial_\nu - \partial_\nu\partial_\sigma)\mathbf{e}_\mu\,, \tag{21}$$

which determines the curvature tensor $R_{\sigma\nu\mu}{}^{\lambda}$ via the scalar product with $\mathbf{e}^\lambda$. Applying the covariant derivative gives

$$D_\tau\mathbf{R}_{\sigma\nu\mu} = \partial_\tau\mathbf{R}_{\sigma\nu\mu} - \Gamma_{\tau\sigma}{}^{\kappa}\mathbf{R}_{\kappa\nu\mu} - \Gamma_{\tau\upsilon}{}^{\kappa}\mathbf{R}_{\sigma\nu\kappa}\,. \tag{22}$$

Performing cyclic sums over $\tau\sigma\nu$ and taking advantage of the trivial antisymmetry of $\mathbf{R}_{\sigma\nu\mu}$ in $\sigma\nu$ we find

$$D_\tau \mathbf{R}_{\sigma\nu\mu} = \partial_\tau \mathbf{R}_{\sigma\nu\mu} - \Gamma_{\tau\mu}{}^{\kappa} R_{\sigma\nu\kappa} + 2S_{\tau\sigma}{}^{\kappa} R_{\nu\kappa\mu} \,. \tag{23}$$

Now we use

$$\partial_\sigma \partial_\nu \mathbf{e}_\mu = \partial_\sigma \left(\Gamma_{\nu\mu}{}^{\alpha} \mathbf{e}_\alpha\right) = \Gamma_{\nu\mu}{}^{\kappa} \mathbf{e}_\kappa \tag{24}$$

to derive

$$\partial_\tau \partial_\sigma \partial_\nu e_\mu = \partial_\tau \Gamma_{\nu\mu}{}^{\kappa} \partial_\sigma \mathbf{e}_\kappa + (\tau \leftrightarrow \sigma) + \partial_\tau \partial_\sigma \Gamma_{\nu\mu}{}^{\kappa} \mathbf{e}_\alpha + \Gamma_{\nu\mu}{}^{\kappa} \partial_\tau \partial_\sigma \mathbf{e}_\kappa \,. \tag{25}$$

Antisymmetrizing this in $\sigma\tau$ gives

$$\partial_\tau \partial_\sigma \partial_\nu \mathbf{e}_\mu - \partial_\sigma \partial_\tau \partial_\nu \mathbf{e}_\mu = \Gamma_{\nu\mu}{}^{\alpha} \mathbf{R}_{\tau\sigma\alpha} + \left[\, (\partial_\tau \partial_\sigma - \partial_\sigma \partial_\tau) \Gamma_{\nu\mu}{}^{\alpha} \right] \mathbf{e}_\alpha \,. \tag{26}$$

This is the place where we make use of the integrability condition for the connection (18) to drop the last term. Together with (21), we find

$$\partial_\tau \mathbf{R}_{\sigma\nu\mu} - \Gamma_{\nu\mu}{}^{\alpha} \mathbf{R}_{\tau\sigma\alpha} = 0 \,. \tag{27}$$

Inserting this into (23) and multiplying by $\mathbf{e}^\kappa$ we obtain an expression involving the covariant derivative of the curvature tensor

$$D_\tau R_{\sigma\nu\mu}{}^{\kappa} - 2S_{\tau\sigma}{}^{\lambda} R_{\nu\lambda\mu}{}^{\kappa} = 0 \,. \tag{28}$$

This is the second *Bianchi identity*, guaranteeing the integrability of the connection.

The Riemann connection is given by the Christoffel symbol

$$\bar{\Gamma}_{\mu\nu\lambda} \equiv \{\mu\nu, \lambda\} = \tfrac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{29}$$

It forms part of the affine connection (11), as shown by the decomposition

$$\Gamma_{\mu\nu\kappa} = \bar{\Gamma}_{\mu\nu\kappa} + K_{\mu\nu\kappa} \,, \tag{30}$$

in which $K_{\mu\nu\kappa}$ is the *contortion tensor*, a combination of three torsion tensors:

$$K_{\mu\nu\lambda} = S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu} \,. \tag{31}$$

This decomposition follows directly from the trivially rewritten expression (11),

$$\begin{aligned} \Gamma_{\mu\nu\lambda} = {} & \\ & \tfrac{1}{2} \left\{ e_{i\lambda} \partial_\mu e^i{}_\nu + \partial_\mu e_{i\lambda} e^i{}_\nu + e_{i\mu} \partial_\nu e^i{}_\lambda + \partial_\nu e_{i\mu} e^i{}_\lambda - e_{i\mu} \partial_\lambda e^i{}_\nu - \partial_\lambda e_{i\mu} e^i{}_\nu \right\} \\ & + \tfrac{1}{2} \left\{ \left[e_{i\lambda} \partial_\mu e^i{}_\nu - e_{i\lambda} \partial_\nu e^i{}_\mu \right] - \left[e_{i\mu} \partial_\nu e^i{}_\lambda - e_{i\mu} \partial_\lambda e^i{}_\nu \right] \right. \\ & \qquad \left. + \left[e_{i\nu} \partial_\lambda e^i{}_\mu - e_{i\nu} \partial_\mu e^i{}_\lambda \right] \right\} \end{aligned} \tag{32}$$

using $e^i{}_\mu(q)e^i{}_\nu(q) = g_{\mu\nu}(q)$. The contortion tensor is antisymmetric in the last two indices:

$$K_{\mu\nu\lambda} = -K_{\mu\lambda\nu}\,, \tag{33}$$

this being a direct consequence of the antisymmetry of the torsion tensor in the first two indices:

$$S_{\mu\nu\lambda} = -K_{\lambda\mu\nu}\,. \tag{34}$$

It is useful to state in more detail the differences between our multivalued tetrads $e^a{}_\lambda(q)$ and the standard tetrads or vierbein fields $h^\alpha{}_\lambda(q)$ whose mathematics is described in [16]. Such tetrads were introduced a long time ago in gravity theories of spinning particles both in purely Riemann [15] as well as in Riemann–Cartan spacetimes [1–3,5,6]. Their purpose was to define a local Lorentz frame at every point by means of another set of coordinate differentials

$$dx^\alpha = h^\alpha{}_\lambda(q)dq^\lambda, \tag{35}$$

which can be contracted with Dirac matrices γ^α to form locally Lorentz invariant quantities. Local Lorentz frames are reached by requiring the induced metric in these coordinates to be Minkowskian:

$$g_{\alpha\beta} = h_\alpha{}^\mu(q)h_\beta{}^\nu(q)g_{\mu\nu}(q) = \eta_{\alpha\beta}\,. \tag{36}$$

Just like $e^a{}_\mu(q)$ in (8), these vierbeins possess reciprocals

$$h_\alpha{}^\mu(q) \equiv \eta_{\alpha\beta}g^{\mu\nu}(q)h^\beta{}_\nu(q), \tag{37}$$

and satisfy orthonormality and completeness relations as in (9):

$$h_\alpha{}^\mu h^\beta{}_\mu = \delta_\alpha{}^\beta, \qquad h^\alpha{}_\mu h_\alpha{}^\nu = \delta_\mu{}^\nu. \tag{38}$$

They also can be multiplied with each other as in (3) to yield the metric

$$g_{\mu\nu}(q) = h^\alpha{}_\mu(q)h^\beta{}_\mu(q)\eta_{\alpha\beta}\,. \tag{39}$$

Thus they constitute another "square root" of the metric. The relation between these square roots

$$e^a{}_\mu(q) = e^a{}_\alpha(q)h^\alpha{}_\mu(q) \tag{40}$$

is necessarily given by a local Lorentz transformation

$$\Lambda^a{}_\alpha(q) = e^a{}_\alpha(q), \tag{41}$$

since this matrix connects the two Minkowski metrics (2) and (36) with each other:

$$\eta_{ab}\Lambda^a{}_\alpha(q)\Lambda^b{}_\beta(q) = \eta_{\alpha\beta}\,. \tag{42}$$

The different local Lorentz transformations allow us to choose different local Lorentz frames which distinguish fields with definite spin by the irreducible representations of these transformations. The physical consequences of the theory must be independent of this local choice, and this is the reason why the presence of spinning fields requires the existence of an additional gauge freedom under local Lorentz transformations, in addition to Einstein's invariance under general coordinate transformations. Since the latter may be viewed as local translations, the theory with spinning particles are locally Poincaré invariant.

The vierbein fields $h^\alpha{}_\mu(q)$ have in common with ours that both violate the integrability condition as in (5), thus describing nonholonomic coordinates dx^α for which there exists only a differential relation (35). However, they differ from ours by being single-valued fields satisfying the integrability condition

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)h^\alpha{}_\lambda(q) = 0, \tag{43}$$

in contrast to our multivalued tetrads $e^a{}_\lambda(q)$ in eq. (7).

In the local coordinate system dx^α, curvature arises from a violation of the integrability condition of the local Lorentz transformations (41), which looks similar to (5).

Equation (15) for the torsion tensor in terms of the multivalued tetrads $e^a{}_\lambda(q)$ must be contrasted with a similar-looking, but geometrically quite different, quantity formed from the vierbein fields $h^\alpha{}_\lambda(q)$ and their reciprocals, the objects of anholonomy [16]:

$$\Omega_{\alpha\beta}{}^\gamma(q) = \tfrac{1}{2}h_\alpha{}^\mu(q)h_\beta{}^\nu(q)\,[\partial_\mu h^\gamma{}_\nu(q) - \partial_\nu h^\gamma{}_\mu(q)\,]. \tag{46}$$

A combination of these similar to (31),

$$\overset{h}{K}_{\alpha\beta}{}^\gamma(q) = \Omega_{\alpha\beta}{}^\gamma(q) - \Omega_\beta{}^\gamma{}_\alpha(q) + \Omega^\gamma{}_{\alpha\beta}(q), \tag{47}$$

appears in the *spin connection*

$$\Gamma_{\alpha\beta}{}^\gamma = h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu(K_{\mu\nu}{}^\lambda - \overset{h}{K}_{\mu\nu}{}^\lambda), \tag{48}$$

which is needed to form a covariant derivative of local vectors

$$v_\alpha(q) = v_\mu(q)h_\alpha{}^\mu(q), \qquad v^\alpha(q) = v^\mu(q)h^\alpha{}_\mu(q). \tag{49}$$

The spin connection (48) is derived in Section 13, where we shall find that the covariant derivative of $v_\beta(q)$ is given by

$$\begin{aligned} D_\alpha v_\beta(q) &= \partial_\alpha v_\beta(q) - \Gamma_{\alpha\beta}{}^\gamma(q) v_\gamma(q), \\ D_\alpha v^\beta(q) &= \partial_\alpha v^\beta(q) + \Gamma_{\alpha\gamma}{}^\beta(q) v^\gamma(q). \end{aligned} \tag{50}$$

In spite of the similarity between the defining equations (15) and (46), the tensor $\Omega_{\alpha\beta}{}^\gamma(q)$ bears no relation to torsion, and $\overset{h}{K}_{\alpha\beta}{}^\gamma(q)$ is independent of the contortion $K_{\alpha\beta}{}^\gamma$. In fact, the objects of anholonomy $\Omega_{\alpha\beta}{}^\gamma(q)$ are in general nonzero in the absence of torsion,[3] and may even be nonzero in flat spacetime, where the matrices $h^\alpha{}_\mu(q)$ degenerate to local Lorentz transformations. The orientation of the local Lorentz frames are characterized by $\overset{h}{K}_{\alpha\beta}{}^\gamma(q)$.

The nonholonomic coordinates dx^α transform the metric to a Minkowskian form at the point q^μ. They correspond to a small "falling elevator" of Einstein in which the gravitational forces vanish only at the center of mass, the neighborhood still being subject to tidal forces. In contrast, the nonholonomic coordinates dx^a flatten the spacetime *in an entire neighborhood* of the point. This is at the expense of producing defects in spacetime (like those produced when flattening an orange peel by stepping on it), as will be explained in Section 4. The affine connection $\Gamma_{ab}{}^c(q)$ in the latter coordinates dx^a vanishes identically.

The difference between our multivalued tetrads and the usual vierbeins is illustrated in the diagram of Fig. 2.

$dx^a = e^a{}_\mu dq^\mu$	$\Gamma_{ab}{}^c = 0$

$dx^\alpha = h^\alpha{}_\mu dq^\mu$	$\Gamma_{\alpha\beta}{}^\gamma = h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu K_{\mu\nu}{}^\lambda - \overset{h}{K}_{\alpha\beta}{}^\gamma$

dq^μ	$\Gamma_{\mu\nu}{}^\lambda = e_a{}^\lambda \partial_\mu e^a{}_\nu$

Figure 2. The coordinate system q^μ and the two sets of local nonholonomic coordinates dx^α and dx^a. The coordinates dx^α have a Minkowski metric only at the point q, the coordinates dx^a in an entire small neighborhood (at the cost of a closure failure).

[3] These differences are explained in detail in pp. 1400–1401 of [6].

A long time ago it was pointed out by Kondo [17] that a crystal with dislocations and disclinations may be described geometrically as a Riemann–Cartan spacetime with curvature and torsion. Turning the argument around, active nonholonomic mappings which are used to produce defects in crystals may be used to carry us from a flat spacetime to a Riemann–Cartan spacetime. This is in contrast to passive nonholonomic coordinate transformation of Cartesian coordinates, which are simply an awkward and highly unrecommendable redescription of flat spacetime.

In the sequel, we shall use the word "space" for spaces as well as spacetimes, for brevity.

In order to show that active nonholonomic transformations can be well-defined, let us first get some exercise in using them by studying some completely analogous but much simpler mathematical structures in magnetostatics.

3. MULTIVALUED FIELDS IN MAGNETISM

To set the stage for the discussion, recall first the standard treatment of magnetism. Since there are no magnetic monopoles, a magnetic field $\mathbf{B}(\mathbf{x})$ satisfies the identity $\boldsymbol{\nabla}\cdot\mathbf{B}(\mathbf{x})=0$, implying that only two of the three field components of $\mathbf{B}(\mathbf{x})$ are independent. To account for this, one usually expresses a magnetic field $\mathbf{B}(\mathbf{x})$ in terms of a vector potential $\mathbf{A}(\mathbf{x})$, setting $\mathbf{B}(\mathbf{x})=\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x})$. Then Ampère's law, which relates the magnetic field to the electric current density $\mathbf{j}(\mathbf{x})$ by $\boldsymbol{\nabla}\times\mathbf{B}=\mathbf{j}(\mathbf{x})$ (in natural units with $c=1$), becomes a second-order differential equation for the vector potential $\mathbf{A}(\mathbf{x})$ in terms of an electric current

$$\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x})]=\mathbf{j}(\mathbf{x}). \tag{51}$$

The vector potential $\mathbf{A}(\mathbf{x})$ is a *gauge field*. Given $\mathbf{A}(\mathbf{x})$, any locally gauge-transformed field

$$\mathbf{A}(\mathbf{x})\to\mathbf{A}'(\mathbf{x})=\mathbf{A}(\mathbf{x})+\boldsymbol{\nabla}\Lambda(\mathbf{x}) \tag{52}$$

yields the same magnetic field $\mathbf{B}(\mathbf{x})$. This reduces the number of physical degrees of freedom in the gauge field $\mathbf{A}(\mathbf{x})$ to two, just as those in $\mathbf{B}(\mathbf{x})$. In order for this to hold, the transformation function must be single-valued, i.e., it must have commuting derivatives

$$(\partial_i\partial_j-\partial_j\partial_i)\Lambda(\mathbf{x})=0\,. \tag{53}$$

The equation for absence of magnetic monopoles $\boldsymbol{\nabla}\cdot\mathbf{B}=0$ is ensured if the vector potential has commuting derivatives

$$(\partial_i\partial_j-\partial_j\partial_i)\mathbf{A}(\mathbf{x})=0\,. \tag{54}$$

This integrability property makes $\boldsymbol{\nabla}\cdot\mathbf{B}=0$ the *Bianchi identity* in this gauge field representation of the magnetic field.

In order to solve (51), we remove the gauge ambiguity by choosing a particular gauge, for instance the *transverse gauge* $\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{x})=0$ in which $\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x})]=-\boldsymbol{\nabla}^2\mathbf{A}(\mathbf{x})$, and obtain

$$\mathbf{A}(\mathbf{x})=\frac{1}{4\pi}\int d^3x'\,\frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,. \tag{55}$$

The associated magnetic field is

$$\mathbf{B}(\mathbf{x})=\frac{1}{4\pi}\int d^3x'\frac{\mathbf{j}(\mathbf{x}')\times\mathbf{R}'}{R'^3}\,,\qquad \mathbf{R}'\equiv\mathbf{x}'-\mathbf{x}. \tag{56}$$

This standard representation of magnetic fields is not the only possible one. There exists another one in terms of a scalar potential $\Lambda(\mathbf{x})$, which must, however, be multivalued to account for the two physical degrees of freedom in the magnetic field.

3.1. Gradient representation of magnetic field of current loop

Consider an infinitesimally thin closed wire carrying an electric current I along the line L. It corresponds to a current density

$$\mathbf{j}(\mathbf{x})=I\boldsymbol{\delta}(\mathbf{x};L), \tag{57}$$

where $\boldsymbol{\delta}(\mathbf{x};L)$ is the δ-function on the line L:

$$\boldsymbol{\delta}(\mathbf{x};L)=\int_L d\mathbf{x}'\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{58}$$

From eq. (55) we obtain the associated vector potential

$$\mathbf{A}(\mathbf{x})=\frac{I}{4\pi}\int_L d\mathbf{x}'\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,, \tag{59}$$

yielding the magnetic field

$$\mathbf{B}(\mathbf{x})=\frac{I}{4\pi}\int_L\frac{d\mathbf{x}'\times\mathbf{R}'}{R'^3}\,,\qquad \mathbf{R}'\equiv\mathbf{x}'-\mathbf{x}. \tag{60}$$

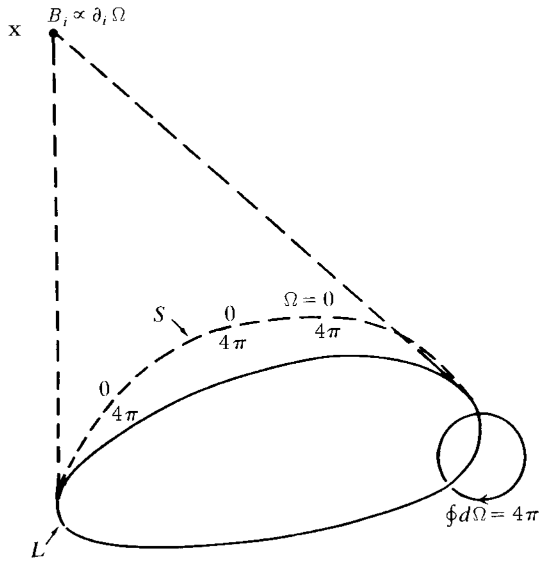

Figure 3. Infinitesimally thin closed current loop L. The magnetic field $\mathbf{B}(\mathbf{x})$ at the point $\mathbf{x}$ is proportional to the solid angle $\Omega(\mathbf{x})$ under which the loop is seen from $\mathbf{x}$. In any single-valued definition of $\Omega(\mathbf{x})$, there is some surface S across which $\Omega(\mathbf{x})$ jumps by 4π. In the multivalued definition, this surface is absent.

Let us now derive the same result from a multivalued scalar field. Let $\Omega(\mathbf{x})$ be the solid angle under which the current loop L is seen from the point $\mathbf{x}$ (see Fig. 3). If S denotes an arbitrary smooth surface enclosed by the loop L, and $d\mathbf{S}'$ a surface element, then $\Omega(\mathbf{x})$ can be calculated from the surface integral

$$\Omega(\mathbf{x}) = \int_S \frac{d\mathbf{S}' \cdot \mathbf{R}'}{R'^3}\,. \tag{61}$$

We form the vector field

$$\mathbf{b}(\mathbf{x}) = \frac{I}{4\pi}\,\boldsymbol{\nabla}\Omega(\mathbf{x}). \tag{62}$$

which is equal to

$$\mathbf{b}(\mathbf{x}) = \frac{I}{4\pi}\int_S dS_i'\boldsymbol{\nabla}\frac{R'_i}{R'^3}\,. \tag{63}$$

Using $\partial_k(R_k'/R'^3) = -\delta^{(3)}(\mathbf{x}-\mathbf{x}')$, it can be rewritten as

$$b_i(\mathbf{x}) = \frac{I}{4\pi}\left[\int_S \left(dS_k'\,\partial_i\frac{R_k'}{R'^3} - dS_i'\,\partial_k\frac{R_k'}{R'^3}\right) - \int_S d\mathbf{S}'\delta^{(3)}(\mathbf{x}-\mathbf{x}')\right]. \tag{64}$$

With the help of Stokes' theorem

$$\int_S (dS_k\partial_i - dS_i\partial_k)f(\mathbf{x}) = \epsilon_{kil}\int_L dx_l f(\mathbf{x}), \tag{65}$$

this becomes

$$\mathbf{b}(\mathbf{x}) = \frac{I}{4\pi}\left[\int_L \frac{d\mathbf{x}'\times\mathbf{R}'}{R'^3} - \int_S d\mathbf{S}'\delta^{(3)}(\mathbf{x}-\mathbf{x}')\right]. \tag{66}$$

The first term is recognized to be precisely the magnetic field (60) of the current I. The second term is the singular magnetic field of an infinitely thin magnetic dipole layer lying on the arbitrarily chosen surface S enclosed by L.

This term is a consequence of the fact that the solid angle $\Omega(\mathbf{x})$ was defined by the surface integral (61). If $\mathbf{x}$ crosses the surface S, the solid angle jumps by 4π. There exists, however, another possibility of defining the solid angle $\Omega(\mathbf{x})$, namely by its analytic continuation from one side of the surface to the other. This removes the jump, albeit at the cost of making $\Omega(\mathbf{x})$ a *multivalued function* defined only modulo 4π. From this multivalued function, the magnetic field (60) can be obtained as a gradient:

$$\mathbf{B}(\mathbf{x}) = \frac{I}{4\pi}\,\boldsymbol{\nabla}\Omega(\mathbf{x}). \tag{67}$$

Ampère's law (51) implies that the multivalued solid angle $\Omega(\mathbf{x})$ satisfies the equation

$$(\partial_i\partial_j - \partial_j\partial_i)\Omega(\mathbf{x}) = 4\pi\epsilon_{ijk}\delta_k(\mathbf{x};L). \tag{68}$$

Thus, as a consequence of its multivaluedness, $\Omega(\mathbf{x})$ violates the Schwarz integrability condition as in (6). This makes it an unusual mathematical object to deal with. It is, however, perfectly suited to describe the physics.

In order to see explicitly how eq. (68) is fulfilled by $\Omega(\mathbf{x})$, let us go to two dimensions where the loop corresponds to two points (in which the loop intersects a plane). For simplicity, we move one of them to infinity, and place the other at the coordinate origin. The role of the solid angle $\Omega(\mathbf{x})$ is now played by the azimuthal angle $\phi(\mathbf{x})$ of the point $\mathbf{x}$:

$$\phi(\mathbf{x}) = \arctan\frac{x^2}{x^1}\,. \tag{69}$$

The function $\arctan(x^2/x^1)$ is usually made unique by cutting the $\mathbf{x}$-plane from the origin along some line C to infinity, preferably along a straight line to $\mathbf{x} = (-\infty, 0)$, and assuming $\phi(\mathbf{x})$ to jump from π to $-\pi$ when crossing the cut. The cut corresponds to the magnetic dipole surface S in the integral (61). In contrast to this, we shall take $\phi(\mathbf{x})$ to be the *multivalued* analytic continuation of this function. Then the derivative ∂_i yields

$$\partial_i\phi(\mathbf{x}) = -\epsilon_{ij}\frac{x_j}{(x^1)^2+(x^2)^2}\,. \tag{70}$$

With the single-valued definition of $\partial_i\phi(\mathbf{x})$, there would have been a δ-function $\epsilon_{ij}\delta_j(C;\mathbf{x})$ across the cut C, corresponding to the second term in (66). When integrating the curl of (70) across the surface s of a small circle c around the origin, we obtain by Stokes' theorem

$$\int_s d^2x(\partial_i\partial_j - \partial_j\partial_i)\phi(\mathbf{x}) = \int_c dx_i\partial_i\phi(\mathbf{x}), \tag{71}$$

which is equal to 2π in the multivalued definition of $\phi(\mathbf{x})$. This result implies the violation of the integrability condition as in (78) below:

$$(\partial_1\partial_2 - \partial_2\partial_1)\phi(\mathbf{x}) = 2\pi\delta^{(2)}(\mathbf{x}), \tag{72}$$

whose three-dimensional generalization is (68). In the single-valued definition with the jump by 2π across the cut, the right-hand side of (71) would vanish, making $\phi(\mathbf{x})$ satisfy the integrability condition (6).

The azimuthal angle $\phi(\mathbf{x})$ solving the differential equation (72) can be used to construct a Green function for solving the corresponding differential equation with an arbitrary source, which is a superposition of infinitesimally thin line-like currents piercing the two-dimensional space at the points $\mathbf{x}_n$:

$$j(\mathbf{x}) = \sum_n I_n\delta(\mathbf{x} - \mathbf{x}_n), \tag{73}$$

where I_n are currents. We may then easily solve the differential equation

$$(\partial_1\partial_2 - \partial_2\partial_1)f(\mathbf{x}) = j(\mathbf{x}). \tag{74}$$

with the help of the Green function

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\,\phi(\mathbf{x} - \mathbf{x}'). \tag{75}$$

The solution of (74) is obviously

$$f(\mathbf{x}) = \int d^2\mathbf{x}'\,G(\mathbf{x},\mathbf{x}')j(\mathbf{x}). \tag{76}$$

The gradient of $f(\mathbf{x})$ yields the magnetic field of an arbitrary set of line-like currents vertical to the plane under consideration.

It must be pointed out that the superposition of line-like currents cannot be smeared out into a continuous distribution. The integral (76) yields the superposition of multivalued functions

$$f(\mathbf{x}) = \frac{1}{2\pi}\sum_n I_n \arctan\frac{x^2 - x_n^2}{x^1 - x_n^1}\,, \tag{77}$$

which is properly defined only if one can clearly continue it analytically into all the parts of the composite Riemann sheets defined by the endpoints of the cut at the origin. If we were to replace the sum by an integral, this possibility would be lost. Thus it is, strictly speaking, impossible to represent arbitrary continuous magnetic fields as gradients of superpositions of scalar potentials $\Omega(\mathbf{x})$. This, however, is not a severe disadvantage of this representation since any current can be approximated by a superposition of line-like currents with any desired accuracy, and the same will be true for the associated magnetic fields.

The arbitrariness of the shape of the jumping surface is the origin of a further interesting gauge structure which will be exploited in Section 8.

3.2. Generating magnetic fields by multivalued gauge transformations

After this first exercise in multivalued functions, we now turn to another example in magnetism which will lead directly to our intended geometric application. We observed before that the local gauge transformation (52) produces the same magnetic field $\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})$ only, as long as the function $\Lambda(\mathbf{x})$ satisfies the Schwarz integrability criterion (6)

$$(\partial_i \partial_j - \partial_j \partial_i)\Lambda(\mathbf{x}) = 0\,. \tag{78}$$

Any function $\Lambda(\mathbf{x})$ violating this condition would change the magnetic field by

$$\Delta B_k(\mathbf{x}) = \epsilon_{kij}(\partial_i \partial_j - \partial_j \partial_i)\Lambda(\mathbf{x}) \tag{79}$$

thus being no proper gauge transformation. The gradient of $\Lambda(\mathbf{x})$,

$$\mathbf{A}(\mathbf{x}) = \boldsymbol{\nabla}\Lambda(\mathbf{x}), \tag{80}$$

would be a *nontrivial* vector potential.

In analogy with the multivalued coordinate transformations violating the integrability conditions of Schwarz as in (6), the function $\Lambda(\mathbf{x})$ will be called nonholonomic gauge function.

Having just learned how to deal with multivalued functions we may change our attitude towards gauge transformations and decide to generate *all* magnetic fields approximately in a field-free space by such improper gauge transformations $\Lambda(\mathbf{x})$. By choosing for instance

$$\Lambda(\mathbf{x}) = \frac{\Phi}{4\pi}\,\Omega(\mathbf{x}), \tag{81}$$

we see from (68) that this generates a field

$$B_k(\mathbf{x}) = \epsilon_{kij}(\partial_i \partial_j - \partial_j \partial_i)\Lambda(\mathbf{x}) = \Phi\delta_k(\mathbf{x}; L). \tag{82}$$

This is a magnetic field of total flux Φ inside an infinitesimal tube. By a superposition of such infinitesimally thin flux tubes analogous to (76) we can obviously generate a discrete approximation to any desired magnetic field in a field-free space.

3.3. Magnetic monopoles

Multivalued fields have also been used to describe magnetic monopoles [18–20]. A monopole charge density $\rho_{\rm m}(\mathbf{x})$ is the source of a magnetic field $\mathbf{B}(\mathbf{x})$ as defined by the equation

$$\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{x}) = \rho_{\rm m}(\mathbf{x}). \tag{83}$$

If $\mathbf{B}(\mathbf{x})$ is expressed in terms of a vector potential $\mathbf{A}(\mathbf{x})$ as $\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})$, eq. (83) implies the noncommutativity of derivatives in front of the vector potential $\mathbf{A}(\mathbf{x})$:

$$\tfrac{1}{2}\epsilon_{ijk}(\partial_i\partial_j - \partial_j\partial_i)A_k(\mathbf{x}) = \rho_{\rm m}(\mathbf{x}). \tag{84}$$

Thus $\mathbf{A}(\mathbf{x})$ must be multivalued. Dirac in his famous theory of monopoles [21] made the field single-valued by attaching to the world line of the particle a jumping world surface, whose intersection with a coordinate plane at a fixed time forms the *Dirac string*, along which the magnetic field of the monopole is imported from infinity. This world surface can be made physically irrelevant by quantizing it appropriately with respect to the charge. Its shape in space is just as irrelevant as that of the jumping surface S in Fig. 3. The invariance under shape deformations constitute once more a second gauge structure of the type to be discussed in Section 8 [18].

Once we allow ourselves to work with multivalued fields, we may easily go one step further and express also $\mathbf{A}(\mathbf{x})$ as a gradient of a scalar field as in (80). Then the condition becomes

$$\epsilon_{ijk}\partial_i\partial_j\partial_k\Lambda(\mathbf{x}) = \rho_{\rm m}(\mathbf{x}). \tag{85}$$

There exists by now a well-developed quantum field theory for many other systems described by multivalued fields [6,22,23].

3.4. Minimal magnetic coupling of particles from multivalued gauge transformations

Multivalued gauge transformations are the ideal tool to minimally couple electromagnetism to any type of matter. Consider for instance a free nonrelativistic point particle with a Lagrangian

$$L = \tfrac{1}{2}\dot{\mathbf{x}}^2. \tag{86}$$

The equations of motion are invariant under a gauge transformation

$$L \to L' = L + \boldsymbol{\nabla}\Lambda(\mathbf{x})\dot{\mathbf{x}}, \tag{87}$$

since this changes the action $\mathcal{A} = \int_{t_1}^{t_2} dt\, L$ merely by a surface term:

$$\mathcal{A}' \to \mathcal{A} = \mathcal{A} + \Lambda(\mathbf{x}_2) - \Lambda(\mathbf{x}_1). \tag{88}$$

The invariance is absent if we take $\Lambda(\mathbf{x})$ to be a multivalued gauge function. In this case, a nontrivial vector potential $\mathbf{A}(\mathbf{x}) = \boldsymbol{\nabla}\Lambda(\mathbf{x})$ (working in natural units with $e = 1$) is created in the field-free space, and the nonholonomically gauge-transformed Lagrangian corresponding to (87),

$$L' = \tfrac{1}{2}\dot{\mathbf{x}}^2 + \mathbf{A}(\mathbf{x})\dot{\mathbf{x}}, \tag{89}$$

describes correctly the dynamics of a free particle in an external magnetic field.

The coupling derived by multivalued gauge transformations is automatically invariant under additional ordinary single-valued gauge transformations of the vector potential

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \boldsymbol{\nabla}\Lambda(\mathbf{x}), \tag{90}$$

since these add to the Lagrangian (89) once more the same pure derivative term which changes the action by an irrelevant surface term as in (88).

The same procedure leads in quantum mechanics to the minimal coupling of the Schrödinger field $\psi(\mathbf{x})$. The Lagrange density is (in natural units with $\hbar = 1$)

$$\mathcal{L} = \psi^*(\mathbf{x})\,(i\partial_t + \tfrac{1}{2}\boldsymbol{\nabla}^2)\psi(\mathbf{x}). \tag{91}$$

The physics described by a Schrödinger wave function $\psi(\mathbf{x})$ is invariant under arbitrary $U(1)$ phase changes

$$\psi(\mathbf{x}, t) \to \psi'(\mathbf{x}) = e^{i\Lambda(\mathbf{x})}\psi(\mathbf{x}, t). \tag{92}$$

This implies that the Lagrange density (91) may equally well be replaced by the gauge-transformed one

$$\mathcal{L} = \psi^*(\mathbf{x}, t)\,(i\partial_t + \tfrac{1}{2}\mathbf{D}^2)\psi(\mathbf{x}, t), \tag{93}$$

where $\mathbf{D} \equiv \boldsymbol{\nabla} - i\boldsymbol{\nabla}\Lambda(\mathbf{x})$. By allowing for nonholonomic gauge functions $\Lambda(\mathbf{x})$ whose gradient is the vector potential as in (80), the operator $\mathbf{D}$ turns into

$$\mathbf{D} = \boldsymbol{\nabla} - i\mathbf{A}(\mathbf{x}), \tag{94}$$

which describes correctly the magnetic coupling in quantum mechanics.

As in the classical case, the coupling derived by multivalued gauge transformations is automatically invariant under ordinary single-valued gauge transformations under which the vector potential $\mathbf{A}(\mathbf{x})$ changes as in (90), whereas the Schrödinger wave function undergoes a local $U(1)$-transformation (92). This invariance is a direct consequence of the simple transformation behavior of $\mathbf{D}\psi(\mathbf{x}, t)$ under gauge transformations (90) and (92) which is

$$\mathbf{D}\psi(\mathbf{x}, t) \to \mathbf{D}\psi'(\mathbf{x}, t) = e^{i\Lambda(\mathbf{x})}\mathbf{D}\psi(\mathbf{x}, t). \tag{95}$$

Thus $\mathbf{D}\psi(\mathbf{x}, t)$ transforms just like $\psi(\mathbf{x}, t)$ itself, and for this reason $\mathbf{D}$ is called gauge-covariant derivative.

The generation of magnetic fields by a multivalued gauge transformation is the simplest example for the power of the nonholonomic mapping principle.

We are now prepared to introduce the same mathematics into differential geometry, where the role of gauge transformations is played by reparametrizations of the space coordinates. If spins are present, we must formulate the theory such as to accommodate also local Lorentz transformations.

4. INFINITESIMAL CURVATURE AND TORSION FROM ACTIVE MULTIVALUED COORDINATE TRANSFORMATIONS

We are now going to study the properties of a space at which we can arrive from a flat space using multivalued tetrad fields ${e_a}^\mu$ and ${e^a}_\mu$ which are close to unit matrices ${\delta_a}^\mu$ and ${\delta^a}_\mu$, respectively. It is easy to see that these correspond to a geometric analog ofinfinitesimal gauge transformations in magnetostatics with multivalued gauge functions of the type (81). Because of the nonlinearity of all geometric quantities, we shall restrict ourselves to *infinitesimal* Einstein transformations

$$x^a \underset{E}{\to} q^\mu = x^{\mu=a} - \xi^\mu(x), \tag{96}$$

which play the role of infinitesimal local translations. According to (4), the associated multivalued tetrad fields are

$$\begin{aligned} {e_a}^\mu &= {\delta_a}^\mu - \partial_a \xi^\mu \\ {e^a}_\mu &= {\delta^a}_\mu + \partial_\mu \xi^a. \end{aligned} \tag{97}$$

Thus they are transformed by a gradient of the functions $\xi^\mu(x)$ in complete analogy with the magnetic vector potential in (52). The metric (3) induced by the infinitesimal local translations (96) is

$$g_{\mu\nu} = \eta_{\mu\nu} + (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu). \tag{98}$$

For small transformation functions $\xi^\mu(x)$, the affine connection (11) becomes

$$\Gamma_{\mu\nu}{}^{\lambda} = \partial_\mu \partial_\nu \xi^\lambda. \tag{99}$$

For multivalued transformation functions $\xi_\mu(x)$, the metric and the affine connection are, in general, also multivalued. This could cause difficulties in performing consistent length measurements and parallel displacements. In order to avoid this, Einstein postulated that the metric $g_{\mu\nu}$ and the affine connection $\Gamma_{\mu\nu}{}^{\lambda}$ should be single-valued and smooth enough to be differentiated twice. Because of the single-valuedness, derivatives in front of $g_{\mu\nu}$ and $\Gamma_{\mu\nu}{}^{\lambda}$ should commute with each other [see (18)], implying the infinitesimal integrability conditions

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)(\partial_\lambda \xi_\kappa + \partial_\kappa \xi_\lambda) = 0, \tag{100}$$

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\partial_\sigma \partial_\lambda \xi_\kappa = 0\,. \tag{101}$$

Since ξ^μ are infinitesimal, we can lower the index in both equations (with a mistake which is only of the order of ξ^2 and thus negligible) so that (14) and (17) yield

$$S_{\mu\nu\lambda} = \tfrac{1}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\xi_\lambda, \tag{102}$$

$$R_{\mu\nu\lambda\kappa} = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\partial_\lambda \xi_\kappa\,. \tag{103}$$

Note that the curvature tensor is antisymmetric in the last two indices, as an immediate consequence of the integrability condition (101). This antisymmetry is therefore a Bianchi identity of the gauge field representation of the curvature tensor for infinitesimal deviations from flat space, where it constitutes the fundamental or second identity in Schouten's nomenclature [16].

Let us also calculate the Riemann part (29) of the infinitesimal connection (99). Inserting (98) into (29), we find

$$\bar{\Gamma}_{\mu\nu\lambda} = \tfrac{1}{2}[\partial_\mu(\partial_\nu \xi_\lambda + \partial_\lambda \xi_\nu) + \partial_\nu(\partial_\mu \xi_\lambda + \partial_\lambda \xi_\mu) - \partial_\lambda(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)\,]. \tag{104}$$

The affine connection (99) can then be decomposed as in (30), with the contortion tensor

$$\begin{aligned} K_{\mu\nu\lambda} &= \tfrac{1}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\xi_\lambda - \tfrac{1}{2}(\partial_\nu \partial_\lambda - \partial_\lambda \partial_\nu)\xi_\mu + \tfrac{1}{2}(\partial_\lambda \partial_\mu - \partial_\mu \partial_\lambda)\xi_\nu \\ &= \tfrac{1}{2}[\partial_\mu(\partial_\nu \xi_\lambda - \partial_\lambda \xi_\nu) + \partial_\lambda(\partial_\nu \xi_\mu + \partial_\mu \xi_\nu) - \partial_\nu(\partial_\lambda \xi_\mu + \partial_\mu \xi_\lambda)\,], \end{aligned} \tag{105}$$

the first line being the combination (31) of torsion tensors. By inserting the infinitesimal Riemann connection (104) into (45), we find the associated Riemann curvature tensor

$$\begin{aligned}&\bar{R}_{\mu\nu\lambda\kappa}\\&= \tfrac{1}{2}\partial_\mu[\partial_\nu(\partial_\lambda\xi_\kappa+\partial_\kappa\xi_\lambda)+\partial_\lambda(\partial_\nu\xi_\kappa+\partial_\kappa\xi_\nu)-\partial_\kappa(\partial_\nu\xi_\lambda+\partial_\lambda\xi_\nu)]\\&= -\tfrac{1}{2}\partial_\nu[\partial_\mu(\partial_\lambda\xi_\kappa+\partial_\kappa\xi_\lambda)+\partial_\lambda(\partial_\nu\xi_\kappa+\partial_\kappa\xi_\nu)-\partial_\kappa(\partial_\mu\xi_\lambda+\partial_\lambda\xi_\lambda)].\end{aligned} \tag{106}$$

Averaging the two equal right-hand sides, the integrability condition (101) for the metric removes the two first parentheses, and we obtain

$$\bar{R}_{\mu\nu\lambda\kappa}=\tfrac{1}{2}\{\,[\partial_\mu\partial_\lambda(\partial_\nu\xi_\kappa+\partial_\kappa\xi_\nu)-(\mu\leftrightarrow\nu)\,]-[\lambda\leftrightarrow\kappa]\,\}. \tag{107}$$

Multivalued coordinate transformations of the type (96) appear naturally in the theory of topological defects in three-dimensional crystals. There one considers infinitesimal displacements of atoms

$$x_i \to x'_i = x_i + u_i(\mathbf{x}), \qquad (i=1,2,3). \tag{108}$$

where x'_i are the shifted positions, as seen from an ideal reference crystal. If we change the point of view to an intrinsic description, i.e., if we measure coordinates by counting the number of atomic steps *within* the distorted crystal, then the atoms of the ideal reference crystal are displaced by

$$x_i \ \to\ x'_i = x_i - u_i(\mathbf{x}). \tag{109}$$

The displacement field is defined only modulo lattice spacings. This makes it intrinsically multivalued, having noncommuting derivatives which contain information on the crystalline topological defects. The physical coordinates of material points x^i for $i=1,2,3$ are identified with the previous spatial coordinates x^a for $a=1,2,3,4$[4] and $\partial_a=\partial/\partial x^a (a=i)$ with the previous derivatives ∂_i. The infinitesimal translation fields in (96) are equal to the displacements $u_i(\mathbf{x})$ such that the multivalued tetrads are

$$e_a^{\ i}=\delta_a^{\ i}-\partial_a u_i\,, \qquad e^a{}_i=\delta^a{}_i+\partial_i u_a\,, \tag{110}$$

and all geometric quantities are defined as before.

[4] When working with four-vectors, it is conventional to consider the upper indices as physical components. In purely three dimensional calculations one usually employs the metric $\eta_{ab}=\delta_{ab}$ such that $x^{a=i}$ and x_i are the same.

In a crystal, one likes to specify the deformation by a strain tensor

$$u_{kl} = \tfrac{1}{2}(\partial_k u_l + \partial_l u_k), \tag{111}$$

and a local rotation tensor

$$\omega_{kl} = \tfrac{1}{2}(\partial_k u_l - \partial_l u_k). \tag{112}$$

For these, the integrability conditions (100),(101) imply that

$$(\partial_i\partial_j - \partial_j\partial_i)(\partial_k u_l + \partial_l u_k) = 0, \tag{113}$$
$$(\partial_i\partial_j - \partial_j\partial_i)\partial_n(\partial_k u_i + \partial_l u_k) = 0, \tag{114}$$
$$(\partial_i\partial_j - \partial_j\partial_i)\partial_k(\partial_k u_i - \partial_l u_k) = 0, \tag{115}$$

stating that the strain tensor

$$u_{kl} = \tfrac{1}{2}(\partial_k u_l + \partial_l u_k), \tag{116}$$

its derivative, and the derivative of the local rotation tensor

$$\omega_{kl} = \tfrac{1}{2}(\partial_k u_l - \partial_l u_k), \tag{117}$$

are all twice-differentiable single-valued functions everywhere. In three dimensions one often uses the rotation vector

$$\omega_j = \tfrac{1}{2}\epsilon_{jmn}\,\omega_{mn} = \tfrac{1}{2}\epsilon_{jmn}\,\partial_m u_n \tag{118}$$

instead of the tensor field (117).

A single-valued distortion field $u_i(\mathbf{x})$ corresponds to an *elastic* deformation, a multivalued field to a *plastic* deformation of the crystal.

The local vector field ω_j has noncommuting derivatives, as measured by the tensor

$$G_{ji} = \epsilon_{ikl}\partial_k\partial_l\omega_j\,. \tag{119}$$

This is the *Einstein curvature* tensor of the Riemann–Cartan geometry. Since the derivative of the local rotation tensor has commuting derivatives, the Einstein tensor is divergenceless:

$$\partial_i G_{ji} = 0\,. \tag{120}$$

This corresponds to the famous original Bianchi identity (the first identity) of Riemann spaces which has served as a prototype for all identities expressing the single-valuedness of physical fields.

Let us prove that G_{ji} coincides with the Einstein tensor in the common definition as the combination of Ricci tensor and scalar curvature:

$$G_{ji} = R_{ji} - \tfrac{1}{2} g_{ji} R_{kk} \,. \tag{121}$$

Returning to the notation $\xi_i(q)$ for the infinitesimal translations, and taking advantage of the integrability condition (101), we write the curvature tensor (30) as

$$R_{ijkl} = (\partial_i \partial_j - \partial_j \partial_i) \tfrac{1}{2} (\partial_k \xi_l - \partial_l \xi_k) = (\partial_i \partial_j - \partial_j \partial_i) \epsilon_{klm} \omega_m(q). \tag{122}$$

In three dimensions, the antisymmetry in ij and kl suggests the introduction of a second-rank tensor

$$G_{ji} \equiv \tfrac{1}{4} \epsilon_{ikl} \epsilon_{jmn} R^{klmn}. \tag{123}$$

In the full nonlinear Riemann geometry, the ϵ-tensors are simply replaced by their generally covariant versions

$$e_{ijk} = \sqrt{g}\, \epsilon_{ijk} = g_{ii'} g_{jj'} g_{kk'} e^{i'j'k'} = g_{ii'} g_{jj'} g_{kk'} \left(\frac{1}{\sqrt{g}}\, \epsilon^{i'j'k'} \right). \tag{124}$$

If we now insert the identity

$$\begin{aligned} e_{ikl} e_{jmn} = {} & g_{ij} g_{km} g_{ln} + g_{im} g_{kn} g_{lj} + g_{in} g_{kj} g_{lm} \\ & - g_{ij} g_{lm} g_{kn} - g_{im} g_{kn} g_{kj} - g_{in} g_{lj} g_{km} \,, \end{aligned} \tag{125}$$

into the fully covariant version of (123) we recover (121).

In four dimensions, the combination (121) can be rewritten as

$$G^{\nu\mu} = \tfrac{1}{4} e^{\mu\alpha\beta\gamma} e^{\nu}{}_{\alpha}{}^{\delta\tau} R_{\beta\gamma\delta\tau} \,,$$

a direct generalization of (123).

Inserting (122) into (123), we find for small displacements

$$G_{ij} = \epsilon_{ikl} \partial_k \partial_l (\tfrac{1}{2} \epsilon_{jmn} \partial_m \xi_n), \tag{126}$$

which coincides with (119), as we wanted to prove.

Let us also form the Einstein tensor $\bar{G}_{ij}$ associated with the Riemannian curvature tensor $\bar{R}_{ijkl}$. Using (107) we find

$$\bar{G}_{ji} = \epsilon_{ikl} \epsilon_{jmn} \partial_k \partial_m \tfrac{1}{2} (\partial_l \xi_n + \partial_n \xi_l) = \epsilon_{ikl} \epsilon_{jmn} \partial_k \partial_m \xi_{ln} \,. \tag{127}$$

In the theory of crystalline topological defects one introduces the following measures for the noncommutativity of derivatives. The dislocation density

$$\alpha_{ij} = \epsilon_{ikl} \partial_k \partial_l \xi_j \, , \tag{128}$$

the disclination density

$$\Theta_{ij} = \epsilon_{ikl} \partial_k \partial_l \omega_j \, , \tag{129}$$

and the defect density

$$\eta_{ij} = \epsilon_{ikl} \epsilon_{jmn} \partial_k \partial_n \xi_{lm} \, . \tag{130}$$

Comparison with eq. (107) shows that α_{ij} is directly related to the torsion tensor $S_{kl}{}^i = \frac{1}{2}(\Gamma_{kl}{}^i - \Gamma_{lk}{}^i)$:

$$\alpha_{ij} \equiv \epsilon_{ikl} \Gamma_{klj} \equiv \epsilon_{ikl} S_{klj} \, . \tag{131}$$

Hence torsion is a measure of the translational defects contained in singular coordinated transformations. We can also use the decomposition (31) and express this in terms of the contortion tensor as

$$\alpha_{ij} = \epsilon_{ikl} K_{klj} \, . \tag{132}$$

In terms of the strain tensor $\xi_{kj} = \frac{1}{2}(\partial_k \xi_j + \partial_j \xi_k)$ and the rotation field ω_l, the contortion tensor becomes

$$\begin{aligned} K_{ijk} &= \tfrac{1}{2}\partial_j(\partial_j \xi_k - \partial_k \xi_j) - \tfrac{1}{2}[\partial_j(\partial_k \xi_j + \partial_i \xi_k) - (j \leftrightarrow k)\,] \\ &= \partial_i \omega_{jk} - [\partial_j \xi_{ki} - (j \leftrightarrow k)\,]. \end{aligned} \tag{133}$$

Since K_{ijk} is antisymmetric in lj, it is useful to introduce the tensor of second rank, called Nye's contortion tensor,

$$K_{ln} = \tfrac{1}{2} K_{klj} \epsilon_{ljn} \, . \tag{134}$$

Inserting this into (132) we see that

$$\alpha_{ij} = -K_{ji} + \delta_{ij} K_{ll} \, . \tag{135}$$

For Nye's contortion tensor, the decomposition (133) takes the form

$$K_{il} = \partial_i \omega_l - \epsilon_{lkj} \partial_j \xi_{kj} \, . \tag{136}$$

Consider now the disclination density Θ_{ij}. Comparing (130) with (119) we see that it coincides exactly with the Einstein tensor G_{jl} formed from the full curvature tensor

$$\Theta_{ij} \equiv G_{ji}\,. \tag{137}$$

The defect density (130), finally, coincides with the Einstein tensor formed from the Riemannian curvature tensor:

$$\eta_{ij} = \bar{G}_{ij}\,. \tag{138}$$

5. EXPLICIT MULTIVALUED TRANSFORMATIONS PRODUCING CURVATURE AND TORSION

Let us give explicit multivalued functions $\xi^\mu(q)$ generating infinitesimal pointlike curvature and torsion in an otherwise flat space. We may restric ourselves to two dimensions. The generalization to D dimensions is straightforward — we may simply deal with each of the $D(D-1)/2$ coordinate plains separately, and compose the results at the end. In each coordinate plane, we now write down transformation functions which correspond to the fundamental topological defects pictures in Fig. 1.

5.1. Torsion

Consider first the upper example in Fig. 1, where a dislocation is generated by a Volterra process in which a layer of atoms is added or removed. The active nonholonomic transformation may be described differentially by

$$dx^i = \begin{cases} dq^1 & \text{for } i = 1, \\ dq^2 + \epsilon\partial_\mu\phi(q)dq^\mu & \text{for } i = 2, \end{cases} \tag{139}$$

where ϵ is a small parameter, and $\phi(q)$ the multivalued function (69). In the two-dimensional subspace under consideration, the tetrads are dyads with components

$$\begin{aligned} e^1{}_\mu(q) &= \delta^1{}_\mu\,, \\ e^2{}_\mu(q) &= \delta^2{}_\mu + \epsilon\partial_\mu\phi(q), \end{aligned} \tag{140}$$

yielding for the torsion tensor the components

$$S_{\mu\nu}{}^1(q) = 0, \qquad S_{\mu\nu}{}^2(q) = \frac{\epsilon}{4\pi}\,(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\phi(q). \tag{141}$$

Using the noncommutativity (72), we obtain a torsion localized at the origin:

$$S_{12}{}^2(q) = \frac{\epsilon}{2}\,\delta^{(2)}(q). \tag{142}$$

The mapping introduces no curvature. When encircling a dislocation along a closed path C, its counter image C' in the ideal crystal does not form a closed path. The closure failure is called the *Burgers vector*

$$b^i \equiv \oint_{C'} dx^i = \oint_C dq^\mu e^i{}_\mu \,. \tag{143}$$

It specifies the direction and thickness of the layer of additional atoms. With the help of Stokes' theorem, it is seen to measure the torsion contained in any surface S spanned by C:

$$b^i = \oint_S d^2 s^{\mu\nu} \partial_\mu e^i{}_\nu = \oint_S d^2 s^{\mu\nu} e^i{}_\lambda S_{\mu\nu}{}^\lambda, \tag{144}$$

where $d^2 s^{\mu\nu} = -d^2 s^{\nu\mu}$ is the projection of an oriented infinitesimal area element onto the plane $\mu\nu$. The above example has the Burgers vector

$$b^i = (0, \epsilon). \tag{145}$$

A corresponding closure failure appears when mapping a closed contour C in the ideal crystal into a crystal containing a dislocation. This defines a Burgers vector:

$$b^\mu \equiv \oint_{C'} dq^\mu = \oint_C dx^i e_i{}^\mu. \tag{146}$$

By Stokes' theorem, this becomes a surface integral

$$\begin{aligned} b^\mu &= \oint_S d^2 s^{ij} \partial_i e_j{}^\mu = \oint_S d^2 s^{ij} e_i{}^\nu \partial_\nu e_j{}^\mu \\ &= -\oint_S d^2 s^{ij} e_i{}^\nu e_j{}^\lambda S_{\nu\lambda}{}^\mu, \end{aligned} \tag{147}$$

with the last step following from (15).

Different pointlike torsions (142) can be used to generate a torsion as an arbitrary superposition of infinitesimal point-like torsions,

$$S_{12}{}^2(q) = \frac{\epsilon_n}{2} \sum_n \delta(q - q_n). \tag{148}$$

We simply have to choose the angular function $\phi(q)$ in (140) in analogy to (77) as

$$\phi^f(q) = \sum_n \epsilon_n \arctan \frac{q^2 - q_n^2}{q^1 - q_n^1} \,. \tag{149}$$

As in the magnetic case, one is not allowed to replace the sum by an integral over a continuous distribution of these functions, since the endpoints of the cuts of the Riemann surfaces must remain clearly distinguishable [see the discussion after eq. (77)]. In crystal physics, this means that there is no mathematically well-defined way of setting up a continuous theory of defects. Fortunately, this need not bother us since defects in crystals are discete objects anyhow. It is curious to see how theorists of plastic deformations have tried to escape this problem verbally.

When applied to spacetime of gravitational physics, this implies that it is impossible to generate, even infinitesimally, a space with a smooth torsion. We can only generate a space carrying a superposition of discrete torsion lines (or surfaces in four spacetime dimensions). This is similar to the geometry generated by the Regge calculus [24]. For the arguments to be presented in the sequel, however, this problem will be irrelevant. We merely need to be sure that a flat space can be transformed into spaces with arbitrary discrete superpositions of infinitesimal line- or surface-like curvatures and torsions. Once we know the transformed laws of nature for such superpositions, we may generalize them to arbitrary infinitesimal curvature and torsion. These can always be approximated discretely to any desired degree of accuracy.

By removing a vertical layer of atoms in Fig. 1, we obtain the same result with the superscript 1 exchanged by 2. By going through the same procedure in all coordinate planes, removing a layer of atoms in each spatial direction, and forming superpositions, we can generate an arbitrary superposition of discrete infinitesimal torsions in the initially flat space. This procedure can be extended to three and four spacetime dimensions in an obvious way.

5.2. Curvature

The second example is the nonholonomic mapping in the lower part of Fig. 1, generating a disclination which corresponds to an entire section of angle Ω missing in an ideal atomic array. For an infinitesimal angle Ω, this may be described, in two dimensions, by the differential mapping

$$x^i = \delta^i{}_\mu \left[q^\mu - \frac{\Omega}{2\pi} \epsilon^\mu{}_\nu q^\nu \phi(q) \right], \tag{150}$$

with the multivalued function (69). The symbol $\epsilon_{\mu\nu}$ denotes the antisymmetric Levi-Civita tensor. The transformed metric

$$g_{\mu\nu} = \delta_{\mu\nu} + \frac{\Omega}{\pi} \frac{1}{q^\sigma q_\sigma} \epsilon_{\mu\lambda}\epsilon_{\nu\kappa} q^\lambda q^\kappa. \tag{151}$$

is single-valued and has commuting derivatives. The torsion tensor vanishes since $(\partial_1\partial_2 - \partial_2\partial_1)x^{1,2}$ is proportional to $q^{2,1}\delta^{(2)}(q) = 0$. The local rotation field $\omega(q) \equiv \frac{1}{2}[\partial_1 x^2(q) - \partial_2 x^1(q)]$, on the other hand, is equal to the multivalued function $\Omega\phi(q)$, thus having the noncommuting derivatives:

$$(\partial_1\partial_2 - \partial_2\partial_1)\omega(q) = \Omega\delta^{(2)}(q). \tag{152}$$

To lowest order in Ω, this determines the curvature tensor, which in two dimensions possesses only one independent component, for instance R_{1212}. From eqs. (122),(152) we see that

$$R_{1212} = (\partial_1\partial_2 - \partial_2\partial_1)\omega(q) = \Omega\delta^{(2)}(q). \tag{153}$$

As in the case of torsion, we may write perform the active nonholonomic coordinate transformation with a superposition of point-like curvatures, inserting into (150) the angular field

$$\phi^f(q) = \sum_n \Omega_n \arctan\frac{q^2 - q_n^2}{q^1 - q_n^1}, \tag{154}$$

and obtain

$$R_{1212} = \sum_n \Omega_n \delta^{(2)}(q - q_n). \tag{155}$$

This forms an approximation to an arbitrary infinitesimal continuous curvature in the 12-plane. Again, we cannot take the continuum limit, but for the derivation of structure of the physical laws, the restricted point-like distributions of curvature and torsion are perfectly sufficient.

By cutting a sector of atoms from all possible coordinate planes and choosing different directions of the sector we can generate a four-dimensional spacetime with an arbitrary superposition of discrete infinitesimal curvatures from an initially flat space.

We conclude: A space with infinitesimally small torsion and curvature can be generated from a flat space via multivalued coordinate transformations, and is completely equivalent to a crystal which has undergone plastic deformation and is filled with dislocations and disclinations. The nonholonomic mapping principle has produced a Riemann–Cartan space with infinitesimal line-like curvature and torsion from a flat space.

We must emphasize the infinitesimal nature of the line-like torsion and curvature. It is mathematically inconsistent to generate the structure to a full geometry of defects as proposed in [12–14]. The reason is that this would produce higher powers of δ-functions (142),(155), which are mathematically undefined.

In a Minkowski space, trajectories of free point particles are straight lines. A space with curvature and torsion may be viewed as a "world crystal" with topological defects. In it, the preferred paths are no longer straight since defects may lie in their way. Translating this into Einstein's theory, mass points in a gravitational field will run along the geometrically preferred path in the space with defects. The defects in the "world crystal" explain all gravitational effects.

In subsection 6.2 we shall demonstrate that the nonholonomic mapping principle will turn straight lines in flat space into the correct particle trajectories. These are autoparallel, forming the straightest possible paths in the metric-affine space.

The natural length scale of gravity is the Planck length

$$l_{\rm p} = \left(\frac{c^3}{8\pi k\hbar}\right)^{-1/2} \tag{156}$$

where c is the light velocity ($\approx 3\times 10^{10}$cm/s), $\hbar$ is Planck's constant ($\approx 1.05459\times 10^{-27}$erg/s), and k is Newton's gravity constant ($\approx 6.673\times 10^{-8}cm^3/g\cdot$s^2). The Planck length is an extremely small quantity ($\approx 8.09\times 10^{-33}$cm) which at present is beyond any experimental resolution. This may be imagined as the lattice constant of the world crystal with defects.

6. THE NEW ACTION PRINCIPLE IN THE PRESENCE OF TORSION

In 1993, Fiziev and the present author [25] applied the nonholonomic mapping principle to the variational derivation of equations of motion from the extremum of an action. We observed that variations of paths in spaces with torsion should reflect the closure failure of parallelograms and can therefore not be performed with both ends of the paths simultaneously held fixed. This has the important and surprising consequence that an action involving only the metric can produce equations of motion containing a torsion force.

The new variational procedure was simplified by Pelster and myself [26] by introducing modified variations which do not commute with the proper-time derivative of the trajectory. The simplified procedure has the advantage of being applicable to a larger variety of actions, in particular to particles in external fields.

6.1. Minkowski spacetime

Starting point is the standard action principle for the free motion of a spinless point particle of mass M in a flat space with Minkowski metric

η_{ab}. Introducing some parameter τ to describe the path $x^a(\tau)$ of the point particle, the infinitesimal proper distance ds is given by

$$ds(\tau) = \sqrt{dx^2} = \sqrt{\eta_{ab} dx^a(\tau) dx^b(\tau)}\,. \tag{157}$$

The associated time $d\sigma = ds/c$ is the proper time. The action of the point particle

$$\mathcal{A}[x^a(\tau)] = \int_{\tau_1}^{\tau_2} d\tau L(\dot{x}^a(\tau)) \tag{158}$$

is proportional to the proper time spent by the particle moving from τ_1 to τ_2, i.e., the Lagrangian reads [15]

$$L(\dot{x}^a) = -M\sqrt{\dot{x}^2}\,. \tag{159}$$

By construction, the action (158) is invariant with respect to arbitrary reparametrizations $\tau \to \tau' = \tau'(\tau)$.

The Hamiltonian action principle states that the physically realized trajectory is found by extremization, requiring the vanishing of the variation

$$\delta\mathcal{A}[x^a(\tau)] = 0 \tag{160}$$

with respect to all variations $\delta x^a(\tau)$ which vanish at the end points τ_1 and τ_2:

$$\delta x^a(\tau_1) = \delta x^a(\tau_2) = 0\,. \tag{161}$$

The geometric meaning of a variation implies that they are independent of changes in the τ-parameter, i.e., that they satisfy the following commutation relation with the derivative $d_\tau \equiv d/d\tau$:

$$\delta d_\tau x^a(\tau) - d_\tau \delta x^a(\tau) = 0\,. \tag{162}$$

Under such variations, the extremization of the action (158) leads immediately to the Euler–Lagrange equation

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^a(\tau)} = 0\,. \tag{163}$$

Inserting the Lagrangian (159), and remembering the proper distance ds in eq. (157), we end up with the equation of motion

$$\ddot{x}^a(\tau) = f(\tau)\,\dot{x}^a(\tau)\,, \tag{164}$$

where $f(\tau)$ is determined by a relation between the proper distance s and the trajectory parameter τ:

$$f(\tau) = \ddot{s}(\tau)/\dot{s}(\tau). \tag{165}$$

Just as with the action (118), the equation of motion (164) is invariant with respect to arbitrary reparametrizations $\tau \to \tau' = \tau'(\tau)$. Under these,

$$f(\tau) \;\to\; f'(\tau') = \ddot{s}(\tau')/\dot{s}(\tau')\,. \tag{166}$$

The particular reparametrization

$$\tau\,'(\tau) = \int^{\tau} du\, \exp\left[\int^{u} dv f(v)\right] \tag{167}$$

leads to a vanishing of $f\,'(\tau\,')$, implying that τ' coincides with the proper time $\sigma = s/c$. Then the equation of motion (164) simply reduces to

$$\ddot{x}^a(\sigma) = 0\,. \tag{168}$$

It is useful to realize that the above relativistic treatment can be reduced to a nonrelativistically looking procedure by not using (159) as a Lagrangian but, instead, the completely equivalent one

$$L(\dot{x}) = -\frac{M}{2\rho(\tau)}\,\dot{x}^2(\tau) - \frac{M}{2}\,\rho(\tau). \tag{169}$$

This contains the particle orbit quadratically, looking like a free nonrelativistic Lagrangian action, but at the expense of an extra dimensionless variable $\rho(\tau)$. At the extremum, the new action coincides with the initial one (158). Indeed, extremizing $\mathcal{A}$ in $\rho(\tau)$ gives the relation

$$\rho(\tau) = \sqrt{\dot{x}^2(\tau)}\;/c. \tag{170}$$

Inserting this back into $\mathcal{A}$ renders the classical action

$$\mathcal{A} = -M\int_{\tau_1}^{\tau_2} d\tau\sqrt{\dot{x}^2(\tau)}\;, \tag{171}$$

which is the same as (158).

The new action shares with the old action (158) the reparametrization invariance $\tau \to \tau' = \tau'(\tau)$. We only have to assign an appropriate transformation behavior to the extra variable $\rho(\tau)$. If τ is replaced by a new parameter $\bar{\tau} = f(\tau)$, the action remains invariant, if $\rho(\tau)$ is simultaneously changed as follows:

$$\rho \ \rightarrow \ \rho/f'. \tag{172}$$

For the proper time

$$\tau = \sigma, \tag{173}$$

the extremal variable $\rho(s)$ is identically equal to unity. Thus we can use the Lagrangian (169) for $\rho \equiv 1$ to find the correct relativistic particle trajectories parametrized with the proper time σ. Moreover, as long as we do not need the numerical value of the action but only its functional dependence on the paths $x(s)$, we may drop the trivial constant last term and the action looks exactly like a nonrelativistic one, except for the overall sign:

$$\mathcal{A} = -\int_{\sigma_1}^{\sigma_2} d\sigma \frac{M}{2}\, \dot{x}^2(\sigma). \tag{174}$$

The negative sign ensures that the spatial part of $\dot{x}^2(\sigma)$ appears with the usual positive sign.

6.2. Riemann–Cartan spacetime

In subsection 3.4 we have learned how to find the action of a point particle in the presence of a magnetic field by simply applying a nonholonomic gauge transformation to the field-free action. In the presence of curvature and torsion, the nonholonomic mapping principle instructs us to transform the action (171), or equivalently, the actions (171) and (174) via the infinitesimal coordinate transformations (96) to curvilinear coordinates. After this we assume the transformation functions $\xi^\lambda(q)$ to be multivalued. For finite transformations we use the mapping (4) to transform the action to an arbitrary metric affine space. For the paths of the particles, this implies the mapping

$$\dot{q}^\mu = e_i{}^\mu(q)\dot{x}^i, \tag{175}$$

by which the Lorentz-invariant proper time increment (157) is mapped into

$$ds = \sqrt{g_{\mu\nu}(q(\tau))\dot{q}^\mu(\tau)\dot{q}^\nu(\tau)}\ . \tag{176}$$

The action (158) and (159) becomes therefore

$$\mathcal{A} = -M\int_{\tau_1}^{\tau_2} d\tau \sqrt{g_{\mu\nu}(q(\tau))\dot{q}^\mu(\tau)\dot{q}^\nu(\tau)}\ . \tag{177}$$

whereas the nonrelativistic-looking form (174) goes over into

$$\mathcal{A} = -\int_{\sigma_a}^{\sigma_b} d\sigma \frac{M}{2} \, g_{\mu\nu}(q(\sigma))\dot{q}^{\mu}(\sigma)\dot{q}^{\nu}(\sigma). \tag{178}$$

Before proceeding with our main argument we first observe a general feature of all actions generated from flat-space actions by means of nonholonomic transformations: They are trivially invariant under ordinary holonomic coordinate transformations. In the context of multivalued gauge transformations in magnetostatics, this was seen before in subsection 3.4, where gauge invariance was automatic. For the actions (177) and (178) the coordinate invariance is obvious: Under a coordinate transformation $q^{\mu} \to q'^{\mu}$, the differentials transform like

$$dq^{\mu} \;\to\; dq'^{\mu} = \alpha^{\mu}{}_{\nu} dq^{\nu}, \qquad \alpha^{\mu}{}_{\nu} \equiv \frac{\partial q'^{\mu}}{\partial q^{\nu}}\,, \tag{179}$$

$$dq_{\mu} \;\to\; dq'_{\mu} = \alpha_{\mu}{}^{\nu} dq_{\nu}\,, \qquad \alpha_{\mu}{}^{\nu} \equiv \frac{\partial q'_{\mu}}{\partial q_{\nu}}\,, \tag{180}$$

where

$$\alpha^{\nu}{}_{\lambda}\alpha_{\nu}{}^{\mu} = \delta_{\lambda}{}^{\mu}, \qquad \alpha_{\nu}{}^{\mu}\alpha^{\lambda}{}_{\mu} = \delta_{\nu}{}^{\lambda}. \tag{181}$$

The new coordinate differentials are related to flat ones by a relation like (4):

$$dx^{a} = e'^{a}{}_{\lambda}(q')\, dq'^{\lambda}\,. \tag{182}$$

Inserting (179), we obtain the transformation law for the multivalued tetrads:

$$\begin{aligned} e_{a}{}^{\mu}(q) = \frac{\partial q^{\mu}}{\partial x^{a}} \;&\to\; e'_{a}{}^{\mu}(q') \equiv \frac{\partial q'^{\mu}}{\partial x^{a}} = \frac{\partial q'^{\mu}}{\partial q^{\nu}}\frac{\partial q^{\nu}}{\partial x^{a}} = \alpha^{\mu}{}_{\nu}(q) e_{a}{}^{\nu}(q),\\ e^{a}{}_{\mu}(q) = \frac{\partial x^{a}}{\partial q^{\mu}} \;&\to\; e'^{a}{}_{\mu}(q') \equiv \frac{\partial x^{a}}{\partial q'^{\mu}} = \frac{\partial q^{\nu}}{\partial q'^{\mu}}\frac{\partial x^{a}}{\partial q^{\nu}} = \alpha_{\mu}{}^{\nu}(q) e^{a}{}_{\nu}(q). \end{aligned} \tag{183}$$

Inserting this into (3), we find the corresponding transformation law for the metric tensor

$$g_{\mu\nu}(q) \;\to\; g'_{\mu'\nu'}(q') = \alpha_{\mu'}{}^{\mu}(q)\alpha_{\nu'}{}^{\nu}(q) g_{\mu\nu}(q). \tag{184}$$

Using this and (179),(180), we readily prove the invariance of the proper time increment (176), and thus of the actions (177) and (178) under arbitrary coordinate transformations.

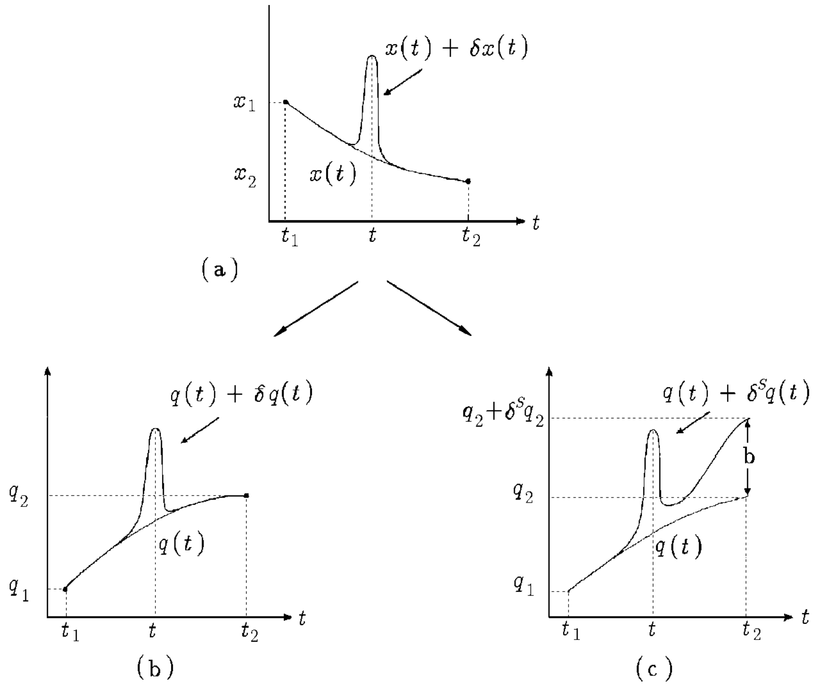

Figure 4. Images under a holonomic and a nonholonomic mapping of a fundamental path variation. In the holonomic case, the paths $x(\tau)$ and $x(\tau)+\delta x(\tau)$ in (a) turn into the paths $q(\tau)$ and $q(\tau)+\delta q(\tau)$ in (b). In the nonholonomic case with $S_{\mu\nu}{}^{\lambda} \neq 0$, they go over into $q(\tau)$ and $q(\tau)+\delta^S q(\tau)$ shown in (c) with a closure failure $\delta^S q_2 = b^\mu$ at τ_2 analogous to the Burgers vector b^μ in a solid with dislocations.

An arbitrary vector field $v_\mu(q)$ transforms like

$$\begin{aligned} v_\mu(q) &\to v'_{\mu'}(q') = \alpha_{\mu'}{}^{\mu}(q) v_\mu(q), \\ v^\mu(q) &\to v'^{\mu'}(q') = \alpha^{\mu'}{}_{\mu}(q) v^\mu(q), \end{aligned} \tag{185}$$

as follows directly from a comparison of the two local representations for a vector field in flat space $v_a(q) = e_a{}^\mu(q) v_\mu(q) = e'_a{}^\mu(q') v_\mu(q')$ and $v^a(q) = e^a{}_\mu(q) v^\mu(q) = e'^a{}_\mu(q') v^\mu(q')$.

As announced before, the closure failure of parallelograms in a space with torsion forces us to reexamine the variational procedure in the action principle for spinless point particles. To be consistent, the same nonholonomic mapping which generates the Riemann–Cartan space requires that the variations in the transformed q^μ-coordinates are performed as gauge images of the variations in the euclidean x^i-space, to be found via (175). It is easy to see that the images of variations $\delta x^i(\tau)$ are quite different from ordinary variations as illustrated in Fig. 4(a). The variations of the Cartesian coordinates $\delta x^i(\tau)$ are performed at fixed end points of the paths. Thus they form *closed paths* in the x^i-space. Their images, however, lie in a space with defects and thus possess a closure failure indicating the amount of torsion introduced by the mapping. This property will be emphasized by writing the images $\delta^S q^\mu(\tau)$ and calling them *nonholonomic variations*.

Let us calculate them explicitly. The paths in the two spaces are related by the integral equation

$$q^\mu(\tau) = q^\mu(\tau_1) + \int_{\tau_1}^{\tau} d\tau' \, e_i{}^\mu(q(\tau'))\dot{x}^i(\tau'). \tag{186}$$

Note that the left-hand side is well defined even though $e_i{}^\mu(q(\tau'))$ is a multivalued function. When performing the integral along a specific path $q^\mu(s)$, we may continue $e_i{}^\mu(q(\tau'))$ analytically through any jumping surface of the type sketched in Fig. 3.

If a path $x^i(\tau)$-space is varied by $\delta x^i(\tau)$, eq. (186) determines the associated change in the image path $q^\mu(\tau)$ by

$$\begin{aligned} \delta^S q^\mu(\tau) &= \int_{\tau_1}^{\tau} d\tau' \delta^S \left[e_i{}^\mu(q(\tau'))\dot{x}^i(\tau') \right] \\ &= \int_{\tau_1}^{\tau} d\tau' \{ \left[\delta^S e_i{}^\mu(q(\tau')) \right] \dot{x}^i(\tau') + e_i{}^\mu(q(\tau'))\delta\dot{x}^i(\tau') \}, \end{aligned} \tag{187}$$

which will be referred to as a *nonholonomic variation* of the image path $q^\mu(\tau)$. The superscript S indicates that the properties of this change depend crucially on the torsion in q-space. A comparison with (175) shows that the variations $\delta^S q^\mu$ and the τ-derivative of q^μ are independent of each other

$$\delta^S \dot{q}^\mu(\tau) = \frac{d}{d\tau} \, \delta^S q^\mu(\tau), \tag{188}$$

just as for ordinary variations δx^i [recall (162)].

It will be useful to introduce in addition a further quantity to be called *auxiliary nonholonomic variation* in q^μ-space by the relation

$$\bar{\delta} q^\mu(\tau) \equiv e_i{}^\mu(q(\tau))\delta x^i(\tau). \tag{189}$$

In contrast to $\delta^S q^\mu(\tau)$, these do vanish at the endpoints:

$$\bar{\delta} q(\tau_1) = \bar{\delta} q(\tau_2) = 0, \tag{190}$$

i.e., they form closed paths in q^μ-space.

With the help of (189) we derive from (187) the relation

$$\frac{d}{d\tau}\,\delta^S q^\mu(\tau) = \delta^S[e_i{}^\mu(q(\tau))\,]\dot{x}^i(\tau) + e_i{}^\mu(q(\tau))\delta\dot{x}^i(\tau)$$

$$= \delta^S[e_i{}^\mu(q(\tau))\,]\dot{x}^i(\tau) + e_i{}^\mu(q(\tau))\frac{d}{d\tau}\,[e^i{}_\nu(q(\tau))\bar{\delta}q^\nu(\tau)\,]. \quad (191)$$

After inserting

$$\delta^S e_i{}^\mu = -\Gamma_{\lambda\nu}{}^\mu\delta^S q^\lambda e_i{}^\nu, \qquad \frac{d}{d\tau}e^i{}_\nu = \Gamma_{\lambda\nu}{}^\mu\dot{q}^\lambda e^i{}_\mu\,, \quad (192)$$

this becomes

$$\frac{d}{d\tau}\,\delta^S q^\mu = -\Gamma_{\lambda\nu}{}^\mu\delta^S q^\lambda\dot{q}^\nu + \Gamma_{\lambda\nu}{}^\mu\dot{q}^\lambda\bar{\delta}q^\nu + \frac{d}{d\tau}\bar{\delta}q^\mu. \quad (193)$$

It is useful to introduce the difference between the nonholonomic variation $\delta^S q^\mu$ and the auxiliary nonholonomic variation $\bar{\delta}q^\mu$:

$$\delta^S b^\mu \equiv \delta^S q^\mu - \bar{\delta}q^\mu. \quad (194)$$

Then we can rewrite (193) as a first-order differential equation for $\delta^S b^\mu$:

$$\frac{d}{d\tau}\,\delta^S b^\mu = -\Gamma_{\lambda\nu}{}^\mu\delta^S b^\lambda\dot{q}^\nu + 2S_{\lambda\nu}{}^\mu\dot{q}^\lambda\bar{\delta}q^\nu. \quad (195)$$

Under an arbitrary nonholonomic variation $\delta^S q^\mu = \bar{\delta}q^\mu + \delta^S b^\mu$, the action (178) changes by

$$\delta^S\mathcal{A} = -M\int_{\sigma_1}^{\sigma_2} d\sigma(g_{\mu\nu}\dot{q}^\nu\delta^S\dot{q}^\mu + \tfrac{1}{2}\partial_\mu g_{\lambda\kappa}\delta^S q^\mu\,\dot{q}^\lambda\dot{q}^\kappa), \quad (196)$$

where σ is now the proper time. Using (188) and (190) we partially integrate of the $\delta\dot{q}$-term, and apply the identity $\partial_\mu g_{\nu\lambda} \equiv \Gamma_{\mu\nu\lambda} + \Gamma_{\mu\lambda\nu}$, which follows from the definitions $g_{\mu\nu} \equiv e^i{}_\mu e^i{}_\nu$ and $\Gamma_{\mu\nu}{}^\lambda \equiv e_i{}^\lambda\partial_\mu e^i{}_\nu$, to obtain

$$\delta^S\mathcal{A} = -M\int_{\sigma_1}^{\sigma_2} d\sigma\left[-g_{\mu\nu}\left(\ddot{q}^\nu + \bar{\Gamma}_{\lambda\kappa}{}^\nu\dot{q}^\lambda\dot{q}^\kappa\right)\bar{\delta}q^\mu\right.$$

$$\left.+\left(g_{\mu\nu}\dot{q}^\nu\frac{d}{d\sigma}\delta^S b^\mu + \Gamma_{\mu\lambda\kappa}\delta^S b^\mu\dot{q}^\lambda\dot{q}^\kappa\right)\right]. \quad (197)$$

To derive the equation of motion we first vary the action in a space without torsion. Then we have $\delta^S b^\mu(\sigma) \equiv 0$, and we obtain

$$\delta^S \mathcal{A} = \delta \mathcal{A} = M \int_{\sigma_1}^{\sigma_2} d\sigma \, g_{\mu\nu}(\ddot{q}^\nu + \bar{\Gamma}_{\lambda\kappa}{}^\nu \dot{q}^\lambda \dot{q}^\kappa) \delta q^\nu . \tag{198}$$

Thus, the action principle $\delta^S \mathcal{A} = 0$ produces the equation for the geodesics

$$\ddot{q}^\nu + \bar{\Gamma}_{\lambda\kappa}{}^\nu \dot{q}^\lambda \dot{q}^\kappa = 0 \, . \tag{199}$$

This describes the correct particle trajectories in the absence of torsion.

In the presence of torsion where $\delta^S b^\mu \neq 0$, the equation of motion receives a contribution from the second parentheses in (197). After inserting (195), the terms proportional to $\delta^S b^\mu$ cancel and the total nonholonomic variation of the action becomes

$$\begin{aligned} \delta^S \mathcal{A} &= M \int_{\sigma_1}^{\sigma_2} d\sigma g_{\mu\nu} \, [\ddot{q}^\nu + (\bar{\Gamma}_{\lambda\kappa}{}^\nu + 2S^\nu{}_{\lambda\kappa}) \dot{q}^\lambda \dot{q}^\kappa \,] \delta q^\mu \\ &= M \int_{\sigma_1}^{\sigma_2} d\sigma g_{\mu\nu} (\ddot{q}^\nu + \Gamma_{\lambda\kappa}{}^\nu \dot{q}^\lambda \dot{q}^\kappa) \delta q^\mu . \end{aligned} \tag{200}$$

The second line follows from the first after using the identity $\Gamma_{\lambda\kappa}{}^\nu = \bar{\Gamma}_{\{\lambda\kappa\}}{}^\nu + 2S^\nu{}_{\{\lambda\kappa\}}$. The curly brackets indicate the symmetrization of the enclosed indices. Setting $\delta^S \mathcal{A} = 0$ and using (190) gives the autoparallel equation of motion

$$\ddot{q}^\nu + \Gamma_{\lambda\kappa}{}^\nu \dot{q}^\lambda \dot{q}^\kappa = 0 \, . \tag{201}$$

Physically, autoparallel trajectories are a manifestation of inertia, which makes particles run along the straightest lines rather than the shortest ones. In the absence of torsion, the two types of curves happen to coincide from mathematical reasons. In the presence of torsion the autoparallel trajectory is more natural than the geodesic. It is hard to conceive, how a particle should know where to go to make the trajectory the shortest curve to a distant point. This seems to contradict our concepts of locality.

In order appreciate the geometric significance of the differential equation (195), we introduce the matrices

$$G^\mu{}_\lambda(\tau) \equiv \Gamma_{\lambda\nu}{}^\mu(q(\tau)) \dot{q}^\nu(\tau) \tag{202}$$

and

$$\Sigma^\mu{}_\nu(\tau) \equiv 2S_{\lambda\nu}{}^\mu(q(\tau)) \dot{q}^\lambda(\tau), \tag{203}$$

and rewrite eq. (195) as a differential equation for a vector

$$\frac{d}{d\tau}\,\delta^S b = -G\delta^S b + \Sigma(\tau)\,\bar{\delta} q^\nu(\tau). \tag{204}$$

The solution is

$$\delta^S b(\tau) = \int_{\tau_1}^{\tau} d\tau' U(\tau,\tau')\,\Sigma(\tau')\;\bar{\delta} q(\tau'), \tag{205}$$

with the matrix

$$U(\tau,\tau') = T\exp\left[-\int_{\tau'}^{\tau} d\tau'' G(\tau'')\right]. \tag{206}$$

In the absence of torsion, $\Sigma(\tau)$ vanishes identically and $\delta^S b(\tau) \equiv 0$, and the variations $\delta^S q^\mu(\tau)$ coincide with the holonomic $\bar{\delta} q^\mu(\tau)$ [see Fig. 4(b)]. In a space with torsion, the variations $\delta^S q^\mu(\tau)$ and $\bar{\delta} q^\mu(\tau)$ are different from each other [see Fig. 4(c)].

The above variational treatment of the action is somewhat complicated and calls for a simpler procedure [26]. The extra term arising from the second parenthesis in the variation (197) can traced to a simple property of the auxiliary nonholonomic variations (189). To find this we form the τ-derivative $d_\tau \equiv d/d\tau$ of the defining equation (189) and find

$$d_\tau \bar{\delta} q^\mu(\tau) = \partial_\nu e_a{}^\mu(q(\tau))\,\dot{q}^\nu(\tau)\delta x^a(\tau) + e_a{}^\mu(q(\tau)) d_\tau \delta x^a(\tau). \tag{207}$$

Let us now perform variation $\bar{\delta}$ and τ-derivative in the opposite order and calculate $d_\tau \bar{\delta} q^\mu(\tau)$. From (4) we have the relation

$$d_\tau q^\lambda(\tau) \;= e_a{}^\lambda(q(\tau))\, d_\tau x^a(\tau)\,. \tag{208}$$

Varying this gives

$$\bar{\delta} d_\tau q^\mu(\tau) = \partial_\nu e_a{}^\mu(q(\tau))\,\bar{\delta} q^\nu d_\tau x^a(\tau) + e_a{}^\mu(q(\tau))\bar{\delta} d_\tau x^a. \tag{209}$$

Since the variation in x^a-space commute with the τ-derivatives [recall (162)], we obtain

$$\begin{aligned}&\bar{\delta} d_\tau q^\mu(\tau) - d_\tau \bar{\delta} q^\mu(\tau)\\ &\qquad = \partial_\nu e_a{}^\mu(q(\tau))\,\bar{\delta} q^\nu d_\tau x^a(\tau) - \partial_\nu e_a{}^\mu(q(\tau))\,\dot{q}^\nu(\tau)\delta x^a(\tau).\end{aligned} \tag{210}$$

After re-expressing $\delta x^a(\tau)$ and $d_\tau x^a(\tau)$ back in terms of $\bar{\delta} q^\mu(\tau)$ and $d_\tau q^\mu(\tau) = \dot{q}^\mu(\tau)$, this becomes using (11)

$$\bar{\delta} d_\tau q^\mu(\tau) - d_\tau \bar{\delta} q^\mu(\tau) = 2S_{\nu\lambda}{}^\mu \dot{q}^\nu(\tau) \bar{\delta} q^\lambda(\tau). \tag{211}$$

Thus, due to the closure failure in spaces with torsion, the operations d_τ and $\bar{\delta}$ do not commute in front of the path $q^\mu(\tau)$, implying that in contrast to variations δ, the auxiliary nonholonomic variations $\bar{\delta}$ of velocities $\dot{q}^\mu(\tau)$ no longer coincide with the velocities of variations.

This property is responsible for shifting the trajectory from geodesics to autoparallels. Indeed, let us vary an action

$$\mathcal{A} = \int_{\tau_1}^{\tau_2} d\tau L(q^\mu(\tau), \dot{q}^\mu(\tau)) \tag{212}$$

directly by $\bar{\delta} q^\mu(\tau)$ and impose (211), we find

$$\bar{\delta}\mathcal{A} = \int_{\tau_1}^{\tau_2} d\tau \left\{ \frac{\partial L}{\partial q^\mu} \bar{\delta} q^\mu + \frac{\partial L}{\partial \dot{q}^\mu} \frac{d}{d\tau} \bar{\delta} q^\mu + 2\, S^\mu{}_{\nu\lambda} \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\nu \bar{\delta} q^\lambda \right\}. \tag{213}$$

After a partial integration of the second term using the vanishing $\bar{\delta} q^\mu(\tau)$ at the endpoints, we obtain the Euler–Lagrange equation

$$\frac{\partial L}{\partial q^\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^\mu} = -2S_{\mu\nu}{}^\lambda \dot{q}^\nu \frac{\partial L}{\partial \dot{q}^\lambda}\,. \tag{214}$$

This differs from the standard Euler–Lagrange equation by an additional contribution due to the torsion tensor. For the action (178) with the proper time σ as a path parameter, we thus obtain the equation of motion

$$M[\ddot{q}^\mu(\sigma) + g^{\mu\kappa}(\partial_\nu g_{\lambda\kappa} - \tfrac{1}{2}\, \partial_\kappa g_{\nu\lambda}) - 2S^\mu{}_{\nu\lambda}]\dot{q}^\nu(\sigma)\dot{q}^\lambda(\sigma) = 0, \tag{215}$$

which is once more eq. (201) for autoparallels.

7. COMPATIBILITY WITH CONSERVATION LAW OF ENERGY MOMENTUM TENSOR

An important consistency check for the correct equations of motion is based on their rederivation from the covariant conservation law for the energy momentum tensor which, in turn, is a general property of any theory which is invariant under arbitrary (single-valued) coordinate transformations (179),(180).

To derive this law, we express the reparametrization invariance once more in another way by studying the behavior of the relativistic action (177) under infinitesimal versions of the coordinate transformation (180), which we shall write as local translations

$$q^\mu \;\to\; q'^\mu(q) = q^\mu - \xi^\mu(q). \tag{216}$$

This looks like the previous infinitesimal transformations (96), but now we deal with ordinary coordinate transformation, where the transformation functions $-\xi^\mu(q)$ are single-valued and possess commuting derivatives. As a further difference, the initial space possesses curvature and torsion.

Inserting (216) into (179) and (180), we have

$$\begin{aligned} \alpha^\lambda{}_\nu &\approx \delta^\lambda \nu - \partial_\nu \xi^\lambda(q), \\ \alpha_\mu{}^\nu &\approx \delta_\mu{}^\nu + \partial_\mu \xi^\nu(q), \end{aligned} \tag{217}$$

and find from (183) and (185) the infinitesimal transformations of the multivalued tetrads $e_a^\mu(q)$:

$$e_a{}^\mu(q) \;\to\; e'_a{}^\mu(q) + \xi^\lambda \partial_\lambda e_a{}^\mu(q) - \partial_\lambda \xi^\mu e_a{}^\mu(q), \tag{218}$$

$$e^a{}_\mu(q) \;\to\; e'^a{}_\mu(q) + \xi^\lambda \partial_\lambda e^a{}_\mu(q) + \partial_\mu \xi^\lambda e^a{}_\lambda(q). \tag{219}$$

To save parentheses, differential operators are supposed to act only on the expression after it. Inserting (219) into (3), we obtain the corresponding transformation law for the metric tensor

$$g_{\mu\nu}(q) \;\to\; g'_{\mu\nu}(q) + \xi^\lambda \partial_\lambda g_{\mu\nu}(q) + \partial_\mu \xi^\lambda g_{\lambda\nu}(q) + \partial_\nu \xi^\lambda g_{\mu\lambda}(q). \tag{220}$$

For an arbitrary vector field $v_\mu(q)$, the transformation laws (185) become

$$\begin{aligned} v_\mu(q) &\;\to\; v'_\mu(q) + \xi^\lambda \partial_\lambda v_\mu(q) + \partial_\mu \xi^\lambda v_\lambda(q), \\ v^\mu(q) &\;\to\; v'^\mu(q) + \xi^\lambda \partial_\lambda v^\mu(q) - \partial_\lambda \xi^\mu v^\lambda(q). \end{aligned} \tag{221}$$

Recalling (29), the change of the metric can be rewritten as

$$\delta_E g_{\mu\nu}(q) = \bar{D}_\mu \xi_\nu(q) + \bar{D}_\nu \xi_\mu(q), \tag{222}$$

where $\bar{D}_\mu$ are covariant derivatives defined as in (10), but with the Riemann connection (29) instead of the affine connections:

$$\bar{D}_\mu v_\nu = \partial_\mu v_\nu - \bar{\Gamma}_{\mu\nu}{}^\lambda v_\lambda, \qquad \bar{D}_\mu v^\lambda = \partial_\mu v^\lambda + \bar{\Gamma}_{\mu\nu}{}^\lambda v^\nu. \tag{223}$$

The subscript of δ_E indicates that these are the general coordinate transformations introduced by Einstein. With this notation, the change of a vector field is

$$\begin{aligned}\delta_E v_\mu(q) &= \xi^\lambda \partial_\lambda v_\mu(q) + \partial_\mu \xi^\lambda v_\lambda(q),\\ \delta_E v^\mu(q) &= \xi^\lambda \partial_\lambda v^\mu(q) - \partial_\lambda \xi^\mu v^\lambda(q).\end{aligned} \tag{224}$$

Inserting for $v^\mu(q)$ the coordinate q^μ themselves, we see that

$$\delta_E q^\mu = -\xi^\mu(q), \tag{225}$$

which is the initial transformation (216) in this notation.

We now calculate the change of the action (177) under infinitesimal Einstein transformations:

$$\delta_E \mathcal{A} = \int d^4q \frac{\delta \mathcal{A}}{\delta g_{\mu\nu}(q)}\, \delta_E g_{\mu\nu}(q) + \int d\sigma \frac{\delta \mathcal{A}}{\delta q^\mu(\sigma)}\, \delta_E q^\mu(\sigma). \tag{226}$$

The functional derivative $\delta\mathcal{A}/\delta g_{\mu\nu}(q)$ is the general definition of the energy momentum tensor of a system:

$$\frac{\delta \mathcal{A}}{\delta g_{\mu\nu}(q)} \equiv -\frac{1}{2}\sqrt{-g}\, T^{\mu\nu}(q), \tag{227}$$

where g is the determinant of $g_{\mu\nu}$. For the spinless particle at hand, the energy momentum tensor becomes

$$T^{\mu\nu}(q) = \frac{1}{\sqrt{-g}}\, M \int d\sigma\, \dot{q}^\mu(\sigma) q^\nu(\sigma)\, \delta^{(4)}(q - q(\sigma)), \tag{228}$$

where σ is the proper time. This and the explicit variations (222) and (225), bring (226) to the form

$$\begin{aligned}\delta_E \mathcal{A} = &-\frac{1}{2}\int d^4q \sqrt{-g}\, T^{\mu\nu}(q)[\bar{D}_\mu \xi_\nu(q) + \bar{D}_\nu \xi_\mu(q)]\\ &- \int d\tau \frac{\delta \mathcal{A}}{\delta q^\mu(\tau)}\, \xi^\mu(q(\tau)).\end{aligned} \tag{229}$$

A partial integration of the derivatives yields (neglecting boundary terms at infinity)

$$\begin{aligned}\delta_E \mathcal{A} = &\int d^4q \left\{ \partial_\nu[\sqrt{-g}\, T^{\mu\nu}(q)] + \sqrt{-g}\, \bar{\Gamma}_{\nu\lambda}{}^{\mu}(q) T^{\lambda\nu}(q) \right\} \xi_\mu(q)\\ &- \int d\tau \frac{\delta \mathcal{A}}{\delta q^\mu(\tau)}\, \xi^\mu(\tau).\end{aligned} \tag{230}$$

Because of the manifest invariance of the action under general coordinate transformations, the left-hand side has to vanish for arbitrary infinitesimal functions $\xi^\mu(\tau)$. We therefore obtain

$$\{\partial_\nu[\sqrt{-g}\,T^{\mu\nu}(q)] + \sqrt{-g}\,\bar{\Gamma}_{\nu\lambda}{}^{\mu}T^{\lambda\nu}(q)\,\}\xi_\mu(q) \\ -\int d\tau \frac{\delta\mathcal{A}}{\delta q^\mu(\tau)}\,\delta^{(4)}(q-q(\tau))\xi^\mu(\tau) = 0\,. \tag{231}$$

To find the physical content of this equation we consider first a space without torsion. On a particle trajectory, the action is extremal, so that the second term vanishes, and we obtain the covariant conservation law

$$\partial_\nu[\sqrt{-g}\,T^{\mu\nu}(q)] + \sqrt{-g}\,\bar{\Gamma}_{\nu\lambda}{}^{\mu}(q)T^{\lambda\nu}(q) = 0\,. \tag{232}$$

Inserting (228), this becomes

$$M\int d\sigma[\,\dot{q}^\mu(\sigma)\dot{q}^\nu(\sigma)\partial_\nu\delta^{(4)}(q-q(\sigma)) \\ +\bar{\Gamma}_{\nu\lambda}{}^{\mu}(q)\dot{q}^\nu(\sigma)\dot{q}^\lambda(\sigma)\,\delta^{(4)}(q-q(\sigma))\,] = 0\,. \tag{233}$$

A partial integration turns this into

$$M\int d\sigma\,[-\ddot{q}^\mu(\sigma) + \bar{\Gamma}_{\nu\lambda}{}^{\mu}(q)\dot{q}^\nu(\sigma)\dot{q}^\lambda(\sigma)]\,\delta^{(4)}(q-q(\sigma)) = 0\,. \tag{234}$$

Integrating this over a small volume around any trajectory point $q^\mu(s)$, we obtain eq. (199) for the geodesic trajectory.

This technique was used by Hehl in his derivation of particle trajectories in the presence of torsion. Since torsion does not appear in the action, he found the trajectories to be geodesic.

The conservation law (232) can be written more covariantly as

$$\sqrt{-g}\,\bar{D}_\nu T^{\mu\nu}(q) = 0\,. \tag{235}$$

This follow directly from the identity

$$\frac{1}{\sqrt{-g}}\,\partial_\nu\sqrt{-g} = \frac{1}{2}\,g^{\lambda\kappa}\partial_\nu g_{\lambda\kappa} = \bar{\Gamma}_{\nu\lambda}{}^{\kappa}, \tag{236}$$

and is a consequence of the rule of partial integration applied to (229), according to which a covariant derivative can be treated in a volume integral $\int d^4\sqrt{-g}\,f(q)\bar{D}g(q)$, just like an ordinary derivative in an euclidean integral $\int d^4x f(x)\partial_a g(x)$ [see Appendix A]. After a partial integration, neglecting surface terms, eq. (229) goes over into

$$\delta_E \mathcal{A} = \frac{1}{2}\int d^4q\sqrt{-g}\,[\bar{D}_\nu T^{\mu\nu}(q)\xi_\nu(q) + (\mu\leftrightarrow\nu)\,] - \int d\tau \frac{\delta\mathcal{A}}{\delta q^\mu(\tau)}\,\xi^\mu(q(\tau)). \tag{237}$$

whose vanishing for all $\xi^\mu(q)$ yields directly (235).

Our theory does not lead to this conservation law. In the presence of torsion, the particle trajectory does not satisfy $\delta\mathcal{A}/\delta q^\mu(\tau) = 0$, but according to (214),

$$\frac{\delta\mathcal{A}}{\delta q^\mu(\tau)} = \frac{\partial L}{\partial q^{\,\mu}} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}^\mu} = 2S_{\mu\nu}{}^\lambda \dot{q}^\nu \frac{\partial L}{\partial \dot{q}^\lambda}\,. \tag{238}$$

For the Lagrangian in the action (177), parametrized with the the proper time σ, the right-hand side becomes

$$2S_{\mu\nu}{}^\lambda \dot{q}^\nu \frac{\partial L}{\partial \dot{q}^\lambda} = -M\,2S_{\mu\nu\lambda}\dot{q}^\nu(\sigma)\dot{q}^\lambda(\sigma). \tag{239}$$

Inserting this into (237), eq. (234) receives an extra term and becomes

$$M\int d\sigma\{-\ddot{q}^\mu(\sigma)+[\bar{\Gamma}_{\nu\lambda}{}^\mu(q)+2S^\mu{}_{\nu\lambda}(q)\,]\dot{q}^\nu(\sigma)\dot{q}^\lambda(\sigma)\}\delta^{(4)}(q-q(\sigma)) = 0, \tag{240}$$

yielding the correct autoparallel trajectories (201) for spinless point particles.

Observe that the extra term (239) can be expressed in terms of the energy momentum tensor (228) as

$$\sqrt{-g}\,2S^\mu{}_{\nu\lambda}T^{\lambda\nu}(q)\xi_\mu(q). \tag{241}$$

We may therefore rewrite the change of the action (229) as

$$\delta_E\mathcal{A} = -\frac{1}{2}\int d^4q\sqrt{-g}\,T^{\mu\nu}(q)\,[\bar{D}_\mu\xi_\nu(q) + \bar{D}_\nu\xi_\mu(q) - 4S^\lambda{}_{\mu\nu}\xi_\lambda(q)]. \tag{242}$$

The quantity in brackets will be denoted by $\bar{\delta}_E g_{\mu\nu}$, and is equal to

$$\bar{\delta}_E g_{\mu\nu} = D_\mu\xi_\nu(q) + D_\nu\xi_\mu(q), \tag{243}$$

where D_μ is the covariant derivative (10) involving the full affine connection. Thus we have

$$\delta_E \mathcal{A} = -\int d^4q \sqrt{-g}\, T^{\mu\nu}(q) D_\nu \xi_\mu(q). \tag{244}$$

Integrals over invariant expressions containing the covariant derivative D_μ can be integrated by parts according to a rule very similar to that for the Riemann covariant derivative $\bar{D}_\mu$, which is derived in Appendix A. After neglecting surface terms we find

$$\delta_E \mathcal{A} = \int d^4q \sqrt{-g}\, D^*_\nu T^{\mu\nu}(q) \xi_\mu(q), \tag{245}$$

where $D^*_\nu = D_\nu + 2S_{\nu\lambda}{}^{\lambda}$. Thus, due to the closure failure in spaces with torsion, the energy-momentum tensor of a free spinless point particles satisfies the conservation law

$$D^*_\nu T^{\mu\nu}(q) = 0\,. \tag{246}$$

This is to be contrasted with the conservation law (235). The difference between the two laws can best be seen by rewriting (235) as

$$D^*_\nu T^{\mu\nu}(q) + 2S_{\kappa}{}^{\mu}{}_{\lambda}(q) T^{\kappa\lambda}(q) = 0\,. \tag{247}$$

This is the form in which the conservation law has usually been stated in the literature [1–3,5,6]. When written in the form (235) it is obvious that (247) is satisfied only by geodesic trajectories, in contrast to (246) which is satisfied by autoparallels.

The variation $\delta_E g_{\mu\nu}(q)$ plays a similar role in deriving the new conservation law (247) as the nonholonomic variation $\delta q(s)$ of eq. (187) does in deriving equations of motion for point particles. Indeed, we may rewrite the transformation (226) formally as

$$\delta_E \mathcal{A} = \int d^4q \frac{\delta \mathcal{A}}{\delta g_{\mu\nu}(q)}\, \delta_E g_{\mu\nu}(q) + \int d\tau \frac{\delta \mathcal{A}}{\delta q^\mu(\tau)}\, \delta_E q^\mu(\tau). \tag{248}$$

Now the last term vanishes according to the new action principle $\delta \mathcal{A} = 0$ from which we derived the autoparallel trajectory (215) by setting (213) equal to zero.

The question arises whether the new conservation law (246) allows for the construction of an extension of Einstein's field equation

$$\bar{G}^{\mu\nu} = \kappa T^{\mu\nu} \tag{249}$$

to spaces with torsion, where $\bar{G}^{\mu\nu}$ is the Einstein tensor formed from the Ricci tensor $\bar{R}_{\mu\nu} \equiv \bar{R}_{\lambda\mu\nu}{}^{\lambda}$ in Riemannian spacetime [$\bar{R}_{\mu\nu\lambda}{}^{\kappa}$ being the same covariant curl of $\bar{\Gamma}_{\mu\nu}{}^{\lambda}$ as $R_{\mu\nu\lambda}{}^{\kappa}$ is of $\Gamma_{\mu\nu}{}^{\lambda}$ in eq. (44)]. The standard extension of (249) to spacetimes with torsion replaces the left-hand side by the Einstein–Cartan tensor $G^{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_{\sigma}{}^{\sigma}$ and becomes

$$G^{\mu\nu} = \kappa T^{\mu\nu}. \tag{250}$$

The Einstein–Cartan tensor $G^{\mu\nu}$ satisfies a Bianchi identity

$$D^*_{\nu}G_{\mu}{}^{\nu} + 2S_{\lambda\mu}{}^{\kappa}G_{\kappa}{}^{\lambda} - \tfrac{1}{2}S^{\lambda}{}_{\kappa}{}^{;\nu}R_{\mu\nu\lambda}{}^{\kappa} = 0, \tag{251}$$

where $S^{\lambda}{}_{\kappa}{}^{;\nu}$ is the Palatini tensor defined by

$$S_{\lambda\kappa}{}^{;\nu} \equiv 2(S_{\lambda\kappa}{}^{\nu} + \delta_{\lambda}{}^{\nu}S_{\kappa\sigma}{}^{\sigma} - \delta_{\kappa}{}^{\nu}S_{\lambda\sigma}{}^{\sigma}). \tag{252}$$

It is then concluded that the energy momentum tensor satisfies the conservation law

$$D^*_{\nu}T_{\mu}{}^{\nu} + 2S_{\lambda\mu}{}^{\kappa}T_{\kappa}{}^{\lambda} - \frac{1}{2\kappa}\,S^{\lambda}{}_{\kappa}{}^{;\nu}R_{\mu\nu\lambda}{}^{\kappa} = 0\,. \tag{253}$$

For standard field theories of matter, this is indeed true if the Palatini tensor satisfies the second Einstein–Cartan field equation

$$S^{\lambda\kappa;\nu} = \kappa\Sigma^{\lambda\kappa;\nu}, \tag{254}$$

where $\Sigma^{\lambda\kappa;\nu}$ is the canonical spin density of the matter fields. A spinless point particle contributes only to the first two terms in (253), in accordance with (247).

What tensor will stand on the left-hand side of the field equation (250) if the energy momentum tensor satisfies the conservation law (246) instead of (247)? At present, we can give an answer [27] only for the case of a pure gradient torsion which has the general form [4]

$$S_{\mu\nu}{}^{\lambda} = \tfrac{1}{2}[\delta_{\mu}{}^{\lambda}\partial_{\nu}\sigma - \delta_{\nu}{}^{\lambda}\partial_{\mu}\sigma]. \tag{255}$$

Then we may simply replace (250) by

$$e^{\sigma}G^{\mu\nu} = \kappa T^{\mu\nu}. \tag{256}$$

Note that for gradient torsion, $G^{\mu\nu}$ is symmetric as can be deduced from the fundamental identity (which expresses merely the fact that the Einstein–Cartan tensor $R_{\mu\nu\lambda}{}^{\kappa}$ is the covariant curl of the affine connection)

$$D^*{}_{\lambda}S_{\mu\nu}{}^{;\lambda} = G_{\mu\nu} - G_{\nu\mu}\,. \tag{257}$$

Indeed, inserting (255) into (252), we find the Palatini tensor

$$S_{\lambda\mu}{}^{;\kappa} \equiv -2[\delta_\lambda{}^\kappa \partial_\mu \sigma - (\lambda \leftrightarrow \mu)]. \tag{258}$$

This has a vanishing covariant derivative

$$\begin{aligned} D^*{}_\lambda S_{\mu\nu}{}^{;\lambda} &= -2[D^*{}_\mu \partial_\nu \sigma - D^*{}_\nu \partial_\mu \sigma] \\ &= 2[S_{\mu\nu}{}^\lambda \partial_\lambda \sigma - 2S_{\mu\lambda}{}^\lambda \partial_\nu \sigma + 2S_{\nu\lambda}{}^\lambda \partial_\mu \sigma], \end{aligned} \tag{259}$$

since the terms on the right-hand side cancel after using (255) and $S_{\mu\lambda}{}^\lambda \equiv S_\mu = -\frac{3}{2}\partial_\mu \sigma$. Now we insert (255) into the Bianchi identity (251), with the result

$$\bar{D}^*_\nu G_\lambda{}^\nu + \partial_\lambda \sigma G_\kappa{}^\kappa - \partial_\nu \sigma G_\lambda{}^\nu + 2\partial_\nu \sigma R_\lambda{}^\nu = 0\,. \tag{260}$$

Inserting here $R_{\lambda\kappa} = G_{\lambda\kappa} - \frac{1}{2} g_{\lambda\kappa} G_\nu{}^\nu$, this becomes

$$D^*_\nu G_\lambda{}^\nu + \partial_\nu \sigma G_\lambda{}^\nu = 0\,. \tag{261}$$

Thus we find the Bianchi identity

$$D^*_\nu (e^\sigma G_\lambda{}^\nu) = 0\,. \tag{262}$$

This makes the left-hand side of the new field equation (256) compatible with the covariant new conservation law (246), just as in Einstein's theory.

The field equation for the σ-field is still unknown.

8. GAUGE FIELD REPRESENTATION OF PARTICLE ORBITS

In Section 3 we have given two examples for the use of multivalued fields in describing magnetic phenomena. Up to now, we have only transferred the second example in subsection 3.2 to geometry by generating nontrivial gauge fields from multivalued gauge transformations.

The exists an equally important geometric version also for the mathematical structure in the first example in 3.1, the gradient representation of the magnetic field, as we shall elaborate in this section.

8.1. Current loop with magnetic forces

To prepare the grounds for this we pose ourselves the problem of calculating the magnetic energy of current loop from the gradient representation of the magnetic field. Since this will provide us with an example for the construction of field actions, we shall consider the energy as a euclidean action and denote it by $\mathcal{A}$. In this sense, the magnetic "action" reads

$$\mathcal{A} = \frac{1}{2}\int d^3x\, \mathbf{B}^2(\mathbf{x}). \tag{263}$$

Remembering the gradient representation (60) of the magnetic field, this becomes

$$\mathcal{A} = \frac{I^2}{2(4\pi)^2}\int d^3x\, [\boldsymbol{\nabla}\Omega(\mathbf{x})]^2. \tag{264}$$

This holds for the multivalued solid angle $\Omega(\mathbf{x})$. In order to perform field theoretic calculations, we go over to the single-valued representation used in eqs. (61) and (62). Recalling (66), the action becomes

$$\mathcal{A} = \frac{I^2}{2(4\pi)^2}\int d^3x\, [\boldsymbol{\nabla}\Omega(\mathbf{x}) - 4\pi\boldsymbol{\delta}(\mathbf{x};S)]^2, \tag{265}$$

where we have expressed the integral over the magnetic dipole surface in (66) with the help of the δ-function on the surface S:

$$\boldsymbol{\delta}(\mathbf{x};S) \equiv \int_S d\mathbf{S}'\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{266}$$

The δ-function is essential in removing the unphysical field energy on the artificial magnetic dipole layer on S which is only serves to make the solid angle single-valued. Its unphysical nature can be exhibited in the action (265) as follows: Suppose we move the surface S to a new location S', while keeping its boundary anchored on the current loop L. Under this move, the δ-function on the surface changes as follows (see Refs. 22,23):

$$\boldsymbol{\delta}(\mathbf{x};S) \;\rightarrow\; \boldsymbol{\delta}(\mathbf{x};S') = \boldsymbol{\delta}(\mathbf{x};S) + \boldsymbol{\nabla}\delta(\mathbf{x};V). \tag{267}$$

Here

$$\delta(\mathbf{x};V) = \int d^3x'\delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{268}$$

is the δ-function on the volume V over which S has swept in moving to S'. Thus the δ-function on the surface S is a *gauge field of the current loop*, and (267) is a gauge transformation which leaves the boundary of L unchanged. The action (265) is also invariant, since the gradient of the δ-function in (267) can be absorbed into $\Omega(\mathbf{x})$:

$$\Omega(\mathbf{x}) \;\rightarrow\; \Omega'(\mathbf{x}) = \Omega(\mathbf{x}) + 4\pi\delta(\mathbf{x};V). \tag{269}$$

The gauge invariance makes the field energy independent of the position of the artificial magnetic dipole layer for a current flowing along the fixed loop L. This gauge invariance has its root in the fact that Ω is defined only up to integer multiples of 4π — it is a *cyclic* field.

We are now ready to calculate the magnetic field energy of the current loop. For this we rewrite the action (265) in terms of an *auxiliary* vector field $\mathbf{B}(\mathbf{x})$ as

$$\mathcal{A} = \int d^3x \left\{ -\frac{1}{2}\,\mathbf{B}^2(\mathbf{x}) + \mathbf{B}(\mathbf{x}) \cdot [\boldsymbol{\nabla}\Omega(\mathbf{x})/4\pi - I\boldsymbol{\delta}(\mathbf{x};S)] \right\}, \tag{270}$$

A partial integration brings the middle term to

$$-\int d^3x [\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{x})]\Omega(\mathbf{x})/4\pi.$$

Extremizing this in $\Omega(\mathbf{x})$ yields the equation

$$\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{x}) = 0, \tag{271}$$

implying that the field lines of $\mathbf{B}(\mathbf{x})$ form closed loops. This equation may be enforced identically (as a Bianchi identity) by expressing $\mathbf{B}(\mathbf{x})$ as a curl of an auxiliary vector potential $\mathbf{A}(\mathbf{x})$, setting

$$\mathbf{B}(\mathbf{x}) \equiv \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x}). \tag{272}$$

With this ansatz, the equation which brings the action (270) to the form

$$\mathcal{A} = \int d^3x \left\{ -\frac{1}{2}\,[\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})]^2 - [\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})] \cdot I\boldsymbol{\delta}(\mathbf{x};S) \right\}. \tag{273}$$

A further partial integration leads to

$$\mathcal{A} = \int d^3x \left\{ -\frac{1}{2}\,[\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x})]^2 - \mathbf{A}(\mathbf{x}) \cdot I[\boldsymbol{\nabla} \times \boldsymbol{\delta}(\mathbf{x};S) \right\}, \tag{274}$$

and we identify in the linear term in $\mathbf{A}(\mathbf{x})$ the *auxiliary current*

$$\mathbf{j}(\mathbf{x}) \equiv I\,\boldsymbol{\nabla} \times \boldsymbol{\delta}(\mathbf{x};S). \tag{275}$$

This current is conserved for closed loops L. This follows from the property of the δ-function on an arbitrary line L connecting the points $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\delta}(\mathbf{x};L) = \delta(\mathbf{x}_2) - \delta(\mathbf{x}_1). \tag{276}$$

For closed loops, the right-hand side vanishes.

We now observe that Stokes' theorem can be rewritten as an identity for δ-functions,

$$\boldsymbol{\nabla} \times \boldsymbol{\delta}(\mathbf{x}; S) = \boldsymbol{\delta}(\mathbf{x}; L). \tag{277}$$

This shows that the auxiliary current (275) is equal to (57). The field equation following from the action (273) is Ampère's law (51). Thus the auxiliary quantities $\mathbf{B}(\mathbf{x})$, $\mathbf{A}(\mathbf{x})$, and $\mathbf{j}(\mathbf{x})$ coincide with the usual magnetic quantities with the same name.

By inserting the explicit solution (55) of Ampère's law into the energy, we obtain the *Biot–Savart* energy for an arbitrary current distribution

$$\mathcal{A} = \frac{1}{4\pi} \int d^3x \, d^3x' \, \mathbf{j}(\mathbf{x}) \frac{1}{|\mathbf{x} - \mathbf{x}'|} \mathbf{j}(\mathbf{x}'). \tag{278}$$

The relations (275) implies that the δ-function on the surface S is a gauge field whose curl produces a unit current loop. Thus the action (273) is invariant under two mutually dual gauge transformations, the usual magnetic one (52) by which the vector potential receives a gradient of an arbitrary scalar field, and the transformation gauge transformation (268), by which the irrelevant surface S is moved to another configuration S'.

Thus we have proved the complete equivalence of the gradient representation of the magnetic field to the usual gauge field representation. In the gradient representation, there exists a new type of gauge invariance which expresses the physical irrelevance of the jumping surface appearing when using single-valued solid angles.

The action (274) describes magnetism in terms of a *double gauge theory*, in which the gauge of $\mathbf{A}(\mathbf{x})$ and the shape of S can be changes arbitrarily.

8.2. Particle world lines with gravitational forces

It is possible to transfer the entire double-gauge structure to geometry. In this way we can derive a theory in which not only the gravitational forces are represented by a metric affine geometry, but also the particle orbits. The latter can be re-expressed in terms of particle world lines, more specifically, the Einstein tensor of the second gauge structure becomes the energy momentum tensor of the particle world line. It is the analog of the auxiliary current (275). The conservation law (276) which is satisfied automatically by the Einstein tensor turns into the conservation law of the energy-momentum tensor for the world lines.

We shall present such a construction only for a system without torsion. For simplicity, we assume the world as a crystal in four Riemannian spacetime dimensions. If the crystal is distorted by a displacement field

$$q^\mu \;\to\; q'^\mu = q^\mu + u^\mu(q), \tag{279}$$

it has a strain energy

$$\mathcal{A} = \frac{M}{4}\int d^4q\,\sqrt{-g}\,(\bar{D}_\mu u_\nu + \bar{D}_\nu u_\mu)^2, \tag{280}$$

where M is some elastic modulus. If part of the distortions are of the plastic type, the world crystal contains defects defined by Volterra surfaces, where crystalline layers or sections have been cut out. The displacement field is multivalued, and the action (280) is the analog of the magnetic action (264) in the presence of a current loop. In order to do field theory with this action, we have to make the displacement field single-valued with the help of δ-functions describing the jumps across the Volterra surfaces, in complete analogy with the magnetic energy (265):

$$\mathcal{A} = M\int d^4x\sqrt{-g}\,(u_{\mu\nu} - u^P_{\mu\nu})^2, \tag{281}$$

where $u_{\mu\nu} = (\bar{D}_\mu u_\nu + \bar{D}_\nu u_\mu)/2$ is the elastic strain tensor and $u^P_{\mu\nu}$ the gauge field of plastic deformations describing the Volterra surfaces via δ-functions on these surfaces [6]. The energy density is invariant under the single-valued *defect gauge transformations* [the analogs of (267)]

$$u_{\mu\nu}{}^P \;\to\; u_{\mu\nu}{}^P + (\bar{D}_\mu\lambda_\nu + \bar{D}_\nu\lambda_\mu)/2, \qquad u_\mu \;\to\; u_\mu + \lambda_\mu\,. \tag{282}$$

Physically, they express the fact that defects are not affected by elastic distortions of the crystal. Only multivalued gauge functions λ_μ would change the defect content in $u^P_{\mu\nu}$.

We now introduce an auxiliary symmetric tensor field $G_{\mu\nu}$ and rewrite the action (281) in a first-order form [the analog of (270)] as

$$\mathcal{A} = \int d^3q\sqrt{g}\left[\frac{1}{4\mu}\,G_{\mu\nu}G^{\mu\nu} + iG^{\mu\nu}(u_{\mu\nu} - u^P_{\mu\nu})\right]. \tag{283}$$

After a partial integration and extremization in u_μ, the middle terms yield the equation

$$\bar{D}_\nu G^{\mu\nu} = 0\,. \tag{284}$$

This may be guaranteed identically, as a Bianchi identity, by an ansatz

$$G^{\nu\mu} = e^{\nu\kappa\lambda\sigma}e^{\mu\kappa\lambda'\sigma'}\,\bar{D}_\lambda\bar{D}_{\lambda'}\chi_{\sigma\sigma'}\,. \tag{285}$$

The field $\chi_{\sigma\sigma'}$ plays the role of an elastic gauge field. Inserting this into (283) we obtain the analog of (273):

$$\mathcal{A} = \int d^4 q \sqrt{-g} \left\{ \frac{1}{4M} \left[e^{\nu\kappa\lambda\sigma} e^{\mu\kappa\lambda'\sigma'} \bar{D}_\lambda \bar{D}_{\lambda'} \chi_{\sigma\sigma'} \right]^2 \right. \\ \left. + i e^{\nu\kappa\lambda\sigma} e^{\mu\kappa\lambda'\sigma'} \bar{D}_\lambda \bar{D}_{\lambda'} \chi_{\sigma\sigma'} u^P_{\mu\nu} \right\}. \tag{286}$$

A further partial integration brings this to the form

$$\mathcal{A} = \int d^4 q \sqrt{-g} \left\{ \frac{1}{4M} G_{\mu\nu} G^{\mu\nu} + i \chi_{\mu\nu} T^{\mu\nu} \right\}, \tag{287}$$

where $T_{\mu\nu}$ is the defect density defined in analogy to η_{ij} of eq. (130):

$$T^{\mu\nu} = e^{\nu\kappa\lambda\sigma} e^{\mu\kappa\lambda'\sigma'} \bar{D}_\lambda \bar{D}_{\lambda'} u^P_{\sigma\sigma'} \,. \tag{288}$$

It is invariant under defect gauge transformations (282), and satisfies the conservation law

$$\bar{D}_\nu T^{\mu\nu} = 0 \,. \tag{289}$$

Although we have written (288) and (289) covariantly, they are only applicable in their linearized approximations to infinitesimal defects, as emphasized in the discussion after eq. (155).

By identifying $\chi_{\mu\nu}$ with half an elastic metric field $g_{\mu\nu}$ [generalizing the linearized expression in terms of the strain field in eq. (127), where the metric is $g_{\mu\nu} = \delta_{\mu\nu} + 2\xi_{\mu\nu}$], the tensor $G_{\mu\nu}$ is recognized as the Einstein tensor associated with the elastic metric tensor $g_{\mu\nu}$. The defect density $T_{\mu\nu}$ is formed in the same way from the plastic strain $u^P_{\mu\nu}$.

For small deviations $\chi'_{\mu\nu}$ of $\chi_{\mu\nu}$ from the flat space limit $\eta_{\mu\nu}/2$, we can linearize $G_{\mu\nu}$ in $\chi'_{\mu\nu}$ and find

$$\begin{aligned} \bar{G}_{\mu\nu} &\approx \epsilon^{\nu\kappa\lambda\sigma} \epsilon^{\mu\kappa\lambda'\sigma'} \partial_\lambda \partial_{\lambda'} \chi'_{\sigma\sigma'} \\ &= -(\partial^2 \chi'_{\mu\nu} + \partial_\mu \partial_\nu \chi'_\lambda{}^\lambda - \partial_\mu \partial_\lambda \chi'_\mu{}^\lambda - \partial_\nu \partial_\lambda \chi'_\mu{}^\lambda) \\ &\quad + \eta_{\mu\nu} (\partial^2 \chi'_\lambda{}^\lambda - \partial_\lambda \partial_\kappa \chi'^{\lambda\kappa}). \end{aligned} \tag{290}$$

Introducing the field $\phi_{\mu\nu} \equiv \chi'_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \chi'_\lambda{}^\lambda$, and going to the Hilbert gauge $\partial_\mu \phi^{\mu\nu} = 0$, the Einstein tensor reduces to

$$\bar{G}_{\mu\nu} = -\partial^2 \phi_{\mu\nu} \,, \tag{291}$$

and the interaction energy of an arbitrary distribution of defects [the analog of (278)]

$$\mathcal{A} \approx M \int d^4q d^4q' \, T_{\mu\nu}(q) \, \Delta(q-q') \, T_{\mu\nu}(q'), \tag{292}$$

where

$$\Delta(q) = \int \frac{d^3p}{(2\pi)^4} \frac{e^{ipq}}{(p^2)^2} \tag{293}$$

is the Green function of the differential operator $(\partial^2)^2$.

The interaction (292) gives the elastic energy of matter in the world crystal. The defect density $T_{\mu\nu}(q)$ plays a similar role as the energy-momentum tensor $\overset{m}{T}_{\mu\nu}(q)$ of matter in gravity. Indeed, it satisfies the same conservation law $(A.3)$. The interaction does not, however, coincide with the gravitational energy for which the Green function should be that of the Laplacian ∂^2 rather than $(\partial^2)^2$ to yield Newton's gravitational potential $\propto r^{-1}$ [as in the magnetic Biot–Savart energy (278)].

There is no problem in modifying our world crystal to achieve this. We merely have to replace the action (287) by

$$\mathcal{A} = \int d^4q \sqrt{-g} \left[-\frac{1}{2\kappa} \bar{R} - \frac{1}{2} g_{\mu\nu} T^{\mu\nu} \right], \tag{294}$$

where κ is the gravitational constant. Indeed, the Einstein action in the first term has the linear approximation

$$\frac{1}{4\kappa} \int d^4q \, g_{\mu\nu} G^{\mu\nu} \approx \frac{1}{2\kappa} \int d^4q \, \phi^{\mu\nu}(-\partial^2)\phi_{\mu\nu} \tag{295}$$

which leads to the field equation

$$-\partial^2 \phi^{\mu\nu} = \kappa T^{\mu\nu}, \tag{296}$$

and thus to the correct gravitational interaction energy.

It is easy to verify that the energy (294) is invariant under defect gauge transformations (282), just as the elastic action (286).

A similar construction exists for a full nonlinear Einstein–Cartan theory of gravity (Ref. 28, and Ref. 6, p. 1448–1456).

8.3. Field representation for ensembles of particle world lines

To end this section let us mention that a grand-canonical ensemble of world lines can be transformed into a quantum field theory [22]. In this way, we convert the double gauge theory into a field theory with a single gauge field. This construction may eventually be helpful in finding the correct theory of gravitation with torsion.

9. EMBEDDING

Readers who feel uneasy in dealing with the unfamiliar multivalued tetrads $e^a{}_\mu(q)$ in (4) may be convinced that autoparallels are the correct particle trajectories of spinless point particles in another way: by the special geometric role of autoparallels in a Riemann–Cartan space generated by embedding. It is well known that a D-dimensional space with curvature can be produced by embedding it into a flat space of a sufficiently large dimension $\bar{D} > D$ via some functions $x^A(q)$ $(A = 1, \ldots, \bar{D})$. The metric η_{AB} in this flat space is pseudo-Minkowskian, containing only diagonal elements ± 1. The mapping $x^A(q)$ is smooth, but cannot be inverted to $q^\mu(x)$. Let $\mathbf{E}_A$ be the $\bar{D}$ fixed basis vectors in the embedding space, then the functions $x^A(q)$ define D local tangent vectors to the submanifold:

$$\mathbf{E}_\lambda(q) = \mathbf{E}_A E^A{}_\lambda(q), \qquad E^A{}_\lambda(q) \equiv \frac{\partial x^A(q)}{\partial q^\lambda} . \tag{297}$$

They induce a metric

$$g_{\lambda\kappa}(q) = E^A{}_\lambda(q) E^B{}_\kappa(q) \eta_{AB} , \tag{298}$$

which can be used to define the reciprocals

$$E^{A\lambda}(q) = g^{\lambda\kappa}(q) E^A{}_\kappa(q). \tag{299}$$

Note that in contrast to our multivalued tetrads in (9), the tangent vectors satisfy only the orthogonality relation

$$E^{A\mu}(q) E_{A\nu}(q) = \delta^\mu{}_\nu , \tag{300}$$

but not the completeness relation

$$E^{A\lambda}(q) E_{B\lambda}(q) \neq \delta^A{}_B , \tag{301}$$

the latter being obvious since the sum over $\lambda = 1, \ldots, D < \bar{D}$ is too small to span a $\bar{D}$-dimensional space. The embedding induces an affine connection in q-space

$$\Gamma_{\mu\nu}{}^\lambda(q) \equiv E_A{}^\lambda(q) \partial_\mu E^A{}_\nu(q) = -E^A{}_\nu(q)\, \partial_\mu\, E_A{}^\lambda(q). \tag{302}$$

Since $E^A{}_\lambda(q) \equiv \partial x^A(q)/\partial q^\lambda$ are derivatives of single-valued embedding functions $x^A(q)$, they satisfy a Schwarz integrability condition [in contrast to (5)]:

$$\partial_\mu\, E^a{}_\lambda(q) - \partial_\lambda\, E^A{}_\mu(q) = 0 . \tag{303}$$

The torsion (14) is therefore necessarily zero.

Because of their single-valuedness, derivatices commute in front of the tangent vectors $E^A{}_\lambda(q)$, so there exists no formula of the type (44) to calculate the curvature:

$$R_{\mu\nu\lambda}{}^\kappa(q) \neq E_A{}^\kappa(q)(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)E^A{}_\lambda(q) = 0\,. \tag{304}$$

In order to derive the curvature tensor (45) from (44), we needed the property

$$\partial_\mu e^a{}_\nu = \Gamma_{\mu\nu}{}^\lambda e^a{}_\lambda\,, \tag{305}$$

which was deduced from (11) using the completeness relation $e^a{}_\mu(q)e_b{}^\mu(q) = \delta^a{}_b$. Since such a relation does not exist now [see (301)], we have

$$\partial_\mu E^A{}_\nu(q) \neq \Gamma_{\mu\nu}{}^\lambda(q)E^A{}_\lambda(q), \tag{306}$$

and a formula of the type (304) cannot be used to find $R_{\mu\nu\lambda}{}^\kappa(q)$.

It is possible to introduce torsion in the embedded q-space [31] by allowing the tangent vectors to disobey the Schwarz integrability condition (7). In contrast to the multivalued tetrads $e^a{}_\mu(q)$, however, the functions $E^A{}_\mu(q)$ possess commuting derivatives. This brings them in spirit close to the ordinary tetrads or vierbein fields $h^\alpha{}_\mu(q)$, except that there are more of them. For nonintegrable functions $E^A{}_\mu(q)$, the embedding is not defined pointwise but only differentially:

$$dx^A = E^A{}_\mu(q)dq^\mu. \tag{307}$$

For any curve $x^A(\tau)$, we can find a curve in q-space which is defined up to a free choice of the initial point:

$$\dot q^\mu(\tau) = \dot q^\mu(\tau_1) + \int_{\tau_1}^{\tau} d\tau'\, E_A{}^\mu(q(\tau'))dx^A(\tau'). \tag{308}$$

In contrast to (186), the integrand does not require an analytic continuation through cuts.

A straight line in the embedding x-space has a constant velocity $v^A(s) = \dot x^A(s)$. Its image in the embedded space via the mapping (307) satisfies

$$E^A{}_\nu(q)\ddot q^\nu(s) + \dot E^A{}_\nu(q(s))\dot q^\nu(s) = 0\,. \tag{309}$$

Multiplying this equation by $E_A{}^\mu(q)$ and using the orthogonality relation (300) as well as the defining equation (302), we find eq. (201), so that the straight line goes over into an autoparallel trajectories. Geodesic trajectories, on the other hand, correspond to complicated curves in x^A-space under this mapping, a fact which makes them once more unappealing candidates for physical trajectories of spinless point particles, apart from the inertia argument given after eq. (201).

10. COULOMB SYSTEM AS AN OSCILLATOR IN A SPACE WITH TORSION

As an application of the new action principle with the ensuing autoparallel trajectories, consider the famous Kustaanheimo–Stiefel transformation in celestial mechanics [30,10]. For a spinless point particle orbiting around a central mass in a three-dimensional space, the Lagrangian reads

$$L(x,\dot{x}) = \frac{M}{2}\,\dot{\mathbf{x}}^2 + \frac{\alpha}{r}\,, \qquad r = |\mathbf{x}|, \tag{310}$$

where ($\alpha =$ const), yielding upon variation the equation of motion

$$M\ddot{\mathbf{x}} + \alpha\mathbf{x}/r^3 = 0\,. \tag{311}$$

Let us perform the Kustaanheimo–Stiefel transformation in two steps: First we map the $\mathbf{x}$-space into a four-dimensional $\vec{u}$-space by setting

$$\begin{aligned} x^1 &= 2(u^1u^3 + u^2u^4),\\ x^2 &= 2(u^1u^4 - u^2u^3),\\ x^3 &= (u^1)^2 + (u^2)^2 - (u^3)^2 - (u^4)^2 \end{aligned} \qquad \text{for} \qquad \vec{u} = \begin{pmatrix} u^1\\ u^2\\ u^3\\ u^4 \end{pmatrix}. \tag{312}$$

Every point $\mathbf{x}$ has infinitely many image points $\vec{u}$. We restrict this freedom by extending $\mathbf{x}$-space to four dimensions with the help of an artificial forth coordinate x^4, which we map nonholonomically into $\vec{u}$-space by an equation

$$dx^4 = u^2du^1 - u^1du^2 + u^4du^3 - u^3du^4. \tag{313}$$

The combined anholonomic coordinate transformation reads $dx^a = e^i{}_\mu(\vec{u})du^\mu$, $a,\mu = 1,2,3,4$, with the matrix:

$$e^a{}_\mu(\vec{u}) = \begin{pmatrix} u^3 & u^4 & u^1 & u^2\\ u^4 & -u^3 & -u^2 & u^1\\ u^1 & u^2 & -u^3 & -u^4\\ u^2 & -u^1 & u^4 & -u^3 \end{pmatrix}. \tag{314}$$

The metric induced in $\vec{u}$ space is

$$g_{\mu\nu}(\vec{u}) = \vec{u}^2\delta_{\mu\nu}\,. \tag{315}$$

It is easy to check that derivatives in front of $x^4(\vec{u})$ do not commute:

$$(\partial_{u_1}\partial_{u_2} - \partial_{u_2}\partial_{u_1})x^4 = 2, \qquad (\partial_{u_3}\partial_{u_4} - \partial_{u_4}\partial_{u_3})x^4 = 2, \tag{316}$$

implying the multivaluedness of $x^4(\vec{u})$ and the presence of torsion, whose nonzero components are

$$S_{1\mu 2} = S_{3\mu 4} = 4(u^2, -u^1, u^4, -u^3). \tag{317}$$

The fourth, anholonomic coordinate x^4 is assumed to have a trivial dynamics. By adding only a kinetic term to the original Lagrangian, the new one is defined by

$$L'(\vec{x}, \dot{\vec{x}}) = \frac{M}{2}\,\dot{\vec{x}}^{\,2} + \frac{\alpha}{r}\,, \qquad r = |\mathbf{x}|. \tag{318}$$

By extremizing this we obtain the correct three-dimensional orbits by imposing the constraint $x^4 = \text{const}$. This system is now mapped to $\vec{u}$-space using $dx^a = e^i{}_\mu(\vec{u})du^\mu$, and we obtain the transformed Lagrangian

$$L(\vec{u}, \dot{\vec{u}}) = 2M\vec{u}^{\,2}\dot{\vec{u}}^{\,2} + \frac{\alpha}{\vec{u}^{\,2}}\,. \tag{319}$$

An arbitrary orbit in he four-dimensional x^4-space can now be found by extremizing this action using our modified action principle. This yields the equation

$$\frac{\delta\mathcal{A}}{\delta u^\mu} + \mathcal{F}_\mu = 0 \tag{320}$$

where $\mathcal{F}_\mu$ is the torsion force

$$\mathcal{F}_\mu = M\dot{x}^4\dot{S}_{1\mu 2}\,. \tag{321}$$

This is the equation for an autoparallel in $\vec{u}$-space, which maps correctly back into the equation of motion following from the four-dimensional Lagrangian (318).

When the solutions are restricted by the constraint $\dot{x}^4 = 0$, the torsion force disappears, so that it does not influence the classical orbit at the end. However, fluctuations make $\dot{x}^4$ nonzero so that the solution of the associated quantum system [10] which describes a hydrogen atom via a path integral is sensitive to this force.

11. QUANTUM MECHANICS

As mentioned in the Introduction, the nonholonomic mapping principle was discovered when trying to solve the quantum-mechanical mechanical problem of finding the correct integration measure for the path integral of the hydrogen atom. Let us briefly sketch the result.

In flat space, quantum mechanics may be defined via path integrals as products of ordinary integrals over Cartesian coordinates on a grated time axis:

$$(\mathbf{x}\,t|\mathbf{x}'t') = \frac{1}{\sqrt{2\pi i\epsilon\hbar/M}^D}\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} d\Delta x_n\right]\prod_{n=1}^{N+1} K_0^\epsilon(\Delta\mathbf{x}_n), \tag{322}$$

where $K_0^\epsilon(\Delta\mathbf{x}_n)$ is an abbreviation for the short-time amplitude

$$\begin{aligned} K_0^\epsilon(\Delta\mathbf{x}_n) &\equiv \langle\mathbf{x}_n|\exp\left(-\frac{i}{\hbar}\,\epsilon\hat{H}\right)|\mathbf{x}_{n-1}\rangle \\ &= \frac{1}{\sqrt{2\pi i\epsilon\hbar/M}^D}\exp\left[\frac{i}{\hbar}\,\frac{M}{2}\,\frac{(\Delta\mathbf{x}_n)^2}{\epsilon}\right] \end{aligned} \tag{323}$$

with $\Delta\mathbf{x}_n \equiv \mathbf{x}_n - \mathbf{x}_{n-1}$, $\mathbf{x} \equiv \mathbf{x}_{N+1}$, $\mathbf{x}' \equiv \mathbf{x}_0$. A possible external potential may be omitted since this would contribute in an additive way, uninfluenced by the space geometry.

The path integral may now be transformed directly to spaces with curvature and torsion by applying the nonholonomic mapping formula (4) to the small but finite increments $\Delta\mathbf{x}$ in the action as well as the measure of integration. The correct result is found only by writing the initial measure in the above form, and not in the form

$$\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} dx_n\right], \tag{234}$$

which in flat space is the same thing, but leads to a wrong measure in noneuclidean space.

There is a good reason for having $\Delta\mathbf{x}$ in the flat-space measure at the start of the nonholonomic transformation. According to Huygens' principle of wave optics, each point of a wave front is a center of a new spherical wave propagating from that point. Therefore, in a time-sliced path integral, the differences Δx_n^i play a more fundamental role than the coordinates themselves.

The details have been explained in [10,11], and need not be repeated here. As an important result, we have *derived* for a nonrelativistic point particle of mass m in a purely curved space the Schrödinger equation

$$-\frac{1}{2m}\,D^\mu D_\mu\psi(q,t) = i\hbar\partial_t\psi(q,t), \tag{325}$$

which does not contain an extra R-term as in the earlier literature on this subject.[5] The operator $D^\mu D_\mu$ is equal to $\bar{D}^\mu \bar{D}_\mu - S^\mu \partial^\mu$, where $\bar{D}^\mu \bar{D}_\mu$ coincides with the well-known Laplace–Beltrami differential operator $\Delta = g^{-1/2}\partial_\mu g^{1/2} g^{\mu\nu}\partial_\nu$.

The appearance of the Laplace operator $D_\mu D^\mu$ in the Schrödinger equation (325) is in conflict with the traditional physical scalar product between two wave functions $\psi_1(q)$ and $\psi_2(q)$:

$$\langle \psi_2 | \psi_1 \rangle \equiv \int d^D q \sqrt{g(q)}\, \psi_2^*(q) \psi_1(q). \tag{326}$$

In such a scalar product, only the Laplace–Beltrami operator is a hermitian, not the Laplacian $D_\mu D^\mu$. The bothersome term is the contracted torsion term $-2S^\mu \partial_\mu \psi$ in $D^\mu D_\mu$. This term ruins the hermiticity and thus also the unitarity of the time evolution operator of a particle in a space with curvature and torsion.

For presently known field theories of elementary particles the unitarity problem is fortunately absent. There the torsion field $S_{\mu\nu}{}^\lambda$ is generated by the spin current density of the fundamental matter fields. The requirement of renormalizability restricts these fields to carry spin $\frac{1}{2}$. However, the spin current density of spin-$\frac{1}{2}$ particles happens to be a completely antisymmetric tensor (Ref. 6, vol. II, p. 1432). This property carries over to the torsion tensor. Hence, the torsion field in the universe satisfies $S^\mu = 0$. This implies that for a particle in a universe with curvature and torsion, the Laplacian always degenerates into the Laplace–Beltrami operator, assuring unitarity after all.

The Coulomb system discussed in Section 10 has another way of escaping the unitarity problem. The path integral of this system is solved by a transformation to a space with torsion where the physical scalar product is [10]

$$\langle \psi_2 | \psi_1 \rangle_{\text{phys}} \equiv \int d^D q \sqrt{g}\, w(q) \psi_2^*(q) \psi_1(q). \tag{327}$$

with some scalar weight function $w(q)$. This scalar product is different from the naive scalar product (326). It is, however, reparametrization-invariant, and $w(q)$ makes the Laplacian $D_\mu D^\mu$ a Hermitian operator.

The characteristic property of torsion in this space is that $S_\mu(q)$ can be written as a gradient of a scalar function: $S_\mu(q) = \partial_\mu s(q)$ [see eq. (317)]. The same thing is true for any gradient torsion (255) with

$$s(q) = (1 - D)\sigma(q)/2. \tag{328}$$

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5 Since in previous gravity theories with torsion only spinning particles could be a source, such a possibility was hitherto excluded; see [33].

The weight-function is

$$w(q) = e^{-2s(q)}. \tag{329}$$

Thus, the physical scalar product can be expressed in terms of the naive one as follows:

$$\langle \psi_2 | \psi_1 \rangle_{\rm phys} \equiv \int d^D q \sqrt{g(q)}\, e^{-2\sigma(q)} \psi_2^*(q) \psi_1(q). \tag{330}$$

Within this scalar product, the Laplacian $D_\mu D^\mu$ is, indeed, Hermitian.

To prove this, we observe that within the naive scalar product (327), a partial integration changes the covariant derivative $-D_\mu$ into

$$D_\mu^* \equiv (D_\mu + 2S_\mu). \tag{331}$$

Consider, for example, the scalar product

$$\int d^D q \sqrt{g}\, U^{\mu\nu_1\ldots\nu_n} D_\mu V_{\nu_1\ldots\nu_n}\,. \tag{332}$$

A partial integration of the derivative term ∂_μ in D_μ gives

$$\begin{aligned}\text{surface term} - \int d^D dq \Big[&(\partial_\mu \sqrt{g}\, U^{\mu\nu_1\ldots\nu_n}) V_{\nu_1\ldots\nu_n} \\ &- \sum_i \sqrt{g}\, U^{\mu\nu_1\ldots\nu_i\ldots\nu_n} \Gamma_{\mu\nu_i}{}^{\lambda_i} V_{\nu_1\ldots\lambda_i\ldots\nu_n} \Big].\end{aligned} \tag{333}$$

Now we use

$$\partial_\mu \sqrt{g} = \sqrt{g}\, \bar{\Gamma}_{\mu\nu}{}^{\nu} = \sqrt{g}\,(2S_\mu + \Gamma_{\mu\nu}{}^{\nu}), \tag{334}$$

to rewrite (333) as

$$\begin{aligned}\text{surface term} - \int d^D q \sqrt{g}\, \Big[&(\partial_\mu U^{\mu\nu_1\ldots\nu_n}) V_{\nu_1\ldots\nu_n} \\ &- \sum_i \Gamma_{\mu\nu_i}{}^{\lambda_i} U^{\mu\nu_1\ldots\nu_i\ldots\nu_n} V_{\nu_1\ldots\lambda_i\ldots\nu_n} \\ &- 2S_\mu U^{\mu\nu_1\ldots\nu_n} V_{\nu_1\ldots\nu_n} \Big],\end{aligned} \tag{335}$$

which is equal to

$$\text{surface term} - \int d^D q \sqrt{g}\, (D_\mu^* U^{\mu\nu_1\ldots\nu_n}) V_{\nu_1\ldots\nu_n}\,. \tag{336}$$

In the physical scalar product (330), the corresponding operation is

$$\begin{aligned}\int d^D q\sqrt{g}\, e^{-2\sigma(q)} U^{\mu\nu_1\dots\nu_n} D_\mu V_{\nu_1\dots\nu_n} \\ &= \text{surface term} - \int d^D q\sqrt{g}\, (D^*_\mu e^{-2\sigma(q)} U^{\mu\nu_1\dots\nu_n}) V_{\nu_1\dots\nu_n} \\ &= \text{surface term} - \int d^D q\sqrt{g}\, e^{-2\sigma(q)} (D_\mu \sqrt{g}\, U^{\mu\nu_1\dots\nu_n}) V_{\nu_1\dots\nu_n}\,. \end{aligned} \quad (337)$$

Hence, iD_μ is a Hermitian operator, and so is the Laplacian $D_\mu D^\mu$.

For spaces with an arbitrary torsion, the correct scalar product has yet to be found. Thus the quantum equivalence principle is so far only appicable to spaces with arbitrary curvature and gradient torsion.

12. RELATIVISTIC SCALAR FIELD IN SPACE WITH GRADIENT TORSION

The scalar product in the above quantum mechanical system is the key to the construction of an action for a relativistic scalar field whose particle trajectories are autoparallels. From (328) and (329) we must use a weight factor $w(q) = e^{-3\sigma(q)}$ in the scalar product. This scalar product introduced in [10] has recently become the basis of a series of studies in general relativity [32,34]. In the latter work, the action of a relativistic free scalar field ϕ was set up as follows:

$$\mathcal{A}[\phi] = \int d^4x\, \sqrt{-g}\, e^{-3\sigma} \left(\frac{1}{2}\, g^{\mu\nu} |\nabla_\mu \phi \nabla_\nu \phi| - \frac{m^2}{2}\, |\phi|^2 e^{-2\sigma} \right). \quad (338)$$

The associated Euler–Lagrange equation is

$$D_\mu D^\mu \phi + m^2 e^{-2\sigma(x)} \phi = 0, \quad (339)$$

whose eikonal approximation $\phi(x) \approx e^{i\mathcal{E}(x)}$ yields the following equation for the phase $\mathcal{E}(x)$ [34]:

$$e^{2\sigma(x)} g^{\mu\nu}(x) [\partial_\mu \mathcal{E}(x)]\, [\partial_\nu \mathcal{E}(x)] = m^2. \quad (340)$$

Since $\partial_\mu \mathcal{E}$ is the momentum of the particle, the replacement $\partial_\mu \mathcal{E} \to m\dot{x}_\mu$ shows that the eikonal equation (340) guarantees the constancy of the Lagrangian

$$L = e^{\sigma(x)} \sqrt{g_{\mu\nu}(x))\dot{x}^\mu \dot{x}^\nu} \equiv 1, \qquad \tau = s. \quad (341)$$

From this constancy, in turn, we easily derive the autoparallel equation (215) with the gradient torsion (255), corresponding the the covariant conservation law (246) for the energy-momentum tensor.

13. LOCAL LORENTZ FRAMES VERSUS LOCALLY FLAT LORENTZ FRAMES

For completeness, we clarify in some more detail the difference between the multivalued basis tetrads $e^a{}_\mu(q)$ and the standard single-valued tetrads or vierbein fields $h^\alpha{}_\mu(q)$ introduced in eqs. (4) and (35), respectively. For this derive the minimal coupling of fields of arbitrary spin to gravity via the nonholonomic mapping principle, generalizing the procedure in subsections 3.4 and 6.2. From the lesson learned in subsection 3.4, we simply have to transform the flat-space field theory nonholonomically into the space with curvature and torsion, and this will yield directly the correct field theory in that space. This will produce, in particular, the covariant derivatives (50) needed to make the gradient terms in the Lagrange density invariant under local translations and local Lorentz transformations. We introduce a *fixed* set of Minkowski basis vectors $\mathbf{e}_a$ in the flat space and define intermediate local basis vectors

$$\mathbf{e}_\alpha(q) = \mathbf{e}_a(q)\frac{\partial x^a}{\partial x^\alpha} = \mathbf{e}_a(q)e^a{}_\alpha(q), \tag{342}$$

as well as final ones

$$\mathbf{e}_\mu(q) = \mathbf{e}_a(q)\frac{\partial x^\alpha}{\partial q^\mu} = \mathbf{e}_\alpha(q)h^\alpha{}_\mu(q). \tag{343}$$

An arbitrary vector field $\mathbf{v}(q)$ can be expanded in either of these three bases as follows:

$$\begin{aligned}\mathbf{v} &\equiv \mathbf{e}_a v^a = \mathbf{e}_a e^a{}_\mu v^\mu = \mathbf{e}_a e^a{}_\alpha(h^\alpha{}_\mu v^\mu)\\ &= \mathbf{e}_a e^{a\alpha} h_\alpha{}^\mu v_\mu \equiv \mathbf{e}_a e^a{}_\alpha v^\alpha \equiv \mathbf{e}_a e^{a\alpha} v_\alpha,\end{aligned} \tag{344}$$

the last two expressions containing the local Lorentz components $v_\alpha(q)$ whose contravariant components were introduced in (49) and whose covariant derivatives are (50). By changing the basis in the derivatives $\partial_a v_b$ we find that that the spin connection $\Gamma_{\alpha\beta}{}^\gamma(q)$ is expressed in terms of $e^a{}_\beta(q)$ in the same way as the affine connection was in (11):

$$\Gamma_{\alpha\beta}{}^\gamma \equiv e_a{}^\gamma\partial_\alpha e^a{}_\beta = -e^a{}_\beta\partial_\alpha e_a{}^\gamma. \tag{345}$$

Written out in terms of $e^a{}_\mu$ and $h_a{}^\mu$, it reads

$$\begin{aligned}\Gamma_{\alpha\beta}{}^\gamma &= e_a{}^\lambda h^\gamma{}_\lambda h_\alpha{}^\mu \partial_\mu(e^a{}_\nu h_\beta{}^\nu)\\ &= h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu \Gamma_{\mu\nu}{}^\lambda + h^\gamma{}_\lambda h_\alpha{}^\mu \delta^\lambda{}_\nu \partial_\mu h_\beta{}^\nu\\ &= h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu(\Gamma_{\mu\nu}{}^\lambda + h^\delta{}_\nu\partial_\mu h_\delta{}^\lambda)\\ &= h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu(\Gamma_{\mu\nu}{}^\lambda - \overset{h}{\Gamma}_{\mu\nu}{}^\lambda),\end{aligned} \tag{346}$$

where $\overset{h}{\Gamma}_{\mu\nu}{}^{\lambda}$ is defined in terms of h in the same way as $\Gamma_{\mu\nu}{}^{\lambda}$ is defined in terms of e in (11):

$$\overset{h}{\Gamma}_{\mu\nu}{}^{\lambda}(q) \equiv h_{\alpha}{}^{\lambda}(q)\partial_{\mu}\, h^{\alpha}{}_{\nu}(q) = -h^{\alpha}{}_{\nu}(q)\,\partial_{\mu}\, h_{\alpha}{}^{\lambda}(q). \tag{347}$$

The second line in (346) implies that $h_{\alpha}{}^{\mu}$ satisfies identities like $e_{a}{}^{\mu}$ in (12):

$$D_{\alpha} h_{\beta}{}^{\mu} = 0, \qquad D_{\alpha} h^{\beta}{}_{\mu} = 0, \tag{348}$$

where the covariant derivative involves the connection for the Einstein index as well as the spin connection for the local Lorentz index as in eqs. (10) and (50), respectively:

$$D_{\alpha} h_{\beta}{}^{\mu} = \partial_{\alpha} h_{\beta}{}^{\mu} - \Gamma_{\alpha\beta}{}^{\gamma} h_{\gamma}{}^{\mu} + h_{\alpha}{}^{\kappa}\Gamma_{\kappa\nu}{}^{\mu} h_{\beta}{}^{\nu}. \tag{349}$$

Since the metric is obtained from the tetrads $h^{\alpha}{}_{\mu}$ by means of the relation [see also (39)]

$$g_{\mu\nu}(q) = e^{a}{}_{\mu}(q) e^{b}{}_{\nu}(q)\eta_{ab} \equiv h^{\alpha}{}_{\mu}(q) h^{\beta}{}_{\nu}(q)\eta_{\alpha\beta}\,, \tag{350}$$

the right-hand side of (347) can be rewritten as in (32) replacing $e_{a}{}^{\mu}$ by $h_{\alpha}{}^{\mu}$, so that we obtain a decomposition completely analogous to (30) for the affine connection:

$$\overset{h}{\Gamma}_{\mu\nu}{}^{\lambda} = \bar{\Gamma}_{\mu\nu}{}^{\lambda} + \overset{h}{K}_{\mu\nu}{}^{\lambda} \tag{351}$$

where $\bar{\Gamma}_{\mu\nu}{}^{\lambda}$ is the Riemann connection and

$$\overset{h}{K}_{\mu\nu}{}^{\lambda} = \overset{h}{S}_{\mu\nu}{}^{\lambda} - \overset{h}{S}_{\nu}{}^{\lambda}{}_{\mu} + \overset{h}{S}{}^{\lambda}{}_{\mu\nu} \tag{352}$$

an analog of the contortion tensor (31). The tensor $\overset{h}{S}_{\mu\nu}{}^{\lambda}$ is the antisymmetric part of $\overset{h}{\Gamma}_{\mu\nu}{}^{\lambda}$, and comparison with (46) and (47) shows that it yields the object of anholonomy (46) by a simple transformation of its indices:

$$\Omega_{\alpha\beta}{}^{\gamma} = \tfrac{1}{2}[h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu}\partial_{\mu} h^{\gamma}{}_{\nu} - (\mu \leftrightarrow \nu)] = h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu}\, \overset{h}{S}_{\mu\nu}{}^{\lambda}. \tag{353}$$

If we now insert the decompositions (351) and (30) into (346), the Riemann connections in $\Gamma_{\mu\nu}{}^{\lambda}$ and $\overset{h}{\Gamma}_{\mu\nu}{}^{\lambda}$ cancel each other, and we obtain

$$\Gamma_{\alpha\beta}{}^{\gamma} = h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu}(K_{\mu\nu}{}^{\lambda} - \overset{h}{K}_{\mu\nu}{}^{\lambda}), \tag{354}$$

which is precisely the spin connection (48).

Note that due to (352) and the antisymmetry of $\overset{h}{S}_{\mu\nu}{}^{\lambda}$ in the first two indices, the tensor $\overset{h}{K}_{\mu\nu\lambda}$ is antisymmetric in the last two indices, just as the contortion tensor $K_{\mu\nu\lambda}$ in eq. (33), so that also the spin connection shares this antisymmetry.

It will be convenient to use $h^{\alpha}{}_{\mu}$, $h_{\alpha}{}^{\mu}$ freely for changing indices α into μ, for instance

$$K_{\alpha\beta}{}^{\gamma} \equiv h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} K_{\mu\nu}{}^{\lambda}, \tag{355}$$

$$\overset{h}{K}_{\alpha\beta}{}^{\gamma} = h^{\gamma}{}_{\lambda} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \overset{h}{K}_{\mu\nu}{}^{\lambda}. \tag{356}$$

Observe that by introducing the tetrad fields $h^{\alpha}{}_{\mu}$, $h_{\alpha}{}^{\mu}$, the description of gravity effects in terms of the 10 metric components $g_{\mu\nu}$ and the 24 torsion components $K_{\mu\nu}{}^{\lambda}$ has been replaced by 16 components $h_{\mu}{}^{\alpha}$ and the 24 $K_{\mu\nu}{}^{\lambda}$. The additional 6 components are redundant, and this is the source of the local Lorentz invariance of the theory which arises in addition to the local translations of Einstein's general coordinate transformations. Relation (350) implies that the tetrad fields $h^{\alpha}{}_{\mu}$ can be considered as another "square root" of the metric $g_{\mu\nu}$ different from $e^{a}{}_{\mu}$. Obviously, such a "square root" is defined only up to an arbitrary local Lorentz transformation which accounts for the six additional degrees of freedom of the $h_{\alpha}{}^{\mu}(q)$ with respect to the $g_{\mu\nu}(q)$ description. These six degrees of freedom characterize the local Lorentz transformations $\Lambda_{\alpha}{}^{a}(q) \equiv e_{\alpha}{}^{a}(q)$ by which the intermediate basis vectors $\mathbf{e}_{\alpha}(q)$ differ from the fixed orthonormal basis vectors $\mathbf{e}_{a}$:

$$\mathbf{e}_{\alpha}(q) \equiv \mathbf{e}_{a} \Lambda^{a}{}_{\alpha}\,. \tag{357}$$

The Lorentz properties of $\Lambda^{a}{}_{\alpha}$ follow from the fact that the basis vectors $\mathbf{e}_{\alpha}(q)$ have the same Minkowskian scalar products as $\mathbf{e}_{a}$:

$$\mathbf{e}_{\alpha}(q)\mathbf{e}_{\beta}(q) = \eta_{\alpha\beta}\,. \tag{358}$$

As a consequence, the matrix $\Lambda_{\alpha}{}^{a}(q)$ satisfies [see (42)]

$$\eta_{ab} \Lambda^{a}{}_{\alpha}(q) \Lambda^{b}{}_{\beta}(q) = \eta_{\alpha\beta}\,. \tag{359}$$

which is the defining equation of Lorentz transformations. Since these local Lorentz transformations bring the good vierbein functions $h_{\alpha}{}^{\mu}(q)$ satisfying (43) to the multivalued functions $e_{a}{}^{\mu}(q)$ with noncommuting derivatives (44), the local Lorentz transformations are also multivalued:

$$(\partial_{\mu}\partial_{\nu} - \partial_{\nu}\partial_{\mu})\Lambda_{a}{}^{\alpha}(q) \neq 0\,. \tag{360}$$

Note that, due to (345), both $e^a{}_\alpha(q)$ and $e_a{}^\alpha(q)$ satisfy identities like (12),(348),

$$D_\alpha e^a{}_\beta = 0, \qquad D_\alpha e_a{}^\beta = 0\,. \tag{361}$$

All derivatives in a reparametrization-invariant theory can be recast in terms of nonholonomic coordinates dx^α, in which form it becomes invariant under both local translations and local Lorentz transformations. Since the metric is $\eta^{\alpha\beta}$, the invariant actions have the same form as those in a flat space except that the derivatives are replaced by covariant ones:

$$\partial_\alpha v_\beta \;\rightarrow\; D_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma_{\alpha\beta}{}^\gamma v_\gamma\,. \tag{362}$$

Nonholonomic volume elements are related to true ones by $d^4x = d^4q\sqrt{g(q)}$. An invariant action of a massless vector field is, for example,

$$\mathcal{A} = \int d^4x D_\alpha v_\beta(q) D^\alpha v^\beta(q). \tag{363}$$

It is the nonholonomic form of a generally covariant action. As we said in the beginning, the specification of space points must be made with the q^μ coordinates. For this reason the action is preferably written as

$$\mathcal{A} = \int d^4q\sqrt{-g}\, D_\alpha v_\beta(q) D^\alpha v^\beta(q). \tag{364}$$

Under a general coordinate transformation (179) *à la* Einstein, $dq^\mu \rightarrow dq'^{\mu'} = dq^\mu \alpha_\mu{}^{\mu'}$, the indices α are unchanged. For instance, $h_\alpha{}^\mu$ itself transforms as

$$h_\alpha{}^\mu(q) \underset{E}{\rightarrow} h_\alpha{}^{\mu'}(q') = h_\alpha{}^\mu(q)\alpha_\mu{}^{\mu'}(q). \tag{365}$$

Vectors and tensors with indices $\alpha, \beta, \gamma, \dots$ experience only changes of their arguments $q \rightarrow q + \xi$, so that their infinitesimal substantial changes are

$$\delta_E v_\alpha(q) = \xi^\lambda \partial_\lambda v_\alpha(q), \tag{366}$$

$$\delta_E D_\alpha v_\beta(q) = \xi^\lambda \partial_\lambda D_\alpha v_\beta(q). \tag{367}$$

The freedom in choosing $h_\alpha{}^\mu(q)$ up to a local Lorentz transformation when taking the "square root" of $g_{\mu\nu}(q)$ in (350) implies that the theory should be invariant under

$$\delta_L dx^\alpha = \omega^\alpha{}_\beta(q) dx^\beta, \tag{368}$$

$$\delta_L h_\alpha{}^\mu(q) = \omega_\alpha{}^\beta(q) h_\beta{}^\mu(q). \tag{369}$$

Here $\omega_\alpha{}^{\alpha'}(q)$ are the local versions of the infinitesimal Lorentz parameters.

Indeed the action (364) is automatically invariant if every index α is transformed accordingly:

$$\delta_L v_\alpha(q) = \omega_\alpha{}^{\alpha'}(q) v_{\alpha'}(q), \tag{370}$$

$$\delta_L D_\alpha v_\beta(q) = \omega_\alpha{}^{\alpha'}(q) D_{\alpha'} v_\beta(q) + \omega_\beta{}^{\beta'}(q) D_\alpha v_{\beta'}(q). \tag{371}$$

The variables q^μ are unchanged since (368) refers only to the differentials dx^α and leaves dq^μ unchanged.

It is useful to verify explicitly how the covariant derivatives guarantee local Lorentz invariance. From (370) we see that the derivative $\partial_\alpha v_\beta$ transforms like

$$\begin{aligned}\delta_L \partial_\alpha v_\beta &= (\delta_L \partial_\alpha) v_\beta + \partial_\alpha (\delta_L v_\beta)\\ &= \omega_\alpha{}^{\alpha'} \partial_{\alpha'} v_\beta + \partial_\alpha (\omega_\beta{}^{\beta'} v_{\beta'})\\ &= \omega_\alpha{}^{\alpha'} \partial_{\alpha'} v_\beta + \omega_\beta{}^{\beta'} \partial_\alpha v_{\beta'} + (\partial_\alpha \omega_\beta{}^{\beta'}) v_{\beta'}\,.\end{aligned} \tag{372}$$

The spin connection behaves as follows: Due to the factors $h_\lambda{}^\gamma h_\alpha{}^\mu h_\beta{}^\nu$ in (351), the first piece of $\Gamma_{\alpha\beta}{}^\gamma$, call it $\Gamma'_{\alpha\beta}{}^\gamma$, transforms like a local Lorentz tensor:

$$\delta_L \Gamma'_{\alpha\beta}{}^\gamma = \omega_\alpha{}^{\alpha'} \Gamma'_{\alpha'\beta} + \omega_\beta{}^{\beta'} \Gamma'_{\alpha\beta'}{}^\gamma + \omega^\gamma{}_{\gamma'} \Gamma'_{\alpha\beta}{}^{\gamma'}. \tag{373}$$

But from the second piece $\overset{h}{\Gamma}_{\mu\nu}{}^\lambda$ there is a nontensorial derivative contribution

$$\begin{aligned}\delta_L \overset{h}{\Gamma}_{\mu\nu}{}^\lambda &= (\delta h_\delta{}^\lambda) \partial^\delta{}_\nu + h_\delta{}^\lambda \partial_\mu (\delta h^\delta{}_\nu)\\ &= \omega_\delta{}^{\delta'} h_{\delta'}{}^\lambda \partial_\mu h^\delta{}_\nu + h_\delta{}^\lambda \partial_\mu (\omega^\delta{}_{\delta'} h^{\delta'}{}_\nu)\\ &= \omega_\delta{}^{\delta'} h_{\delta'}{}^\lambda \partial_\mu h^\delta{}_\nu + \omega^\delta{}_{\delta'} h_\delta{}^\lambda \partial_\mu{}^\lambda \partial_\mu h^{\delta'}{}_\nu + \partial_\mu \omega^\delta{}_{\delta'} (h_\delta{}^\lambda h^{\delta'}{}_\nu)\\ &= \partial_\mu \omega^\delta{}_{\delta'} h^{\delta'}{}_\nu = -\partial_\mu \omega_{\delta'}{}^\delta h_\delta{}^\lambda h^{\delta'}{}_\nu\,,\end{aligned} \tag{374}$$

the cancellation in the third line being due to the antisymmetry of $\omega_\delta{}^{\delta'} = -\omega^{\delta'}{}_\delta$. Thus we arrive at

$$\begin{aligned}\delta_L \overset{h}{\Gamma}_{\mu\nu}{}^\lambda &= \partial_\mu \omega^\delta{}_{\delta'}\, h_\delta{}^\lambda h^{\delta'}{}_{\nu'}\,,\\ \delta_L \Gamma_{\alpha\beta}{}^\gamma &= \delta_{L_0} \Gamma_{\alpha\beta}{}^\gamma + \partial_\alpha \omega_\beta{}^\gamma,\end{aligned} \tag{375}$$

where δ_{L_0} abbreviates the three tensor-like transformed terms corresponding to (373):

$$\delta_{L_0} \Gamma_{\alpha\beta}{}^\gamma = \omega_\alpha{}^{\alpha'} \Gamma_{\alpha'\beta} + \omega_\beta{}^{\beta'} \Gamma_{\alpha\beta'}{}^\gamma + \omega^\gamma{}_{\gamma'} \Gamma_{\alpha\beta}{}^{\gamma'}. \tag{376}$$

The last term in (375) is precisely what is required to cancel the last nontensorial piece of (372), when transforming $D_\alpha v_\beta$, so that we indeed obtain the covariant transformation law (371).

Armed with these transformation laws it is now straightforward to introduce spinor fields into a gravity theory. In a local inertial frame (such as a freely falling elevator), a spinor field $\psi(q)$ transforms like

$$\delta_L \psi(q) = -\frac{i}{2}\,\omega^{\alpha\beta}(q)\Sigma_{\alpha\beta}\psi(q), \tag{377}$$

when locally changing from one such frame of reference to another Lorentz transformed one. The matrices $\Sigma_{\alpha\beta}$ represent the local Lorentz group on the ψ-fields. They are antisymmetric in their indices, and have the nonzero commutation relations

$$[\Sigma_{\alpha\beta}, \Sigma_{\alpha\gamma}] = -i\eta_{\alpha\alpha}\Sigma_{\beta\gamma} \qquad \text{no sum over } \alpha. \tag{378}$$

For vector representations, they are given explicitly by

$$(\Sigma_{\alpha\beta})_{\alpha'\beta'} = i(\eta_{\alpha\alpha'}\eta_{\beta\beta'} - (\alpha \leftrightarrow \beta)\,). \tag{379}$$

Replacing ψ by v and writing out the Lorentz indices, eq. (377) reduces to (369):

$$\delta_L v_\alpha = -\frac{i}{2}\,\omega^{\gamma\delta} i(\eta_{\gamma\alpha}\eta_{\delta\beta} - (\gamma \leftrightarrow \delta)\,)v^\beta = \omega_\alpha{}^\beta v_\beta\,. \tag{380}$$

For Dirac fields, the representation matrices $\Sigma_{\alpha\beta}$ are expressed products of Dirac matrices:

$$\Sigma_{\alpha\beta} = \frac{i}{4}\,[\gamma_\alpha, \gamma_\beta]. \tag{381}$$

The derivative of ψ changes as

$$\begin{aligned}\delta_L \partial_\alpha \psi &= \omega_\alpha{}^{\alpha'}\partial_{\alpha'}\psi + \partial_\alpha \delta_L \psi = \omega_\alpha{}^{\alpha'}\partial_{\alpha'}\psi - \frac{i}{2}\,\partial_\alpha(\omega^{\beta\gamma}\Sigma_{\beta\gamma})\psi \\ &= \omega_\alpha{}^{\alpha'}\partial_{\alpha'}\psi - \frac{i}{2}\,\omega^{\beta\gamma}\Sigma_{\beta\gamma}\partial_\alpha\psi - \frac{i}{2}\,(\partial_\alpha\omega^{\beta\gamma})\Sigma_{\beta\gamma}\psi. \end{aligned} \tag{382}$$

The first two terms describe the normal Lorentz behavior of $\partial_\alpha\psi$. The last term accounts for the q-dependence of the angles $\omega^{\beta\gamma}(q)$. It does not appear if we go over to the covariant derivative

$$D_\alpha\psi(q) \equiv \partial_\alpha\psi(q) + \frac{i}{2}\,\Gamma_{\alpha\beta}{}^\gamma\,\Sigma^\beta{}_\gamma\psi(q). \tag{383}$$

Indeed, when forming

$$\delta_L \frac{i}{2}\,\Gamma_{\alpha\beta}{}^{\gamma}\Sigma^{\beta}{}_{\gamma}\psi(q) = \frac{i}{2}\,\delta_L\Gamma_{\alpha\beta}{}^{\gamma}\Sigma^{\beta}{}_{\gamma}\psi + \frac{i}{2}\,\Gamma_{\alpha\beta}{}^{\gamma}\Sigma^{\beta}{}_{\gamma}\delta_L\psi, \tag{384}$$

we obtain two terms. The first of these corresponds to a tensor transformation law, being equal to

$$\delta_{L_0} \frac{i}{2}\,\Gamma_{\alpha\beta}{}^{\gamma}\Sigma^{\beta}{}_{\gamma}\psi = -\frac{i}{2}\,\omega^{\sigma\tau}\Sigma_{\sigma\tau}\left(\frac{i}{2}\,\Gamma_{\alpha\beta}{}^{\gamma}\Sigma^{\beta}{}_{\gamma}\psi\right). \tag{385}$$

It is obtained by inserting into (384) the equations (375) and (377), and applying the commutation rule (378). The second, nontensorial term arises from $\partial_\alpha\omega_\beta{}^\gamma$ in (375):

$$\frac{i}{2}\,\partial_\alpha\omega_\beta{}^{\gamma}\Sigma^{\beta}{}_{\gamma}\psi, \tag{386}$$

and cancels the last term in (382). Thus $D_\alpha\psi$ behaves like

$$\delta_L D_\alpha\psi = \omega_\alpha{}^{\alpha'}(q)D_{\alpha'}\psi - \frac{i}{2}\,\omega^{\beta\gamma}(q)\Sigma_{\beta\gamma}D_\alpha\psi \tag{387}$$

and represents, therefore, a proper covariant derivative which generalizes the standard Lorentz transformation behavior to the case of local transformations $\omega_\alpha{}^\beta(q)$.

We can now immediately construct the spin$-\frac{1}{2}$ action for a Dirac particle in a gravity field:

$$\mathcal{A}_m[h,K,\psi] = \frac{1}{2}\int d^4q\sqrt{-g}\,\bar{\psi}(\gamma^\alpha D_\alpha - m)\psi(q) + \text{h.c.} \tag{388}$$

$$\equiv \frac{1}{2}\int d^4q\sqrt{-g}\,\bar{\psi}\gamma^a\left(\partial_\alpha + \frac{i}{2}\,\Gamma_{\alpha\beta}{}^{\gamma}\Sigma^{\beta}{}_{\gamma}\right)\psi(q) + \text{h.c.} \tag{389}$$

If we wish, we may change the derivatives from ∂_α to ∂_μ by using $\partial_\alpha = h_\alpha{}^\mu\partial_\mu$ and $\gamma^\alpha = h^\alpha{}_\mu(q)\gamma^\mu(q)$ so that $\Sigma_{\alpha\beta}(q) = (i/4)[\gamma_\alpha(q),\,\gamma_\beta(q)]$ and, expressing $\Gamma_{\alpha\beta}{}^\gamma$ by (351), the action reads

$$\begin{aligned}\mathcal{A}_m[h,K,\psi] = \frac{1}{2}\int d^4q\sqrt{-g}\,\bar{\psi}(q)\times \\ \times\left\{\gamma^\mu(q)\left[\partial_\mu + \frac{i}{2}\,(K_{\mu\nu}{}^{\lambda} - \overset{h}{K}_{\mu\nu}{}^{\lambda})\Sigma^{\nu}{}_{\lambda}\right] - m\right\}\psi(q) \\ + \text{h.c.}\end{aligned} \tag{390}$$

Due to the x-dependence of γ^μ and $\Sigma^{\mu\nu}{}_\lambda$, this form is, however, not very convenient for calculations. This minimal type of coupling between spin and gravity can easily be generalized to higher-spin fields if desired.

14. CONCLUSION AND OUTLOOK

We have pointed out that the nonholonomic mapping principle supplies us with a perfect tool for deriving physical laws in spaces with curvature and torsion by means of multivalued coordinate transformations. As mentioned earlier, there are other evidences for the correctness of this principle, one of them being deduced from the solution of the path integral of the Coulomb system. As a particular result we have derived from this principle a new variational procedure for Hamilton's action principle which has led to the surprising result that trajectories of spinless point particles are autoparallels, not geodesics as commonly believed [3,5,7].

Since spinless particles experience a torsion force, we expect them to be also the source of torsion. Under the assumption that torsion propagates we may add to the gravitational action a gradient term involving the torsion, and will be able to derive deviations from Einstein's gravity effects (deflection of light, gravitational red shift, perihelion precession of Mercury) [33]. The experimental smallness of such deviations will provide us with limits on the coupling constant in front of the gradient term.

Up to now, it is doubtful whether the minimally coupled field theories described in Section 13 are physically correct. A proper construction will require a field version of the new variation formula (239) caused by the closure failure. This is in fact necessary and nontrivial even for the Schrödinger action (91). In the semiclassical limit (eikonal approximation), Schrödinger wave functions have to describe autoparallel particle trajectories. Except for the case of gradient torsion discussed in Section 12 it is unknown how to achieve this.

It should be pointed out that conventional gravity in which the torsion field is coupled only to spin has the severe consistency problem that spin and orbital angular momentum are distinguishable which contradicts the universality principle of these in elementary particle theory [35]. Only a modified torsion coupling which leads to autoparallel trajectories has a chance of being consistent.

Another problem with conventional field theory of gravity theory with torsion arises in the context of electroweak gauge theories. In the theory, massless and massive vector bosons are coupled differently to torsion. This, however, is incompatible with the Higgs mechanism according to which the vector meson masses arise from a spontaneous symmetry breakdown of a scalar field theory. As shown recently in [36], only a Higgs field whose particles run along autoparallel trajectories can remove this problem. This problem should be studied in more detail using the field theory of gravity with gradient torsion in [4] where massless vector mesons do couple to torsion without violating gauge invariance.

The motion of particles with spin (see e.g. Ref. 37) will also be altered by a generalization of the arguments given above.

ACKNOWLEDGEMENTS

I am grateful to Dr. A. Pelster for many stimulating discussions, and to Drs. G. Barnich, H. von Borzeszkowski, C. Maulbetsch, and S.V. Shabanov for useful comments.

APPENDIX A. PARTIAL INTEGRATION IN SPACES WITH CURVATURE AND TORSION

Take any tensors $U^{\mu\dots\nu}, V_{\dots\nu\dots}$ and consider an invariant volume integral

$$\int dx\sqrt{-g}\, U^{\mu\dots\nu\dots} D_\mu V_{\dots\nu\dots}\,. \qquad (A.1)$$

A partial integration gives

$$-\int dx \left[\partial_\mu \sqrt{-g}\, U^{\mu\dots\nu\dots} V_{\dots\nu\dots} + \sum_i U^{\mu\dots\nu_i\dots} \Gamma_{\mu\nu_i}{}^{\lambda_i} V_{\dots\lambda_i\dots}\right]$$
$$+ \text{surface terms}, \qquad (A.2)$$

where the sum over i runs over all indices of $V_{\dots\lambda_i\dots}$, linking them via the affine connection with the corresponding indices of $U^{\mu\dots\nu_i\dots}$. Now we use the relation

$$\partial_\mu \sqrt{-g} = \sqrt{-g}\, \bar{\Gamma}_{\mu\kappa}{}^{\kappa} = \sqrt{-g}\, \Gamma_{\mu\kappa}{}^{\kappa} = \sqrt{-g}\, (2S_\mu + \Gamma_{\kappa\mu}{}^{\kappa}) \qquad (A.3)$$

and $(A.2)$ becomes

$$-\int dx\sqrt{-g}\left[(\partial_\mu U^{\mu\dots\lambda_i\dots} - \Gamma_{\kappa\mu}{}^{\kappa} U^{\mu\dots\lambda_i\dots} + \sum_i \Gamma_{\mu\nu_i}{}^{\lambda_i} U^{\mu\dots\nu_i\dots}) V_{\dots\lambda_i\dots}\right.$$
$$\left. + 2S_\mu \sum_i U^{\mu\dots\lambda_i\dots} V_{\dots\lambda_i\dots}\right] + \text{surface terms}. \qquad (A.4)$$

Now, the terms in parentheses are just the covariant derivative of $U^{\mu\dots\nu_i\dots}$ such that we arrive at the rule

$$\int dx\sqrt{-g}\, U^{\mu\dots\nu\dots} D_\mu V_{\dots\nu\dots} = -\int dx\sqrt{-g}\, D_\mu{}^{*} U^{\mu\dots\nu\dots} V_{\dots\nu\dots}$$
$$+ \text{surface terms}, \qquad (A.5)$$

where D^*_μ is defined as

$$D^*_\mu \equiv D_\mu + 2S_\mu\,, \tag{A.6}$$

abbreviating

$$S_\kappa \equiv S_{\kappa\lambda}{}^{\lambda}, \qquad S^\kappa \equiv S^\kappa{}_\lambda{}^\lambda. \tag{A.7}$$

It is easy to show that the operators D_μ and D^*_μ can be interchanged in the rule $(A.5)$, i.e., there is also the rule

$$\int dx\sqrt{-g}\,V_{\dots\nu\dots}D_\mu U^{\mu\dots\nu\dots} = -\int dx\sqrt{-g}\,U^{\mu\dots\nu\dots}D^*_\mu V_{\dots\nu\dots} + \text{surface terms}, \tag{A.8}$$

For the particular case that $V_{\dots\nu\dots}$ is equal to 1, the second rule yields

$$\int dx\sqrt{-g}\,D_\mu U^\mu = -\int dx\sqrt{-g}\,2S_\mu U^\mu + \text{surface terms}. \tag{A.9}$$

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