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FIELD DIMENSION AND GRAVITATIONAL VERTEX

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Abstract: The implications of the dimension of a field on its gravitational vertex function are discussed. As expected from the LSZ theory, the dimension as an intrinsic property of the interpolating field does not enter into on-mass-shell quantities. Thus, contrary to the claim of Kastrup, the slope of the scalar form factor is not determined by the dimension of the field. Only under additional assumptions involving the dimension of the SU(2) × SU(2) breaking is such a determination possible.

1. INTRODUCTION

The dimensional analysis of field operator equations has recently turned out to be a powerful tool in studying the singular behaviour of operator products $\phi(x)\phi(0)$ for small x^μ as well as close to the light cone [1]. It turns out that to any finite order in perturbation theory the dimensions of fields are approximately ** equal to what one expects from canonical theory, a result which agrees with the scaling law found experimentally in deep inelastic scattering.

It is therefore desirable to test this result in other physical processes. Thus predictions have been derived for vector-meson dominance, $K_{\ell 3}$ form factors, and μ pair production in pp collisions [2], and for the high-energy behaviour of propagators [3,4] ***.

The most striking consequence was, however, derived by Kastrup. Without any dynamical assumption, he found that the slope of the *on-shell* form factor of the trace of the energy momentum tensor is given by [5]

$$\frac{\partial \hat{\Gamma}}{\partial t}(0, \mu^2, \mu^2) = -\tfrac{1}{2}(\tfrac{7}{2} - d_\phi) , \tag{1}$$

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** For example, the most singular part of the operator product $\phi(x)\phi(0)$ is given to finite order in perturbation theory by $\phi(x)\phi(0) \xrightarrow[x\to 0]{} (c/x^2)(\log x^2)^\lambda$ which shows that $\phi(x)$ has dimension one except for the logarithmic factor.

*** One of the authors (H.K.) is grateful to Professor Jackiw for sending him an anonymous piece of work entitled 'Canonical and non-Canonical Scale Symmetry Breaking' (MIT preprint, May 1970) in which the equations show his handwriting.

where d_ϕ is the (mass) dimension of the field. From this result he claims to prove that there are at least three $I=0$ s-wave resonances in nature [5, 6].

A result like this is highly surprising for two reasons:

(i) Kastrup's relation is not satisfied by a free massive scalar field (where $d_\phi = 1$ but $\partial \hat{\Gamma}/\partial t(0, \mu^2, \mu^2) = 0$).

(ii) The dimension of an interpolating field can easily be changed without changing the internal quantum numbers, for example by using $\bar{\psi}(x)\psi(x)$ of dimension three instead of $\phi(x)$ of dimension one. The on-mass shell result, however, cannot depend on which of these fields one uses.

Kastrup's result must therefore be wrong. It is shown in this paper that he obtains it on the basis of two errors:

(a) In his Ward identity for the current $j_\mu(x) = x_m \mathcal{K}_{\mu m}(x)$ involving the conformal current density $\mathcal{K}_{\mu\nu}(x)$ ($m = 1, 2, 3$, but no summation implied) he argues that a quantity

$$\lim_{\substack{p^2 \to \mu^2 \\ \boldsymbol{p}=0}} \{(p^2 - \mu^2) \lim_{q \to 0} \int \mathrm{d}x \mathrm{d}y\, \mathrm{e}^{-i(qx-py)} \langle 0 | T\mathcal{K}_{mm}(x)\phi(y)\phi(0) | 0 \rangle\} ,$$

vanishes on the grounds that the self stress of a particle at rest is zero. However, this expression does not depend on the self-stress but contains only second moments of the energy momentum tensor.

(b) He falsely evaluates the remainder of his equation and finds (1) †. If he had proceeded correctly after neglecting the $\mathcal{K}_{mm}(x)$ term, he would have found $\partial \hat{\Gamma}/\partial t(0, \mu^2, \mu^2) = 0$.

It is the purpose of this paper to clarify the situation about the gravitational vertex and to show that a result on the slope $\partial \hat{\Gamma}/\mathrm{d}t(0, \mu^2, \mu^2)$ can be derived for pions.

In order to do so we shall invoke, however, several additional assumptions

(i) The energy density θ_{00} can be split according to $\theta_{00} = \theta^*_{00} + \theta_0 + \theta_4$, where θ^*_{00} conserves both chiral and conformal symmetry, and θ_0, θ_4 are Lorentz scalars breaking conformal symmetry and behaving like $(0,0)$, $(\frac{1}{2}, \frac{1}{2})_4$ under $SU(2) \times SU(2)$ respectively.

(ii) The $SU(2) \times SU(2)$ breaking term θ_4 has a unique dimension d.

(iii) The time components of the $SU(2) \times SU(2)$ currents have dimension 3.

(iv) The divergence of the axial-vector current is in the same $(\frac{1}{2}\frac{1}{2})$ multiplet as the symmetry breaking term.

(v) The propagators $\langle 0 | T(\partial^\mu A^a_\mu(x) \partial^\nu A^a_\nu(0)) | 0 \rangle$ and $\langle 0 | T(\theta^\mu_\mu(x) \theta_4(0)) | 0 \rangle$ are dominated by a pion and σ-meson, respectively.

Notice that the last assumption automatically implies the existence of a $\pi\pi\sigma$ interaction.

† Neglecting sea-gulls which cancel with Schwinger terms of his Ward identity (see ref. [8]).

2. DIMENSION, WARD AND TRACE IDENTITIES

If $\theta_{\mu\nu}(x)$ is the symmetric energy momentum tensor of the world (with finite matrix elements *) we assume (as is true in field theories where the field virial is a divergence) that the current densities of the conformal group can be defined for spinless fields as

$$\mathcal{D}_{\mu}(x) = x^{\nu}\theta_{\mu\nu}(x) \ , \tag{2}$$

$$\mathcal{K}_{\mu\nu}(x) = (2x^{\rho}x_{\nu} - x^{2}g^{\rho}_{\nu})\theta_{\mu\rho}(x) \ . \tag{3}$$

We say that a field $\phi(x)$ has dimension d_{ϕ} if its commutation rules with the corresponding charges

$$D(x_{0}) \equiv \int \mathcal{D}_{0}(x)\mathrm{d}^{3}x \ , \qquad K_{\nu}(x_{0}) \equiv \int \mathcal{K}_{0\nu}\mathrm{d}^{3}x \ , \tag{4}$$

are

$$i[D(x_{0}), \phi(x)] = (x\partial + d_{\phi})\phi(x) \ , \tag{5}$$

$$i[K_{\nu}(x_{0}), \phi(x)] = (2x_{\nu}(x\partial + d_{\phi}) - x^{2}\partial_{\nu})\phi(x) \ . \tag{6}$$

The origin of this dimension can be traced back to the occurrence of a Schwinger term in the commutation rule

$$i[\theta_{0\mu}(x), \phi(y)]_{x_{0}=y_{0}} = \partial_{\mu}\phi(x)\delta^{3}(x-y) - \tfrac{1}{3}d_{\phi}\,g_{\mu}{}^{i}\,\phi(y)\frac{\partial}{\partial x^{i}}\delta^{3}(x-y) \ . \tag{7}$$

As is obvious from the definition, the divergence of $\mathcal{K}_{\mu\nu}(x)$ and $\mathcal{D}_{\mu}(x)$ are connected by

$$\partial^{\mu}\mathcal{K}_{\mu\nu}(x) = 2x_{\nu}\theta^{\mu}_{\mu}(x) = 2x_{\nu}\partial^{\mu}\mathcal{D}_{\mu}(x) \ . \tag{8}$$

Thus, by using relation (7) alone, one can obtain all results contained in (5), (6) and (8) (which are the only relations used by Kastrup).

Consider the T-product $T(\theta^{\mu\nu}(x)\phi(y)\phi(0))$. Applying the methods developed by Gross and Jackiw [8, 4], it can be covariantized by adding the sea-gull terms

$$-iT(S^{\mu\nu}(y)\phi(0))\delta^{4}(x-y) - iT(S^{\mu\nu}(0)\phi(y))\delta^{4}(x) \ ,$$

with

$$S^{\mu\nu}(y) = \tfrac{1}{3}d_{\phi}(g^{\mu\nu} - g^{\mu 0}g^{\nu 0})\phi(y) + \bar{S}^{\mu\nu}(y) \ ,$$

* For renormalizable field theories it can be chosen up to first order by the methods of Callan, Coleman and Jackiw (ref. [7]) and possible improvements thereof by Symanzik (ref. [3]).

and forming the T^* product

$$T^*(\theta^{\mu\nu}(x)\phi(y)\phi(0)) = T(\theta^{\mu\nu}(x)\phi(y)\phi(0))$$
$$- \tfrac{1}{3} i\, d_\phi (g^{\mu\nu} - g^{\mu 0} g^{\nu 0}) T(\phi(y)\phi(0))(\delta^4(x-y) + \delta^4(x))$$
$$- iT(\bar{S}^{\mu\nu}(y)\phi(0))\delta^4(x-y) - iT(\bar{S}^{\mu\nu}(0)\phi(y))\delta^4(x) \,. \qquad (9)$$

Here $\bar{S}^{\mu\nu}(y)$ is a symmetric traceless operator having at most ij components to compensate possible Schwinger terms in the commutator $[\theta_{ij}(x), \phi(y)]$.

For this covariant T^* product the conservation of energy momentum leads to the Ward identity [3, 4]

$$i\,\partial_\mu T^*(\theta^{\mu\nu}(x)\phi(y)\phi(0)) = T(\partial^\nu\phi(y)\phi(0))\delta^4(x-y) + T(\phi(y)\partial^\nu\phi(0))\delta^4(x) \,, \qquad (10)$$

while contraction with $g_{\mu\nu}$ gives the trace identity [3, 4]

$$g_{\mu\nu} T^*(\theta^{\mu\nu}(x)\phi(y)\phi(0)) = T(\theta^\mu_\mu(x)\phi(y)\phi(0))$$
$$- i\, d_\phi T(\phi(y)\phi(0))(\delta^4(x-y) + \delta^4(x)) \,. \qquad (11)$$

Define the gravitational vertex $\Gamma^{\mu\nu}(q,p)$ by taking the two particle propagators out of the three-point function

$$-\Delta(p^2)\Delta((p-q)^2)\Gamma^{\mu\nu}(q,p) \equiv \tau^{\mu\nu}(q,p)$$
$$= \int \mathrm{d}x\,\mathrm{d}y\; \mathrm{e}^{-i(qx-py)} \langle 0|T^*(\theta^{\mu\nu}(x)\phi(y)\phi(0))|0\rangle \,.$$

Similarly define τ and $\Gamma(q^2, p^2, (p-q)^2)$ for the three-point function

$$\langle 0|T(\theta^\mu_\mu(x)\phi(y)\phi(0))|0\rangle \,.$$

Then the Ward identity implies †

$$q_\mu \Gamma^{\mu\nu}(q,p) = -(q-p)^\nu \Delta^{-1}(p^2) - p^\nu \Delta^{-1}((p-q)^2) \,, \qquad \text{(WI)}$$

while the trace identity gives [3, 4]

$$\Gamma^\mu_\mu(q,p) = \Gamma(q^2, p^2, (p-q)^2) - d_\phi(\Delta^{-1}(p^2) + \Delta^{-1}((p-q)^2)) \,. \qquad \text{(TI)}$$

Differentiating the Ward identity (WI) once with respect to q_λ and setting $q = 0$ yields the well-known results ($\mathring{\Delta}(p^2) \equiv (\partial/\partial p^2)\,\Delta(p^2)$)

† The content of the Ward identity is to determine two of the four form factors of $\Gamma_{\mu\nu}$ everywhere in terms of the remaining two and to fix $\Gamma^\mu_\mu(0, \mu^2, \mu^2)$ (eq. (15)).

$$\Gamma_{\mu\nu}(0,p) = -g_{\mu\nu}\Delta^{-1}(p^2) - 2p_\mu p_\nu \Delta^{-2}(p^2)\dot{\Delta}(p^2)\ , \tag{12}$$

$$\Gamma^\mu_\mu(0,p) = -4\Delta^{-1}(p^2) - 2p^2\Delta^{-2}(p^2)\dot{\Delta}(p^2)\ , \tag{13}$$

$$\Gamma(0,p^2,p^2) = -(4-2d_\phi)\Delta^{-1}(p^2) - 2p^2\Delta^{-2}(p^2)\dot{\Delta}(p^2)\ . \tag{14}$$

In eqs. (13) and (14) we can go on the mass shell ($p^2 = \mu^2$) and obtain the equivalence principle

$$\Gamma^\mu_\mu(0,\mu^2,\mu^2) = \Gamma(0,\mu^2,\mu^2) = 2\mu^2 Z^{-1}\ , \tag{15}$$

where Z is the wave function renormalization constant of the field ϕ [$\langle 0|\phi|p\rangle \equiv Z^{\frac{1}{2}}$]. It is therefore often convenient to introduce a properly normalized vertex $\hat{\Gamma} = Z\Gamma$ which becomes $2\mu^2$ on mass shell. Eq. (14) shows us that the dimension of the field d_ϕ enters *only off mass shell* (since $\Delta^{-1}(\mu^2) = 0$). Thus the slope of $\Gamma(q^2 p_1 p_2^2)$ with respect to one of its masses [$\dot{\Gamma} \equiv \partial/\partial p_2^2\,\Gamma$] becomes

$$\dot{\Gamma}(0,p^2,p^2) = \Delta^{-2}(p^2)[(1-d_\phi)\dot{\Delta}(p^2) + 2p^2\Delta^{-1}(p^2)\dot{\Delta}^2(p^2) - p^2\ddot{\Delta}(p^2)]\ . \tag{16}$$

Our ignorance of $\Delta(p^2)$ does not allow us to determine $\dot{\Gamma}$ completely. Let us assume for simplicity that we are dealing with a field that has a pole at $p^2 = \mu^2$ with the next larger singularities being beyond some mass $M^2 \gg \mu^2$. Then we can write for $q^2 \ll M^2$

$$\Delta(p^2) = \frac{f(p^2)}{p^2-\mu^2}\ , \tag{17}$$

with

$$f(\mu^2) = Z\ , \qquad Z^{-1}\dot{f}(\mu^2) = O(\frac{\mu^2}{M^2})\ , \qquad Z^{-1}\dot{f}(0) = O(\frac{\mu^2}{M^2})\ , \tag{18}$$

and we obtain

$$\dot{\hat{\Gamma}}(0,p^2,p^2) = -(1-d_\phi) + O(\frac{\mu^2}{M^2},\frac{p^2}{M^2})\ . \tag{19}$$

If ϕ is a good interpolating field for the pion, μ^2/M^2 is commonly assumed to be small and $\dot{\hat{\Gamma}}$ is determined by the dimension of ϕ. As one expects, eq. (19) states that $\Gamma(0,p^2,p^2)$ becomes smoothest in p^2 if d_ϕ has the canonical value one.

There is no relation one can obtain, at this level, for $\Gamma'(0,p^2,p^2)$, where $' \equiv \partial/\partial q^2$. Let us see how Kastrup was led to his wrong result $\hat{\Gamma}'(0,\mu^2,\mu^2) = -\frac{1}{2}(\frac{7}{2} - d_\phi)$. He considers the Ward identity for the current $j_\mu = x_m \mathcal{K}_{\mu m}(x)$:

$$\partial^\mu \langle 0|T^*(x_m \mathcal{K}_{\mu m}(x)\phi(y)\phi(0))|0\rangle = \langle 0|T^* \mathcal{K}_{mm}(x)\phi(y)\phi(0))|0\rangle$$

$$+ x_m \langle 0|T^*([\mathcal{K}_{om}(x), \phi(y)]\,\phi(0))|0\rangle\,\delta(x^o - y^o)$$

$$+ 2x_m^2 \langle 0|T^*(\theta^\mu_\mu(x)\phi(y)\phi(0))|0\rangle\ , \qquad (20)$$

and obtains for the Fourier transform τ †:

$$2\,\frac{\partial^2}{\partial q^{m\,2}}\,\tau(q^2, p^2, (p-q)^2)\Big|_{\substack{q=0\\ \boldsymbol{p}=0}}$$

$$= -(2\partial_m\partial_m(d_\phi - \partial p) + \partial^2\partial_m p_m)\Delta(p^2)\Big|_{\substack{q=0\\ \boldsymbol{p}=0}} + R(p^2)\ , \qquad (21)$$

where

$$R(p^2) = \int \mathrm{d}x\,\mathrm{d}y\ \mathrm{e}^{-i(qx-py)}\langle 0|T^*(\mathcal{K}_{mm}(x)\phi(y)\phi(0))|0\rangle\Big|_{\substack{q=0\\ \boldsymbol{p}=0}}\ . \qquad (22)$$

He then neglects $\Delta^{-2}(p^2)R(p^2)$ on the mass shell using the following argument: Since the self-stress of a particle at rest vanishes

$$\langle(\mu, \boldsymbol{p}=0)|\theta_{mn}(x)|(\mu, \boldsymbol{p}=0)\rangle = 0\ ,$$

one can conclude that

$$\lim_{\substack{q\to 0\\ \boldsymbol{p}=0}} \int \mathrm{d}x\ \mathrm{e}^{-iqx}\langle p|\mathcal{K}_{mn}(x)|p-q\rangle = 0\ .$$

But this step involves exchanging the limit with the integral, which is not permissible, due to the occurrence of factors x in the integrand (producing second moments of the self-stress). In evaluating the remainder of eq. (21) he finds his result (1). A correct calculation of (21) gives, however,

$$\Delta^2(p^2)[\Gamma'(0,p^2,p^2) + \dot\Gamma(0,p^2,p^2)] + \Delta(p^2)\dot\Delta(p^2)\Gamma(0,p^2,p^2)$$

$$= (d-3)\dot\Delta(p^2) - p^2\ddot\Delta(p^2) + \tfrac{1}{4}R(p^2)\ . \qquad (23)$$

Inserting in this equation the result (16), following from our Ward identity, we find for all p^2

† The T^* product is necessary to make the Schwinger terms occurring in the commutator $[\mathcal{K}_{om}(x), \phi(y)]$ cancel against the sea-gulls covariantizing the remaining terms.

$$4\Delta^2(p^2)\Gamma'(0,p^2,p^2) = R(p^2) \ . \qquad (24)$$

Thus, assuming $\Delta^{-2}(\mu^2)R(\mu^2)$ to vanish amounts to $\Gamma'(0, \mu^2, \mu^2) = 0$ (instead of eq. (1)†). While $\Gamma'(0, \mu^2, \mu^2)$ is true for a free scalar field, it does not hold in general. The σ-model gives for example $\hat{\Gamma}'(0, \mu^2, \mu^2) = 1 - \mu^2/m_\sigma^2$.

As argued in the beginning of this section, every result contained in (5), (6) and (8) can also be derived directly from (7). Indeed, differentiating the Ward identity thrice and setting $q = 0$, $\boldsymbol{p} = 0$ produces again (24).

3. INCLUSION OF CHIRAL SU(2) × SU(2)

While an on-shell result for Γ' cannot be obtained by the methods developed up to here, it is quite simple to derive such a result for pions. For this we have to make additional assumptions and blend conformal properties with chiral information. We assume PCAC and that the SU(2) × SU(2) symmetry of the world is broken in the standard way by‡

$$\theta_{00}(x) = \bar{\theta}_{00}(x) + \theta_4(x) \ , \qquad (25)$$

where $\bar{\theta}_{00}$ is an SU(2) × SU(2) singlet and θ_4 is a scalar field transforming, together with $\partial^\mu A^a_\mu$, as a representation $(\frac{1}{2}\frac{1}{2})$ of chiral SU(2) × SU(2), i.e.

$$[A^a_0(x), \theta_4(y)]_{x_0=y_0} = i\partial^\mu A^a_\mu(x)\delta^3(x-y) \ , \qquad (26)$$

$$[A^a_0(x), \partial^\mu A^b_\mu(y)]_{x_0=y_0} = -i\theta_4(x)\delta^{ab}\,\delta^3(x-y) \ . \qquad (27)$$

Let us now assume that θ_4 has a definite dimension d

$$i[D(x_0), \theta_4(x)]_{x_0=y_0} = (x\partial + d)\theta_4(x) \ . \qquad (28)$$

Assuming that the time components of the SU(2) × SU(2) currents $V^a_0(x)$, $A^a_0(x)$ have dimension three, which is consistent with the current commutation rules, it follows from (27) that $\partial^\mu A^a_\mu(x)$ has the same dimension d.

Finally, we assume that all parts in $\bar{\theta}_{00}$ with dimension not equal to four are Lorentz (and chiral) scalars. Thus, it can be shown that

$$\theta(x) = w(x) + (4-d)\theta_4(x) \ , \qquad (29)$$

† His error is probably due to the fact that he operates with the three point function still containing the over-all momentum δ-function. This makes his calculation corresponding to eq. (21) extremely tedious.

‡ Which can be derived, for example, from the superconvergence of the $I_t = 2$ amplitude in the s-channel.

where $w(x)$ is some SU(2) × SU(2) singlet field [9]. Therefore we find

$$[A_0^a(x), \theta(y)]_{x_0=y_0} = i(4-d)\,\partial^\mu A_\mu^a(x)\delta^3(x-y) \ . \tag{30}$$

The commutation rules (27) and (30) allow us to derive a Ward identity for the gravitational vertex Γ of the three-point function $T(\theta(x)\,\partial^\mu A_\mu^a(y)\,\partial^\nu A_\nu^a(0))$. By Fourier transforming the equation (no sum over a)

$$\partial_y^\mu \langle 0|T(\theta(x)A_\mu^a(y)\partial^\nu A_\nu^a(0))|0\rangle = \langle 0|T(\theta(x)\,\partial^\mu A_\mu^a(y)\partial^\nu A_\nu^a(0))|0\rangle$$
$$+ \langle 0|T(\theta(x)\theta_4(y))|0\rangle\,\delta^4(y) + (4-d)\,\langle 0|T(\partial^\mu A_\mu^a(y)\partial^\nu A_\nu^a(0))|0\rangle\,\delta^4(x-y) \ , \tag{31}$$

and setting $p = 0$ we obtain

$$\Gamma(q^2, 0, q^2) = \Delta^{-1}(0)\,[\Delta_{\theta\theta_4}(q^2)\Delta^{-1}(q^2) - (4-d)] \ , \tag{32}$$

where $\Delta(q^2)$ is now the propagator of the divergence of the axial-vector field. From this we find by differentiating

$$(\Gamma' + \dot\Gamma)(0,0,0) = \Delta^{-2}(0)\,[\dot\Delta_{\theta\theta_4}(0) - \Delta_{\theta\theta_4}(0)\Delta^{-1}(0)\dot\Delta(0)] \ . \tag{33}$$

The derivative $\dot\Gamma$ can be eliminated using our earlier result (17) at $p^2 = 0$:

$$\dot\Gamma(0,0,0) = (1-d)\Delta^{-2}(0)\dot\Delta(0) \ . \tag{34}$$

Hence we obtain

$$\Gamma'(0,0,0) = \Delta^{-2}(0)\,[\dot\Delta_{\theta\theta_4}(0) - \Delta_{\theta\theta_4}(0)\Delta^{-1}(0)\dot\Delta(0) - (1-d)\dot\Delta(0)] \ .$$

This result can be simplified by comparing (32) with (14) at all arguments zero. We find [6] †

$$\Delta_{\theta\theta_4}(0) = d\Delta(0) \ . \tag{36}$$

Inserting this in eq. (34) we arrive at the exact result

$$\Gamma'(0,0,0) = \Delta^{-2}(0)\,[\dot\Delta_{\theta\theta_4}(0) - \dot\Delta(0)] \ . \tag{37}$$

Let us see what this result implies if we assume, for small q^2, PCAC in the form

$$\Delta(q^2) = \frac{(f\mu^2)^2}{q^2 - \mu^2} \ , \tag{38}$$

† Note that this result can be obtained more directly by considering Ward identities for the two-point functions $T(D_\mu(x)\theta_4(0))$ and $T(A_\mu(x)\partial^\nu A_\nu(0))$. One finds in addition to (36) that $\Delta(0) = \langle 0|\theta_4|0\rangle$ (cf., the σ-model where $\Delta(0) = -f^2\mu^2$, $\theta_4 = -f\mu^2\sigma$, and $\langle\sigma\rangle = f$).

where in the following $\mu = m_\pi$, and σ-dominance of $\Delta_{\theta\theta 4}$ as

$$\Delta_{\theta\theta 4}(q^2) = \frac{(f\mu^2)^2 g}{q^2 - m_\sigma^2} . \tag{39}$$

From eq. (36) we determine $g = d(m_\sigma^2/\mu^2)$ and inserting this into eq. (37) gives †

$$\hat{\Gamma}'(0,0,0) = 1 - d\,\frac{\mu^2}{m_\sigma^2} . \tag{40}$$

Eqs. (19) for $\hat{\Gamma}(0,0,0)$ and eq. (40) can be combined to obtain an on-mass shell result for $\Gamma'(0, \mu^2, \mu^2)$. From eq. (32) we see that as a consequence of the σ-pole dominance of $\Delta_{\theta\theta 4}$, also Γ is dominated by a σ-pole. The smoothest Γ-function containing a σ-pole and fulfilling eqs. (15), (19) and (40) is given by

$$\hat{\Gamma}(q^2, p_1^2, p_2^2) = \frac{aq^2 + bm_\sigma^2}{q^2 - m_\sigma^2} , \tag{41}$$

where

$$a = -m_\sigma^2 + (4 - d)\mu^2 ,$$

$$b = -2\mu^2 + (1 - d)(p_1^2 + p_2^2 - 2\mu^2) . \tag{42}$$

From this we obtain our final result

$$\hat{\Gamma}'(0, \mu^2, \mu^2) = 1 + (d - 2)\,\frac{\mu^2}{m_\sigma^2} . \tag{43}$$

We can compare this result with the σ-model where $d = 1$ and $\hat{\Gamma}'(0, \mu^2, \mu^2) = 1 - (\mu^2/m_\sigma^2)$. It is amusing to note that for $d = 1$ the on-shell slope (43) and the off-shell slope (40) are exactly the same as can be checked in the σ-model as well.

4. CONCLUSION

Our result (43) can be used to calculate the $\sigma\pi\pi$ coupling constant $g_{\sigma\pi\pi}$ in terms of the σ gravitational coupling constant $\gamma^{-1} m_\sigma^3 \equiv \langle 0 | \theta | \sigma \rangle$:

$$g_{\sigma\pi\pi} = \gamma \left[1 + (d - 2)\,\frac{\mu^2}{m_\sigma^2} \right] ; \qquad \left(\Gamma_{\sigma\pi\pi} = \tfrac{3}{4} q\,\frac{g_{\sigma\pi\pi}^2}{4\pi} \right) .$$

† Our result holds up to order $O(\mu^2/M^2, m_\sigma^2/M^2)$ and in principle for any possible ratio μ^2/m_σ^2. (Only experimentally it happens that μ^2/m_σ^2 is as small as the terms neglected such that $g_{\sigma\pi\pi} \approx \gamma$.) This approximation is the same as is implied by effective Lagrangians using σ- and π-fields.

In a forthcoming paper we shall obtain independent information on γ and discuss consequences of the dimensional properties of the SU(3) $\times$ SU(3) decomposition of the energy momentum tensor.

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