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# Simple explicit formulas for Gaussian path integrals with time-dependent frequencies

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## Abstract

Quadratic fluctuations require an evaluation of ratios of functional determinants of second-order differential operators. We relate these ratios to the Green functions of the operators for Dirichlet, periodic and antiperiodic boundary conditions on a line segment. This permits us to take advantage of Wronski's construction method for Green functions without knowledge of eigenvalues. Our final formula expresses the ratios of functional determinants in terms of an ordinary  $2 \times 2$  determinant of a constant matrix constructed from two linearly independent solutions of the homogeneous differential equations associated with the second-order differential operators. For ratios of determinants encountered in semiclassical fluctuations around a classical solution, the result can further be expressed in terms of this classical solution. In the presence of a zero mode, our method allows for a simple universal regularization of the functional determinants. For Dirichlet's boundary condition, our result is equivalent to Gelfand–Yaglom's. Explicit formulas are given for a harmonic oscillator with an arbitrary time-dependent frequency. © 1998 Published by Elsevier Science B.V.

## 1. Introduction

Evaluation of Gaussian path integrals is needed in many physical problems, notably in all semiclassical calculations of fluctuating systems. Typically, we are confronted with a ratio of functional determinants of second-order differential operators [1]. For Dirichlet boundary conditions encountered in quantum mechanical fluctuation problems, a general result has been found by Gelfand and Yaglom [2]. Working with time-sliced path integrals, they reduced the evaluation to a simple initial-value problem for the homogeneous second order differential equations associated with the

above operators. The functional determinants are directly given by the value of the solutions at the final point. Unfortunately, Gelfand and Yaglom's method becomes rather complicated for the periodic and antiperiodic boundary conditions of quantum statistics (see Subsection 2.12 in Ref. [1]), and has therefore rarely been used. Several papers have studied the functional determinants of second-order Sturm–Liouville operators with periodic boundary conditions [3–6], and related them to boundary-value problems. The calculated determinants are all singular and were regularized with the help of generalized zeta-functions [7]. This has the disadvantage of a physical quantity depending unnecessarily on the analyticity properties of generalized zeta-functions. Moreover, the auxiliary boundary-value problems were formulated in terms

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of first-order operators, rather than the initial second-order one, making the treatment of a zero mode of operator with periodic boundary conditions unclear, and requiring additional work [8].

In this paper we shall avoid the above drawbacks by developing a simple and systematic method for finding ratios of functional determinants of second-order differential operators with Dirichlet, periodic and antiperiodic boundary conditions. By focusing our attention upon ratios instead of the determinants themselves, we avoid the need of regularization. The main virtue of our method is that it takes advantage of the existence of Wronski's simple construction rule for Green functions. This permits us to reduce the functional determinants to an ordinary constant  $2 \times 2$ -determinant formed from solutions of homogeneous differential equations associated with the differential operators. For semiclassical fluctuations around a classical solution, our final result will be expressed entirely in terms of a classical trajectory. Furthermore, for fluctuation operator with a zero mode, a case frequently encountered in many semiclassical calculations, we find a simple universal expression for the regularized ratio of determinants without the zero mode.

## 2. Basic relations

The typical fluctuation action arising in semiclassical approximations has the form

$$\begin{aligned} \mathcal{A}[x] &= \int_{t_a}^{t_b} dt L(\dot{x}(t), x(t)) \\ &= \int_{t_a}^{t_b} dt \frac{M}{2} [\dot{x}^2 - \Omega^2(t)x^2]. \end{aligned} \quad (1)$$

The time-dependent frequency  $\Omega(t)$  can be expressed in terms of the potential  $V(x)$  of the system as

$$\Omega^2(t) = V''(x_{\text{cl}}(t))/M, \quad (2)$$

where  $x_{\text{cl}}(t)$  is a classical trajectory solving the equation of motion (for examples see Subsection 17.3 of Ref. [1]).

$$M\ddot{x} = -V'(x). \quad (3)$$

The action (1) describes a harmonic oscillator with a time-dependent frequency  $\Omega(t)$ . partition function contain a phase factor  $\exp(i\mathcal{A}_{\text{cl}})$ , where  $\mathcal{A}_{\text{cl}} = \mathcal{A}[x_{\text{cl}}]$  is the action of the classical path  $x_{\text{cl}}(t)$ . The phase factor is multiplied by a fluctuation factor proportional to

$$F(t_b, t_a) \sim \left( \frac{\text{Det } K_1}{\text{Det } \tilde{K}} \right)^{-1/2}, \quad (4)$$

where  $K_1 = -\partial_t^2 - \Omega^2(t) \equiv K_0 - \Omega^2(t)$  is obtained as the operator governing the second variation of the action  $\mathcal{A}[x]$  along the classical path  $x_{\text{cl}}(t)$ :

$$\frac{\delta^2 \mathcal{A}[x_{\text{cl}}]}{\delta x(t) \delta x(t')} = \delta(t - t') K_1. \quad (5)$$

The ratio of determinants (4) arises naturally from the normalization of the path integral [1] and is well defined. The linear operator  $K_1$  acts on the space of twice differentiable functions  $y(t) = \delta x(t)$  on an interval  $t \in [t_a, t_b]$  with appropriate boundary conditions. In the quantum-mechanical fluctuation problem, these are Dirichlet-like with  $y(t_b) = y(t_a) = 0$ , in the quantum-statistical case, they are periodic or antiperiodic with  $y(t_b) = \pm y(t_a)$  and  $\dot{y}(t_b) = \pm \dot{y}(t_a)$ . The operator  $\tilde{K}$  in the denominator of (4) may be chosen as  $K_0$  (Dirichlet case) or  $K_1^\omega \equiv K_0 - \omega^2$  (periodic and antiperiodic cases), respectively, where  $\omega$  is a time-independent oscillator frequency. Then the operator  $\tilde{K}$  is invertible, having the Fredholm property

$$\frac{\text{Det } K_1}{\text{Det } \tilde{K}} = \text{Det } \tilde{K}^{-1} K_1 \quad (6)$$

(a possible multiplicative anomaly being equal to unity [9]). Furthermore, since the operator  $\tilde{K}^{-1} K_1$  is of the form  $I + B$ , with  $B$  an operator of the trace class, it has a well-defined determinant without any regularization.

To calculate  $F(t_b, t_a)$ , we introduce a one-parameter family of operators

$$K_g \equiv -\partial_t^2 - g\Omega^2(t) \quad (7)$$

depending linearly on the parameter  $g \in [0, 1]$ , and reducing to the initial operator  $K_1$  for  $g = 1$ . Then we consider the eigenvalue problem

$$K_g(t) y_n(g; t) = \lambda_n(g) y_n(g; t), \quad (8)$$

with eigenvalues  $\lambda_n(g)$ . The eigenfunctions  $y_n(g; t)$  satisfy the orthonormality and completeness relations

$$\int_{t_a}^{t_b} dt y_n(g; t) y_m(g; t) = \delta_{nm}, \tag{9}$$

$$\sum_n y_n(g; t) y_n(g; t') = \delta(t - t'). \tag{10}$$

The completeness relation permits us to write down immediately a spectral representation for the Green function  $G_g(t, t')$  associated with the differential equation (8). By applying  $K_g(t)$  to

$$G_g(t, t') = \sum_{n=1}^{\infty} \frac{y_n(g; t) y_n(g; t')}{\lambda_n(g)}, \tag{11}$$

and using (9), (10), we verify the validity of the defining differential equation

$$K_g(t) G_g(t, t') = \delta(t - t'). \tag{12}$$

In terms of the eigenvalues  $\lambda_n(g)$ , the determinant (6) would read

$$\text{Det } \tilde{K}^{-1} K_g = C \prod_{n=1}^{\infty} \frac{\lambda_n(g)}{\lambda_n(0)}, \tag{13}$$

where  $C = \text{Det}(\tilde{K}^{-1} K_0)$  is a constant of the  $g$ -integration, which still may depend on  $t_b, t_a$ . Since the infinite product of ratio of the eigenvalues  $\lambda_n$  in Eq. (13) converges uniformly for all  $g \in [0, 1]$ , we can differentiate this equation to obtain

$$\partial_g \log \text{Det } \tilde{K}^{-1} K_g = \sum_{n=1}^{\infty} \frac{\lambda'_n(g)}{\lambda_n(g)}. \tag{14}$$

Differentiating Eq. (8), and using the condition (10) gives for all boundary conditions,

$$\lambda'_n(g) = - \int_{t_a}^{t_b} dt \Omega^2(t) y_n^2(g; t). \tag{15}$$

This may be inserted into (14). Because of the convergence of sum in (14), summation and integration can be interchanged, and using the spectral representation (11) we find the compact formula

$$\begin{aligned} \partial_g \log \text{Det}(\tilde{K}^{-1} K_g) &= -\text{Tr}\{\Omega^2(t) G_g(t, t')\}. \\ &= - \int_{t_a}^{t_b} dt \Omega^2(t) G_g(t, t). \end{aligned} \tag{16}$$

By integrating this equation in  $g$ , we obtain the ratio of functional determinants (6), in the form.

$$\begin{aligned} \text{Det } \tilde{K}^{-1} K_g &= C \exp \left( - \int_0^g dg' \int_{t_a}^{t_b} dt \Omega^2(t) G_{g'}(t, t) \right) \end{aligned} \tag{17}$$

with the same integration constant  $C$  as in Eq. (13). It is fixed calculating the same expression for  $g = 0$  where the left-hand side is well known. In the case of Dirichlet boundary conditions, where  $\tilde{K} = K_0$ , the left-hand side is trivially unity. For periodic and antiperiodic boundary conditions, where we take  $\tilde{K} = K_0 - \omega^2 = -\partial_t^2 - \omega^2$ , the most convenient way to normalize the right-hand side is to go to  $g = 1$  and choose the frequency  $\Omega^2(t)$  to be equal to the constant frequency  $\omega^2$ . The left-hand side is again unity thus fixing  $C$ .

Having determined  $C$  we set  $g = 1$  in Eq. (17) and obtain the final result for the operator  $K_1$ . In the sequel we shall evaluate the right-hand side of formula (17) using explicit Wronski constructions of the Green function for the different boundary conditions.

### 3. Wronski's construction of green functions

The general solution of the differential equation (12) may be expressed in terms of retarded and advanced Green functions which have the general form

$$G_g^{(-)}(t, t') = G_g^{(+)}(t', t) = \Theta_{tt'} \Delta_g(t, t'), \tag{18}$$

where  $\Theta_{tt'} \equiv \Theta(t - t')$  is Heaviside's step function which vanishes for  $t < t'$  and is equal to unity for  $t > t'$ . The function  $\Delta_g(t, t')$  satisfies the homogeneous differential equation corresponding to (12). This is seen by applying the operator  $K_g(t)$  to (18) and making use of the identity  $t\delta'(t) = -\delta(t)$ :

$$\begin{aligned} K_g(t) G_g^{(-)}(t, t') &= \Theta_{tt'} K_g(t) \Delta_g(t, t') \\ &+ \left( \frac{\Delta_g(t, t')}{(t - t')} - 2\partial_t \Delta_g(t, t') \right) \delta(t - t'). \end{aligned} \tag{19}$$

Since the right-hand side must be equal to  $\delta(t - t')$ , the function  $\Delta_g(t, t')$  has to satisfy the homogeneous differential equation

$$K_g(t)\Delta_g(t, t') = 0, \quad \text{for } t > t', \quad (20)$$

while the bracket in (19) must be equal to 1 at  $t = t'$ . Upon expanding  $\Delta_g(t, t')$  around  $t = t'$ , this leads to the conditions

$$\Delta_g(t, t) = 0, \quad \partial_t \Delta_g(t, t')|_{t=t'} = -1. \quad (21)$$

Eq. (20) is solved by a linear combination

$$\Delta_g(t, t') = \alpha_g(t')\eta_g(t) + \beta_g(t')\xi_g(t) \quad (22)$$

of any two independent solutions  $\eta_g(t)$  and  $\xi_g(t)$  of the homogeneous equation

$$K_g(t)h_g(t) = [-\partial_t^2 - g\Omega^2(t)]h_g(t) = 0. \quad (23)$$

Their time-independent Wronski determinant  $W_g = \eta_g \dot{\xi}_g - \dot{\eta}_g \xi_g$  is nonzero, so that we can determine the coefficients in the linear combination (22) from (21) and find

$$\begin{aligned} \Delta_g(t, t') &= \frac{1}{W_g} [\eta_g(t)\dot{\xi}_g(t') - \dot{\xi}_g(t)\eta_g(t')] \\ &= -\Delta_g(t', t). \end{aligned} \quad (24)$$

The right-hand side contains the so-called Jacobi commutator of the two functions  $\eta_g(t)$  and  $\xi_g(t)$ . Here we list a few algebraic properties of  $\Delta_g(t, t')$  which will be useful in the sequel:

$$\Delta_g(t, t') = \frac{\Delta_g(t_b, t)\Delta_g(t', t_a) - \Delta_g(t_b, t')\Delta_g(t, t_a)}{\Delta_g(t_a, t_b)}, \quad (25)$$

$$\begin{aligned} \Delta_g(t_b, t)\partial_{t_b}\Delta_g(t_b, t_a) - \Delta_g(t, t_a) \\ = \Delta_g(t_b, t_a)\partial_t\Delta_g(t_b, t), \end{aligned} \quad (26)$$

$$\begin{aligned} \Delta_g(t, t_a)\partial_{t_b}\Delta_g(t_b, t_a) + \Delta_g(t_b, t) \\ = \Delta_g(t_b, t_a)\partial_t\Delta_g(t, t_a). \end{aligned} \quad (27)$$

Note that the solution (18) is so far not unique, leaving room for an additional general solution of the homogeneous equation (23),

$$G_g(t, t') = \Theta_{t't'}\Delta_g(t, t') + a_g(t')\eta_g(t) + b_g(t')\xi_g(t) \quad (28)$$

with arbitrary coefficients  $a_g(t')$  and  $b_g(t')$ . This ambiguity is removed by appropriate boundary conditions.

Consider first the quantum mechanical fluctuating problem with Dirichlet boundary conditions  $y(g; t_b) = y_g(t_a) = 0$  for the eigenfunctions  $y(g; t)$  of  $K_g$ , implying for the Green function the boundary conditions

$$\begin{aligned} G_g(t_b, t) &= 0, \quad t_b \neq t, \\ G_g(t, t_a) &= 0, \quad t \neq t_a. \end{aligned} \quad (29)$$

Substituting (28) into (29) leads to a simple algebraic pair of equations

$$a_g(t)\eta_g(t_a) + b_g(t)\dot{\xi}_g(t_a) = 0, \quad (30)$$

$$a_g(t)\eta_g(t_b) + b_g(t)\dot{\xi}_g(t_b) = -\Delta(t_b, t). \quad (31)$$

We now define a fundamental matrix  $A_g$  as the constant  $2 \times 2$  matrix,

$$A_g = \begin{pmatrix} \eta_g(t_a) & \dot{\xi}_g(t_a) \\ \eta_g(t_b) & \dot{\xi}_g(t_b) \end{pmatrix}, \quad (32)$$

and observe that under the condition

$$\det A_g = W_g \Delta_g(t_a, t_b) \neq 0, \quad (33)$$

the system (31) has a unique solution, so that the coefficients  $a_g(t)$  and  $b_g(t)$  in the Green function (28) are easily calculated. Making use of identity (25), we obtain Wronski's well-known formula

$$\begin{aligned} G_g(t, t') &= \\ &= \frac{\Theta_{t't'}\Delta_g(t_b, t)\Delta_g(t', t_a) + \Theta_{t't}\Delta_g(t_b, t')\Delta_g(t, t_a)}{\Delta_g(t_a, t_b)}. \end{aligned} \quad (34)$$

For Dirichlet boundary conditions, this equation yields a unique and well-defined Green function assuming the absence of a zero mode of the operator  $K_1$  with these boundary conditions. Such a mode would cause problems since  $\eta_1(t_a) = \eta_1(t_b) = 0$  would make  $\det A_1 = 0$ , thus destroying the property (33) which was necessary to find (34). Indeed, the Wronski expression (24) is undetermined since the boundary condition  $\eta_1(t_a) = 0$  together with (30) imply  $\dot{\xi}_1(t_a) = 0$ , making  $W_1 = \eta_1 \dot{\xi}_1 - \dot{\eta}_1 \xi_1$  vanish at the initial time  $t_a$  and thus identically in  $t$ .

Consider now the quantum statistical fluctuation problem with periodic or antiperiodic boundary conditions  $y(g; t_b) = \pm y(g; t_a)$ ,  $\dot{y}(g; t_b) = \pm \dot{y}(g; t_a)$  for the eigenfunctions  $y(g; t)$  of the operator  $K_g(t)$ . For the Green function

$$G_g^p(t, t'),$$

these imply

$$\begin{aligned} G_g^p(t_b, t') &= \pm G_g^p(t_a, t'), \\ \dot{G}_g^p(t_b, t') &= \pm \dot{G}_g^p(t_a, t'). \end{aligned} \quad (35)$$

In both cases, the frequency  $\Omega(t)$  and the Dirac delta function in Eq. (12) are also assumed to be periodic or antiperiodic in time with the same period. Inserting (28) into (35) gives now the equations

$$\begin{aligned} a(t)(\eta_b \mp \eta_a) + b(t)(\xi_b \mp \xi_a) &= -\Delta(t_b, t), \\ a(t)(\dot{\eta}_b \mp \dot{\eta}_a) + b(t)(\dot{\xi}_b \mp \dot{\xi}_a) &= -\partial_t \Delta(t_b, t). \end{aligned} \quad (36)$$

For brevity, we have omitted the subscripts  $g$  and written  $\xi_{a,b}, \eta_{a,b}$  for  $\xi_g(t_{a,b}), \eta_g(t_{a,b})$ . Defining now the constant  $2 \times 2$  matrices

$$\bar{A}_g^p = \begin{pmatrix} \eta_b \mp \eta_a & \xi_b \mp \xi_a \\ \dot{\eta}_b \mp \dot{\eta}_a & \dot{\xi}_b \mp \dot{\xi}_a \end{pmatrix} \quad (37)$$

the condition analogous to (33)

$$\det \bar{A}_g^p = W_g \bar{\Delta}_g^p(t_a, t_b) \neq 0 \quad (38)$$

with

$$\bar{\Delta}_g^p(t_a, t_b) = 2 \pm \partial_t \Delta_g(t_a, t_b) \pm \partial_t \Delta_g(t_b, t_a) \quad (39)$$

enables us to obtain the unique solution to Eqs. (36). After some algebra using the identities (26) and (27), the expression (28) for Green functions with periodic and antiperiodic boundary conditions (35) can be cast into the form

$$\begin{aligned} G_g^p(t, t') &= G_g(t, t') \\ &\mp \frac{[\Delta_g(t, t_a) \pm \Delta_g(t_b, t)][\Delta_g(t', t_a) \pm \Delta_g(t_b, t')]}{\bar{\Delta}_g^p(t_a, t_b) \Delta_g(t_a, t_b)}. \end{aligned} \quad (40)$$

The right-hand side is well defined unless the operator  $K_1$  has a zero mode with  $\eta_b = \pm \eta_a$ ,  $\dot{\eta}_b = \pm \dot{\eta}_a$ , which would make the determinant of the  $2 \times 2$ -matrix  $\bar{A}_g^p$  vanish.

Note that the Green functions (34) and (40) are both continuous at  $t = t'$ , as is necessary for calculating the associated ratios of functional determinants from formula (17), which we shall now do.

#### 4. Main results and relation to Gelfand–Yaglom’s initial-value problem

Excluding at first zero modes, we evaluate formula (17) for ratios of functional determinants. The temporal integral on the right-hand side can be performed efficiently following Ref. [10]. Here we present an even more direct method, by which we express the result in terms of solutions of Gelfand–Yaglom’s initial-value problem for Dirichlet boundary conditions, and of a dual problem for periodic and antiperiodic boundary conditions.

##### 4.1. Dirichlet case

The Gelfand–Yaglom initial-value problem consists in the search for a function  $D_g(t)$  solving the following equations,

$$K_g(t)D_g(t) = 0, \quad D_g(t_a) = 0, \quad \dot{D}_g(t_a) = 1. \quad (41)$$

By differentiating these three equations with respect to the parameter  $g$ , we obtain for  $D'_g(t) \equiv \partial_g D_g(t)$  the inhomogeneous initial-value problem

$$\begin{aligned} K_g(t)D'_g(t) &= \Omega^2(t)D_g(t), \\ D'_g(t_a) &= 0, \quad \dot{D}'_g(t_a) = 0. \end{aligned} \quad (42)$$

The unique solution of Eqs. (41) can easily be expressed in terms of our arbitrary set of solutions  $\eta_g(t)$  and  $\xi_g(t)$  as follows:

$$D_g(t) = \frac{\eta_g(t_a)\xi_g(t) - \xi_g(t_a)\eta_g(t)}{W_g} = \Delta_g(t_a, t), \quad (43)$$

thus leading to

$$D_g(t_b) = \frac{\text{Det } A_g}{W_g} = \Delta_g(t_a, t_b). \tag{44}$$

In terms of the same functions, the general solution of the inhomogeneous initial-value problem (42) can be seen to have the form

$$D'_g(t) = \int_{t_a}^t dt' \Omega^2(t') \Delta_g(t, t') \Delta_g(t_a, t'). \tag{45}$$

Comparison with (34) shows that at the final point  $t = t_b$ ,

$$D'_g(t_b) = -\Delta_g(t_a, t_b) \int_{t_a}^{t_a} dt \Omega^2(t) G_g(t, t), \tag{46}$$

which together with (44) implies the following simple relation for the Green function (34) with Dirichlet's boundary conditions,

$$\begin{aligned} \text{Tr} [\Omega^2(t) G_g(t, t')] &= -\partial_g \log \left( \frac{\det A_g}{W_g} \right) \\ &= -\partial_g \log D_g(t_b). \end{aligned} \tag{47}$$

Inserting this into (17), we find for the ratio of functional determinants the simple formula

$$\text{Det } K_0^{-1} K_g = C D_g(t_b). \tag{48}$$

The constant of integration is fixed by applying (48) to the trivial case  $g = 0$ , where  $K_0 = -\partial_t^2$  and the solution to the initial-value problem (41) is

$$D_0(t) = t - t_a. \tag{49}$$

At  $g = 0$ , the left-hand side of (50) is unity, determining  $C = (t_b - t_a)^{-1}$  and the final result for  $g = 1$ :

$$\text{Det } K_0^{-1} K_1 = \frac{\det A_1/W_1}{\text{Det } A_0/W_0} = \frac{D_1(t_b)}{t_b - t_a}. \tag{50}$$

This compact formula was first derived by Gelfand and Yaglom [2] via a direct calculation of the determinant arising in a time-sliced path integrals [1].

#### 4.2. Periodic and antiperiodic case

Our technique makes it straight-forward to derive an equally compact formula for periodic and antiperiodic boundary conditions. For this purpose we introduce another homogeneous initial-value problem

whose boundary conditions are dual to Gelfand and Yaglom's in (41):

$$K_g(t) \bar{D}_g(t) = 0, \quad \bar{D}_g(t_a) = 1, \quad \dot{\bar{D}}_g(t_a) = 0. \tag{51}$$

In terms of the previous arbitrary set  $\eta_g(t)$  and  $\xi_g(t)$  of solutions of the homogeneous differential equation, the unique solution of (51) reads

$$\bar{D}_g(t) = \frac{\eta_g(t) \dot{\xi}_g(t_a) - \xi_g(t) \dot{\eta}_g(t_a)}{W_g}. \tag{52}$$

This can be combined with the time derivative of (43) at  $t = t_b$  to yield

$$\dot{D}_g(t_b) + \dot{\bar{D}}_g(t_b) = \pm [2 - \dot{\bar{A}}_g^p(t_a, t_b)]. \tag{53}$$

By differentiating Eqs. (51) with respect to  $g$ , we obtain the following inhomogeneous initial-value problem for  $\bar{D}'_g(t) = \partial_g \bar{D}_g(t)$ ,

$$\begin{aligned} K_g(t) \bar{D}'_g(t) &= \Omega^2(t) \bar{D}'_g(t), \\ \bar{D}'_g(t_a) &= 0, \quad \dot{\bar{D}}'_g(t_a) = 0, \end{aligned} \tag{54}$$

whose general solution reads in analogy to (45)

$$\bar{D}'_g(t) = - \int_{t_a}^t dt' \Omega^2(t') \Delta_g(t, t') \dot{\Delta}_g(t_a, t'), \tag{55}$$

where the dot denotes the time derivative with respect of the first argument of  $\Delta_g(t, t')$ . With the help of identities (26) and (27), the combination  $\dot{D}'(t) + \dot{\bar{D}}'_g(t)$  at  $t = t_b$  can now be expressed in terms of the periodic and antiperiodic Green functions (40), in analogy to (46),

$$\dot{D}'_g(t_b) + \dot{\bar{D}}'_g(t_b) = \pm \dot{\bar{A}}_g^p(t_a, t_b) \int_{t_a}^{t_b} dt \Omega^2(t) G_g^p(t, t). \tag{56}$$

Together with (53), this yields for the temporal integral on the right-hand sides of (16) and (17) the simple expression analogous to (47)

$$\begin{aligned} \text{Tr} [\Omega^2(t) G_g^p(t, t')] &= -\partial_g \log \left( \frac{\det \bar{A}_g^p}{W_g} \right) \\ &= -\partial_g \log [2 \mp \dot{D}_g(t_b) \mp \dot{\bar{D}}_g(t_b)]. \end{aligned} \tag{57}$$

This is inserted into formula (17) yielding for periodic and antiperiodic boundary conditions

$$\text{Det } \tilde{K}^{-1} K_g = C [2 \mp \dot{D}_g(t_b) \mp \bar{D}_g(t_b)], \quad (58)$$

where  $\tilde{K} = K_0 - \omega^2 = -\partial_t^2 - \omega^2$ . The constant of integration  $C$  is fixed in the way described after Eq. (17). We go to  $g = 1$  and set  $\Omega^2(t) \equiv \omega^2$ . For the operator  $K_1^\omega \equiv -\partial_t^2 - \omega^2$ , we can easily solve the Gelfand–Yaglom initial-value problem (41) as well as the dual one (51) by

$$\begin{aligned} D_1^\omega(t) &= \frac{1}{\omega} \sin[\omega(t - t_a)], \\ \bar{D}_1^\omega(t) &= \cos[\omega(t - t_a)], \end{aligned} \quad (59)$$

so that (58) determines  $C$  by

$$\begin{aligned} 1 &= C \times 4 \sin^2[\omega(t_b - t_a)/2] \quad \text{periodic case,} \\ &\times 4 \cos^2[\omega(t_b - t_a)/2] \quad \text{antiperiodic case.} \end{aligned} \quad (60)$$

Hence we obtain the final results for periodic boundary conditions

$$\begin{aligned} \text{Det } (\tilde{K}^{-1} K_1) &= \frac{\det \bar{A}_1^p / W_1}{\text{Det } \bar{A}_1^{\omega p} / W_1^\omega} \\ &= \frac{2 - \dot{D}_1(t_b) - \bar{D}_1(t_b)}{4 \sin^2[\omega(t_b - t_a)/2]}, \end{aligned} \quad (61)$$

and for antiperiodic boundary conditions

$$\begin{aligned} \text{Det } (\tilde{K}^{-1} K_1) &= \frac{\det \bar{A}_1^a / W_1}{\text{Det } \bar{A}_1^{\omega a} / W_1^\omega} \\ &= \frac{2 + \dot{D}_1(t_b) + \bar{D}_1(t_b)}{4 \cos^2[\omega(t_b - t_a)/2]}. \end{aligned} \quad (62)$$

The intermediate expressions in (50), (61), and (62) show that the ratios of functional determinants are ordinary determinants of two arbitrary independent solutions  $\eta_1(t)$  and  $\xi_1(t)$  of the homogeneous differential equation  $K_1 y(t) = [-\partial_t^2 - \Omega^2(t)]y(t) = 0$ . As such, the results are manifestly invariant under arbitrary linear transformations of these functions  $(\eta_1, \xi_1) \rightarrow (\tilde{\eta}_1, \tilde{\xi}_1)$ .

### 5. Expressions in terms of classical trajectory

In semiclassical fluctuation problems, the time-dependent frequency  $\Omega^2(t)$  is determined by the classical solution  $x_{cl}(t)$  of the equation of motion (3) via Eq. (2). In this case, the above results can be made quite explicit by expressing the solutions  $D_1(t)$  and  $\bar{D}_1(t)$  of the initial-value problems (41) and (51) directly in terms the classical trajectory  $x_{cl}(t)$  if this is specified in terms of its initial position  $x_a$  and initial velocity  $\dot{x}_a$  as  $x_{cl}(t, x_a, \dot{x}_a)$ . Given such a trajectory  $x_{cl}(t, x_a, \dot{x}_a) = x_a \bar{D}_1(t) + \dot{x}_a D_1(t)$  the solutions of (41) and (51) can be written in the form

$$\begin{aligned} D_1(t) &= \frac{\partial x_{cl}(t, x_a, \dot{x}_a)}{\partial \dot{x}_a}, \\ \bar{D}_1(t) &= \frac{\partial x_{cl}(t, x_a, \dot{x}_a)}{\partial x_a}. \end{aligned} \quad (63)$$

As an example, take a harmonic oscillator where formulas (63) are given explicitly by the previous expressions (59). For a classical path, we can use the equation of motion (3) and a partial integration to express the action as a surface term

$$\mathcal{A}[x_{cl}] = M(x_b \dot{x}_b - x_a \dot{x}_a) / 2, \quad (64)$$

where

$$\begin{aligned} x_b &= x_a \bar{D}_1(t_b) + \dot{x}_a D_1(t_b), \\ \dot{x}_b &= x_a \dot{\bar{D}}_1(t_b) + \dot{x}_a \dot{D}_1(t_b). \end{aligned} \quad (65)$$

With the help of Eqs. (65), we can write the action (64) as a function of initial and final positions  $x_a$  and  $x_b$ , and of the time difference  $t_b - t_a$ :

$$\begin{aligned} \mathcal{A}_{cl}(x_a, x_b; t_b - t_a) &= \frac{M}{2D_1(t_b)} \\ &\times [\dot{D}_1(t_b)x_b^2 - 2x_b x_a + \bar{D}_1(t_b)x_a^2]. \end{aligned} \quad (66)$$

From this we obtain directly

$$D_1(t_b) = -M \left( \frac{\partial^2 \mathcal{A}_{cl}(x_a, x_b, t_b - t_a)}{\partial x_a \partial x_b} \right)^{-1}, \quad (67)$$

so that the ratio (50) of functional determinants for Dirichlet boundary conditions becomes

Det  $K_0^{-1} K_1$

$$= -M \left( \frac{\partial^2 \mathcal{A}_{cl}(x_a, x_b, t_b - t_a)}{\partial x_a \partial x_b} \right)^{-1} / (t_b - t_a). \tag{68}$$

The right-hand side is known as one-dimensional Van Vleck–Pauli–Morette determinant (see Subsection 4.3 in Ref. [1]).

In the case of periodic and antiperiodic boundary conditions, we find from Eq. (66),

$$\begin{aligned} 2 \mp \dot{D}_1(t_b) \mp \bar{D}_1(t_b) \\ = 2 \pm \left( \frac{\partial^2 \mathcal{A}_{cl}(x_a, x_b, t_b - t_a)}{\partial x_a \partial x_b} \right)^{-1} \\ \times \left( \frac{\partial^2 \mathcal{A}_{cl}(x_a, x_b, t_b - t_a)}{\partial x_a^2} \right. \\ \left. + \frac{\partial^2 \mathcal{A}_{cl}(x_a, x_b, t_b - t_a)}{\partial x_b^2} \right), \end{aligned} \tag{69}$$

which determines the ratio of functional determinants (58) in terms of the classical action, in analogy to (68).

For a harmonic oscillator with the classical action

$$\begin{aligned} \mathcal{A}_{cl}(x_a, x_b, t_b - t_a) = \frac{M\omega}{2 \sin \omega(t_b - t_a)} \\ \times [(x_b^2 + x_a^2) \cos \omega(t_b - t_a) - 2x_b x_a], \end{aligned} \tag{70}$$

and we obtain  $D_1^\omega(t_b) = \omega^{-1} \sin \omega(t_b - t_a)$  as in (59) and

$$\begin{aligned} 2 \mp \dot{D}_1^\omega(t_b) \mp \bar{D}_1^\omega(t_b) = 4 \sin^2[\omega(t_b - t_a)/2], \\ = 4 \cos^2[\omega(t_b - t_a)/2] \end{aligned} \tag{71}$$

in agreement with the previous results (50), (61), and (62).

### 6. Treatment of zero mode

Consider now the often encountered situations that the operator  $K_1$  has a zero mode. In path integrals, such a zero mode arises for example from the translational invariance along the time axis of a classical solution in a potential  $V(x)$ . As in the last section, the squared frequency  $\Omega^2(t)$  is determined by (2).

For simplicity, we shall assume the presence of only a single zero mode, which we choose as one of two independent solutions of the homogeneous differential equation, say  $\eta(t)$ . For Dirichlet boundary conditions, we call this a Dirichlet zero mode, satisfying

$$\eta_b = 0, \quad \eta_a = 0. \tag{72}$$

For periodic and antiperiodic boundary conditions, the zero mode satisfies

$$\eta_b \mp \eta_a = 0, \quad \dot{\eta}_b \mp \dot{\eta}_a = 0, \tag{73}$$

respectively. As pointed out earlier, the Wronski construction for evaluating ratios of functional determinants is not applicable here since the conditions (33) and (38) are violated as a consequence of (72) and (73). In order to enforce (33) and (38), we modify the boundary conditions for eigenfunctions  $y(t)$  of the operator  $K_1$  by a small regulator parameter  $\epsilon > 0$ , and determine new eigenfunctions  $y^\epsilon(t)$  with  $y^\epsilon(t) \rightarrow y(t)$  and  $\lambda^\epsilon \rightarrow \lambda$  for  $\epsilon \rightarrow 0$ . The specific form of regularized boundary conditions will be irrelevant. It is merely required to keep the boundary-value problem self-conjugated. For instance, the Dirichlet boundary may be slightly modified to

$$\eta_a^\epsilon - \epsilon \dot{\eta}_a^\epsilon = 0, \quad \eta_b^\epsilon + \epsilon \dot{\eta}_b^\epsilon = 0, \tag{74}$$

the periodic and antiperiodic ones to

$$\begin{aligned} \eta_a^\epsilon = \pm \cosh \epsilon \eta_b^\epsilon + \sinh \epsilon \dot{\eta}_b^\epsilon, \\ \dot{\eta}_a^\epsilon = \sinh \epsilon \eta_b^\epsilon \pm \cosh \epsilon \dot{\eta}_b^\epsilon. \end{aligned} \tag{75}$$

Whereas the zero mode  $\eta(t)$  satisfies (72) or (73), the modified function  $\eta^\epsilon(t)$  is no longer a zero mode, but has an eigenvalue  $\delta\lambda^\epsilon$  of  $K_1$ , which goes to zero for  $\epsilon \rightarrow 0$ . As long as  $\epsilon$  is nonzero, the Wronski construction provides us with a regularized determinant  $\text{Det } K_1^\epsilon$  which tends to zero in the limit  $\epsilon \rightarrow 0$ . In terms of the independent solutions  $\eta(t)$  and  $\xi(t)$  of  $K_1 y = 0$ , this determinant is given for the regularized Dirichlet boundary conditions (74), to first order in  $\epsilon$ , by

$$\text{Det } K_1^\epsilon = \text{Det } K_1 + \frac{\epsilon}{W} (\eta_a \dot{\xi}_b - \dot{\eta}_a \xi_b + \eta_b \dot{\xi}_a - \dot{\eta}_b \xi_a). \tag{76}$$

The determinant  $\text{Det } K_1$  vanishes, and the constant Wronskian



$$W = \eta_a \dot{\xi}_a - \dot{\eta}_a \xi_a = \eta_b \dot{\xi}_b - \dot{\eta}_b \xi_b \quad (77)$$

is, by (72), equal to

$$W = -\dot{\eta}_a \xi_a = -\dot{\eta}_b \xi_b. \quad (78)$$

Simplifying (76) further with the help of (72), we obtain

$$\text{Det } K_1^\epsilon = -\frac{\epsilon}{W} (\dot{\eta}_a \xi_b + \dot{\eta}_b \xi_a) = \frac{\epsilon}{W^2} \xi_a \xi_b (\dot{\eta}_b^2 + \dot{\eta}_a^2). \quad (79)$$

For the regularized periodic and antiperiodic boundary conditions (75), the determinant reads, to first order in  $\epsilon$ :

$$\text{Det } K_1^\epsilon = \frac{\epsilon}{W} (\eta_b \xi_a - \eta_a \xi_b - \dot{\eta}_b \dot{\xi}_a + \dot{\eta}_a \dot{\xi}_b), \quad (80)$$

with the same Wronski determinant (77) whose constancy implies, together with (73), that

$$\eta_b (\dot{\xi}_b \mp \dot{\xi}_a) - \dot{\eta}_b (\xi_b \mp \xi_a) = 0. \quad (81)$$

Using (73) once more in (80), we find

$$\text{Det } K_1^\epsilon = \mp \frac{\epsilon}{W \eta_b} (\eta_b^2 - \dot{\eta}_b^2) (\xi_b \mp \xi_a). \quad (82)$$

In order to find a finite expression for the functional determinant we must divide out the eigenvalue  $\delta\lambda^\epsilon$  before taking  $\epsilon \rightarrow 0$ . From the regularized eigenvalue equation

$$K_1 \eta^\epsilon(t) = \delta\lambda^\epsilon \eta^\epsilon(t), \quad (83)$$

with  $\eta^\epsilon(t)$  normalized as in (9), we find to first order in  $\epsilon$

$$\eta K_1 \eta^\epsilon = (\eta^\epsilon \dot{\eta} - \dot{\eta}^\epsilon \eta)|_{t_a}^{t_b} \approx \delta\lambda^\epsilon \int_{t_a}^{t_b} dt \eta^2(t) = \delta\lambda^\epsilon.$$

Taking into account the regularized boundary conditions (74) and (75) for  $\eta^\epsilon(t)$ , as well as the conditions (72) and (73) for  $\eta(t)$ , gives for the eigenvalue of the Dirichlet would-be zero mode  $\eta^\epsilon(t)$

$$\delta\lambda^\epsilon = \dot{\eta}_b \eta_b^\epsilon - \dot{\eta}_a \eta_a^\epsilon = -\epsilon (\dot{\eta}_b \dot{\eta}_b^\epsilon + \dot{\eta}_a \dot{\eta}_a^\epsilon), \quad (84)$$

and for periodic (antiperiodic) boundary conditions:

$$\begin{aligned} \delta\lambda^\epsilon &= \dot{\eta}_b (\eta_b^\epsilon \mp \eta_a^\epsilon) - \eta_b (\dot{\eta}_b^\epsilon \mp \dot{\eta}_a^\epsilon) \\ &= \mp \epsilon (\dot{\eta}_b \dot{\eta}_b^\epsilon - \eta_b \dot{\eta}_b^\epsilon). \end{aligned} \quad (85)$$

These equations enable us to remove  $\delta\lambda^\epsilon$  from the regularized determinants (79) and (82). Defining the determinant without the zero mode by (see Subsection 17.5 in Ref. [1]).

$$\text{Det}' K_1 = \lim_{\epsilon \rightarrow 0} \frac{\text{Det } K_1^\epsilon}{\delta\lambda^\epsilon}, \quad (86)$$

we obtain from (79) and (84) for Dirichlet boundary conditions

$$\begin{aligned} \text{Det}' K_1 &= -\frac{\xi_b \xi_a}{W^2} \lim_{\epsilon \rightarrow 0} \frac{\dot{\eta}_b^2 + \dot{\eta}_a^2}{\dot{\eta}_b \dot{\eta}_b^\epsilon + \dot{\eta}_a \dot{\eta}_a^\epsilon} \\ &= -\frac{\xi_b \xi_a}{W^2} = -\frac{1}{\dot{\eta}_b \dot{\eta}_a}. \end{aligned} \quad (87)$$

For periodic and antiperiodic boundary conditions, the result is from (80) and (85):

$$\text{Det}' K_1 = \frac{\xi_b \mp \xi_a}{\eta_b W} \lim_{\epsilon \rightarrow 0} \frac{\eta_b^2 - \dot{\eta}_b^2}{\dot{\eta}_b \dot{\eta}_b^\epsilon - \eta_b \dot{\eta}_b^\epsilon} = -\frac{(\xi_b \mp \xi_a)}{\eta_b W}, \quad (88)$$

which by (81) becomes

$$\text{Det}' K_1 = -\frac{\xi_b \mp \xi_a}{\eta_b W} = -\frac{\dot{\xi}_b \mp \dot{\xi}_a}{\dot{\eta}_b W}. \quad (89)$$

Formulas (87) and (89) are useful for semiclassical calculations of path integrals whose equations of motion possess nontrivial classical solutions such a solitons or instantons [1], as will be illustrated in Section 8.

Note that our final expressions (87) and (89) for the functional determinants are independent of the specific choice of regularization.

### 7. Time-dependent harmonic oscillator

To illustrate the power of the formulas derived in this work consider the time-dependent harmonic oscillator described by the Lagrangian (1). The path integral formalism for such a system with the Dirichlet boundary conditions was studied in several papers [10–12]. Here we rederive their results and generalize them to periodic and antiperiodic boundary conditions. Due to the absence of time-translational invariance of the Lagrangian (1), a zero mode can be

excluded here. For the Wronski construction, we take two independent solutions of Eq. (23) as follows:

$$\eta(t) = q(t) \cos \phi(t), \quad \xi(t) = q(t) \sin \phi(t) \quad (90)$$

with a constant Wronski determinant  $W$ . The solutions  $\eta(t)$  and  $\xi(t)$  are parametrized by two functions  $q(t)$  and  $\phi(t)$  satisfying the constraint

$$\dot{\phi}(t)q^2(t) = W. \quad (91)$$

The function  $q(t)$  is a soliton of the Ermakov–Pinney equation [13]

$$\ddot{q} + \Omega^2(t)q - W^2q^{-3} = 0. \quad (92)$$

For Dirichlet boundary conditions we insert (90) into (50), and obtain the ratio of fluctuation determinants in the form

$$\text{Det } K_0^{-1} K_1 = \frac{1}{W} \frac{q(t_a)q(t_b) \sin[\phi(t_b) - \phi(t_a)]}{t_b - t_a}. \quad (93)$$

For periodic or antiperiodic boundary conditions and  $\Omega(t)$ , the functions  $q(t)$  and  $\phi(t)$  in Eq. (90) do not in general have the same periodicity. This is possible because of the nonlinearity of Eqs. (91) and (92). Moreover, since we assume here the absence of a zero mode with such boundary conditions, it is a *necessary* property of the solutions of the homogeneous Eqs. (23). Substituting (90) into (61) and (62), we obtain the ratios of functional determinants for periodic boundary conditions

$$\begin{aligned} \text{Det } \tilde{K}^{-1} K_1 &= 4 \sin^{-2} \frac{\omega(t_b - t_a)}{2} \\ &\times \left( 4 \sin^2 \frac{[\phi(t_b) - \phi(t_a)]}{2} \right. \\ &\quad \left. - \frac{[\dot{q}(t_b)q(t_a) - \dot{q}(t_a)q(t_b)]}{c} \right. \\ &\quad \times \sin[\phi(t_b) - \phi(t_a)] \\ &\quad \left. - \frac{[q(t_b) - q(t_a)]^2}{q(t_a)q(t_b)} \cos[\phi(t_b) - \phi(t_a)] \right). \end{aligned} \quad (94)$$

For antiperiodic ones, we must interchange  $\sin \rightarrow -\cos$ . By a linear combination of the solutions (90) we can always redefine  $\phi(t)$  such that  $\phi(t_a) = 0$ .

In the literature, only formula (93) for the Dirichlet case appears to be known (see Refs. [10–12]). Formulas (94) for periodic and antiperiodic boundary conditions are new, except for predecessors in a time-sliced formulation (see Subsection 2.12 in Ref. [1]). The present derivation is, however, much simpler than that of the predecessor since we have been able to take full advantage of Wronski's simple construction method for Green functions.

## 8. Fluctuation determinant of instanton

As an application of our formulas we derive the functional determinant of the quadratic fluctuations around an instanton which governs the energy level splitting of a quantum mechanical point particle in a double well. Setting the mass equal to unity, for simplicity, we consider a potential of the form [1]

$$V(x) = \frac{\omega^2}{8a^2} (x^2 - a^2)^2. \quad (95)$$

The tunneling through the central barrier is controlled by the solution of the equation of motion at imaginary time  $\tau = -it$ , which can be integrated once to yield the energy conservation law

$$\frac{1}{2} \dot{x}^2(\tau) = V(x(\tau)) + E, \quad (96)$$

where  $\dot{x}(\tau) \equiv dx(\tau)/d\tau$ , and  $E$  is the integration constant corresponding to the particle energy in the inverted double-well. For the splitting between ground state and first excited state, we must study the path integral for the evolution amplitude over a large but finite time interval  $(\tau_a, \tau_b)$ . In a semiclassical approximation, this is dominated by periodic solutions of with energy  $E \leq 0$ , whose turning points lie close to the minima of the double well. We consider first a single sweep across the central barrier from a turning point at  $x(\tau_a) = x_a$  to  $x(\tau_b) = x_b$ , where the velocities vanish:  $\dot{x}(\tau_a) = \dot{x}(\tau_b) = 0$ , so that the energy is given by

$$E = -V(x_b) = -V(x_a) = -\frac{\omega^2}{8a^2} (x_b^2 - a^2)^2. \quad (97)$$

For a single sweep this implies

$$x_b = -x_a, \quad x_b \leq a. \quad (98)$$

For an infinite time interval  $(\tau_a, \tau_b)$ , the sweep connects the potential minima with each other, in which

case  $x_a = -x_b = a$  and  $E = 0$ . Then Eq. (96) can easily be integrated yielding the well-known kink solution [1] centered around some finite  $\tau_0 = (\tau_a + \tau_b)/2$ :

$$x_{cl}(\tau) = a \tanh\left(\frac{\omega(\tau - \tau_0)}{2}\right). \quad (99)$$

With the explicit energy (97), the equation of motion (96) reads

$$\dot{x}^2(\tau) = \frac{\omega^2}{4a^2}(x_b^2 - x^2)(b^2 - x^2), \quad (100)$$

where  $b^2 \equiv 2a^2 - x_b^2$  and  $x^2 \leq x_b^2 \leq b^2$ . Integrating Eq. (100) gives

$$\int_{x(\tau)}^{x_b} \frac{dt}{\sqrt{(x_b^2 - t^2)(b^2 - t^2)}} = -\frac{\omega}{2a}(\tau - \tau_b). \quad (101)$$

It is useful to introduce a normalized coordinate  $y(\tau) \equiv x(\tau)/x_b$  moving between  $-1$  and  $1$ , and rewrite (101) as

$$\frac{1}{b} \int_0^{y(\tau)} \frac{dt}{\sqrt{(1 - t^2)(1 - m t^2)}} = \frac{\omega}{2a}(\tau - \tau_b) + \frac{\kappa}{b}. \quad (102)$$

The parameter  $m$  is equal to  $x_b^2/b^2 \leq 1$  and determines the constant  $\kappa$  on the right-hand side via the complete elliptic integral of the first kind

$$\kappa = K(m) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - m t^2)}}. \quad (103)$$

This constant fixes the period  $T$  via formula (101) for  $\tau = \tau_a$  as follows:

$$2\kappa = \frac{\omega}{2a} bT. \quad (104)$$

The general solution of Eq. (102) is

$$x_{cl}(\tau, \tau_b, m) = x_b \operatorname{sn}(z(\tau); m), \quad (105)$$

where  $z(\tau) = \omega b(\tau - \tau_b)/2a + \kappa$ , so that  $z_b = \kappa$ ,  $z_a = -\kappa$  and  $\operatorname{sn}(z; m)$  is the elliptic function running from  $-1$  to  $1$  for  $\tau \in (\tau_a, \tau_b)$ , thus ensuring the correct boundary conditions  $x_{cl}(\tau_b) = x_b$ ,  $x_{cl}(\tau_a) = -x_b$ .

According to Eqs. (7) and (2), the fluctuations  $\delta x(\tau) = y(\tau)$  around the solution (105) are governed by the differential operator  $K_1(\tau) = -d^2/d\tau^2 + \omega^2(3x_{cl}^2 - a^2)/2a^2$ . The boundary conditions are eventually irrelevant for the level splitting in the ground state, since this will require taking the limit of an infinite time interval. As an example, we consider here Dirichlet boundary conditions for eigenfunctions of the operator  $K_1(\tau) : y_b = y_a = 0$ . The derivation of fluctuation determinant requires, in general, two independent solutions of the homogeneous differential equation which after going over from the time  $\tau$  to the variable  $z(\tau)$  takes the Lamé's form

$$\ddot{h}(z) + \left(\frac{2a^2}{b^2} - 6m \operatorname{sn}^2(z; m)\right) h(z) = 0. \quad (106)$$

By translational invariance, the first independent solution  $\eta(t)$  to this equation is the derivative  $(\partial/\partial\tau_b)x_{cl}(z(\tau); m)$  of Eq. (105). Normalizing, we have explicitly

$$\begin{aligned} \eta(\tau) &= N \frac{\partial x_{cl}(\tau, \tau_b, m)}{\partial \tau_b} \\ &= -N \frac{\omega}{2a} b x_b \operatorname{cn}(z; m) \operatorname{dn}(z; m), \end{aligned} \quad (107)$$

whose time derivative is

$$\begin{aligned} \dot{\eta}(\tau) &= N \left(\frac{\omega}{2a} b\right)^2 x_b \operatorname{sn}(z; m) \\ &\quad \times [\operatorname{dn}^2(z; m) + m \operatorname{cn}^2(z; m)]. \end{aligned} \quad (108)$$

Here  $\operatorname{cn}(z; m)$  and  $\operatorname{dn}(z; m)$  are the elliptic functions. The normalization factor  $N$  is determined by the condition (9) as follows:

$$N^{-2} = x_b^2 \left(\frac{\omega}{2a} b\right) \int_{-k}^k dz \operatorname{cn}^2 z \operatorname{dn}^2 z. \quad (109)$$

Performing the integral yields

$$\begin{aligned} N^{-2} &= -\frac{4a^2}{3(m+1)} \left(\frac{\omega}{2a} b\right) \\ &\quad \times [(1-m)\kappa - (m+1)\varepsilon], \end{aligned} \quad (110)$$

where  $\varepsilon = E(m)$  is given by the complete elliptic integral of the second kind

$$E(m) = \int_0^1 dt \sqrt{\frac{1 - mt^2}{1 - t^2}}. \quad (111)$$

The solution (107) is a zero mode of the operator  $K_1(\tau)$ , since it satisfies Dirichlet boundary condition

$$\eta_b = \eta_a = -N \frac{\omega}{2a} b x_b \operatorname{cn} \kappa \operatorname{dn} \kappa = 0. \tag{112}$$

Because of the property (112), the Dirichlet determinant (50) vanishes, so that we may only calculate the primed determinant according to formula (87). This does not require the second independent solution  $\xi(\tau)$  of Eq. (106). From Eq. (108) we observe

$$\dot{\eta}_b = -\dot{\eta}_a = N \left( \frac{\omega}{2a} b \right)^2 x_b (1-m). \tag{113}$$

Inserting (113) with (110) into Eq. (87), we find immediately

$$\begin{aligned} \operatorname{Det}' K_1 &= -\frac{1}{\dot{\eta}_b \dot{\eta}_a} \\ &= \frac{4a^2 [(m+1)\varepsilon - (1-m)\kappa]}{3x_b^2 (m+1)(1-m)^2} \left( \frac{\omega}{2a} b \right)^{-3}. \end{aligned} \tag{114}$$

Let us turn now to the limit of an infinite time interval where  $E \rightarrow 0$  and  $x_b$  and  $b$  go to the constant  $a$ , the parameter  $m$  tends to unity as  $(1-m) \rightarrow 16 \exp(-2\kappa)$ , and  $\varepsilon \rightarrow 1$ . Using equation (104), we obtain from (114)

$$\operatorname{Det}' K_1 \rightarrow \frac{e^{4\kappa}}{24\omega^3} = \frac{e^{\omega T}}{24\omega^3}. \tag{115}$$

In the same limit

$$\operatorname{sn} \left[ \frac{\omega}{2a} b(\tau - \tau_b) + k; 1 \right] \rightarrow \tanh \left( \frac{\omega(\tau - \tau_0)}{2} \right), \tag{116}$$

so that (105) reduces to the limiting kink solution (99), for which the fluctuation determinant is (115). The presence of exponentially divergent factor  $e^{\omega T}$  like in Eq. (115) is a special feature of fluctuation determinants in the limit  $T \rightarrow \infty$ . It also appears if one derives the determinant of harmonic differential operator  $\tilde{K} = K_1^\omega = -d^2/d\tau^2 + \omega^2$ , which governs the fluctuations around the trivial constant classical solution  $x_{cl}(\tau) = a$ , with the same Dirichlet boundary conditions. Indeed, inserting of the independent solutions  $\eta(\tau) = \cosh \omega\tau$ ,  $\xi(\tau) = \sinh \omega\tau$  into Eq. (50) yields directly

$$\operatorname{Det} K_1^\omega = \frac{\sinh \omega T}{\omega} \rightarrow \frac{e^{\omega T}}{2\omega}, \tag{117}$$

where the right-hand side being the large- $T$  limit. Certainly, when considering the more relevant ratio of (115) and (117), we obtain the finite result

$$\operatorname{Det}' K_1 / \operatorname{Det} K_1^\omega \rightarrow \frac{1}{12\omega^2}, \tag{118}$$

which agrees, of course, with previous calculation [1].

The primed determinant (115) can also be derived from (87) using only the asymptotic behavior of the independent solutions  $\eta(\tau)$  and  $\xi(\tau)$  at  $T \rightarrow \infty$  [15,1]. For this purpose, we set the particle energy in (97) equal to zero. The elliptic functions degenerate into hyperbolic, simplifying Eq. (106) to

$$\ddot{h}(z) - 2(2 - 3 \cosh^{-2} z)h(z) = 0, \tag{119}$$

where  $z(\tau) = \omega(\tau - \tau_0)/2$  with  $\tau_0 = (\tau_b + \tau_a)/2$ . Now we are looking for the asymptotics of two independent solutions to this equation at  $\tau_b - \tau_a \rightarrow \infty$ . For the solution corresponding to (107) we find

$$\eta(\tau) = -N \frac{a\omega}{2} \cosh^{-2} z \xrightarrow{\tau_b - \tau_a \rightarrow \infty} -2a\omega N e^{-2|z|}, \tag{120}$$

the proper normalization factor being  $N^{-2} = 2a^2\omega/3$ . For the second independent solution, the asymptotic behavior may be deduced from the constancy of Wronskian  $W(\eta, \xi)$  as follows:

$$\xi(\tau) \xrightarrow{\tau_b - \tau_a \rightarrow \infty} \pm e^{2|z|}. \tag{121}$$

Here the normalization is irrelevant since the expressions (87) is independent of it. The solutions (120) and (121) have the Wronskian  $W(\eta, \xi) = -4a\omega^2 N$ , and the asymptotic boundary conditions for  $\eta(\tau)$

$$\begin{aligned} \eta_b = \eta_a &\approx -2a\omega N e^{-\omega(\tau_b - \tau_a)/2}, \\ \dot{\eta}_b = -\dot{\eta}_a &\approx \omega\eta_b, \end{aligned} \tag{122}$$

and for  $\xi(\tau)$

$$\xi_b = -\xi_a \approx e^{\omega(\tau_b - \tau_a)/2}, \quad \dot{\xi}_b = \dot{\xi}_a \approx \omega\xi_b. \tag{123}$$

Inserting this into formula (87), with the right-hand side rewritten as  $-\xi_a \dot{\xi}_b / W^2$ , we obtain once again the result (115).

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