

Crossover from Weak- to Strong-Coupling Superconductivity and to Normal State with Pseudogap.

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Abstract

For an electron gas with a δ -function attraction we investigate the crossover from weak-coupling to strong-coupling superconductivity as well as normal state near the temperature T^* , at which the strong coupling produces a pseudogap in the energy spectrum due to the binding of electron pairs. We present curves for the behavior of the superconductive transition temperature, the gap formation temperature, the gap size and several thermodynamic quantities as functions of coupling strength and temperature, both in two and three dimensions.

1 Introduction

The crossover from BCS superconductors to a Bose-Einstein condensate of tightly bound fermion pairs was first studied many years ago in Refs. [1]–[3]. Sparked by experimental studies of short coherence length cuprate superconductors, it has recently attracted renewed interest [4]– [33], the experiments show an anomalous behavior in the normal phase well *above* the superconductive transition T_c (see the review [34], also [32], [35]–[38]). Anomalous is the temperature dependence of resistivity, specific heat, spin susceptibility, etc.. Moreover, angular-resolved photoemission spectroscopy experiments (ARPES) indicate the existence of a *pseudogap* in the single-particle excitation spectrum [36] - [39] manifesting itself in a significant suppression of low-frequency spectral weight above T_c , this being similar to the complete suppression below T_c in an ordinary superconductor due to the energy gap.

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In essence, such anomalous properties of the normal state of superconductor can be described by a simple model of superconductivity in which electrons are bound to pairs by a δ -function potential. We merely must leave the BCS regime and go to stronger couplings ².

In this paper we present a detailed study of the crossover in such a model. Physically, the most important distinctions between conventional (BCS) and strong-coupling (Bose-Einstein) regime lies in the fact that in the former only a small fraction of the conduction electrons is paired with the superfluid density involving all pairs, whereas in the latter practically all carriers are paired below a certain temperature T^* , although not condensed, effect that results in deviation from the Fermi liquid behavior in the region between T^* and T_c . The temperature has to be lowered further below some critical temperature $T_c < T^*$ to make these pairs condense and establish phase coherence, which leads to superconductive behavior. We shall neglect the coupling to the magnetic vector potential throughout the upcoming discussion, so that the phase coherence below T_c can be of long range, unspoiled by the Meissner effect which would reduce the range to a finite penetration depth.

The pseudogap behavior between T_c and T^* is characterized by short-range pair correlation functions. The common physical origin of the superconductive gap below T_c and the pseudogap above T_c observed in cuprates is suggested by the above-quoted ARPES data, which show that the two gaps have the same magnitude and wave vector dependence. Important experiments on the gap properties are:

1. In experiments on YBCO [40], [41], a significant suppression of in-plane conductivity $\sigma_{ab}(\omega)$ was observed at frequencies below 500 cm^{-1} beginning at temperatures much above T_c . Experiments [42], [43] on underdoped samples revealed deviations from the linear resistivity law. In particular, $\sigma_{ab}(\omega = 0; T)$ increases slightly with decreasing T below a certain temperature.
2. Specific heat experiments [44] also clearly display pseudogap behavior much above

² It should be noted that phase diagram in the case of such complicated materials as High- T_c cuprate superconductors contains large region where anomalous properties of normal state are governed by anti-ferromagnetic correlations.

T_c .

3. NMR and neutrons observations in [45] and [46] show that below temperatures T^* much higher than T_c , spin susceptibility starts decreasing.
4. Experiments on optical conductivity [51], [52] and tunneling exhibit the opening of a pseudogap. A review of actual experimental data confirming the pseudogap behavior of the underdoped and optimally doped cuprates is given in [32], [51].

In the model to be investigated in this paper, the crossover from BCS-type to Bose-type superconductivity and pseudogap state will take place either by varying the coupling strength, or by decreasing the carrier density.

Analytic calculations will be performed using a crossover parameter x_0 . This parameter is directly related to the ratio of chemical potential and gap function at zero temperature. It is a monotonous function of coupling strength and carrier density [to be seen in Figs. 1 and 2 in two and three dimensions, respectively]. It is also a direct measure for the scattering length a_s of the electron-electron interaction, at zero temperature in three dimensions [to be seen in Eq. (14)], and to the binding energy of the electron-electron pairs in two dimensions [to be seen in Eq. (20)].

Our analysis of the model start with a mean-field approximation to the collective pair field theory [53]. It is well known, that mean-field results are reliable at all temperatures for weak coupling strength, i.e. in the BCS regime. As this regime is approached from the strong-coupling side, the temperature T^* where pairs are formed and the temperature T_c where phase coherence sets in merge to the single BCS phase transition temperature T_c . This merging will be described analytically in this paper in two as well as three dimensions.

Mean-field results are also reliable at strong couplings if the temperatures are sufficiently small to suppress fluctuations [see the discussion in [6]].

Apart from that, mean-field results at stronger couplings seem to indicate correctly the *position* of the temperature T^* where pseudogap forms due to precursor pair formation [5]. The model shows the binding of noncondenserd pairs by the appearance a nonzero complex

gap function $\Delta(T)$. The precise temperature behavior near T^* is certainly predicted wrongly as being $\propto (T^* - T)^{1/2}$. This would suggest a second-order phase transition, whereas the experimental data show a smooth crossover phenomenon. Here the effect of fluctuations is too important to be calculable analytically.

In this paper we shall focus attention upon the onset of long-range order when lowering the temperature down from T^* . Due to the strength of the coupling, this regime lies outside the mean-field approximation. It appears, however, that by extracting the lowest gradient terms governing the Gaussian fluctuations around the mean-field we shall obtain sufficient information to study the onset of long-range order and its destruction at $T_c < T^*$.

In three dimensions, this was first investigated in Ref. [2] by summing particle-particle ladder diagrams which correspond to Gaussian fluctuations around the mean field, in the functional integral formalism it was studied in Ref. [5].

As it was noted above, at strong couplings, the formation of a gap does not imply the existence of superconductive currents. Although the modulus of the gap field $\Delta(\mathbf{x}) \equiv |\Delta(\mathbf{x})|$ is nonzero, the phase of Δ will initially fluctuate so violently that long-range order cannot be established. This phenomenon is well-known from XY-models [O(2) classical Heisenberg model], which describe the fluctuations of a field of two-dimensional unit vectors $\mathbf{n} = (\cos \phi, \sin \phi)$. The fluctuations of the phase angle ϕ prevent the existence of a long-range order above a certain temperature.

Much of the previous work on the model with an attractive δ -potential at strong couplings was devoted to zero-temperature or to a study of the critical region near T_c where phase coherence sets in. Apparently, there is no analytic work displaying a complete set of global properties of the pseudogap state above T_c . There only exist some numeric results in Ref. [24] on the paramagnetic susceptibility in the pseudogap phase of two-dimensional superconductors, derived within the same model after adding a paramagnetic term to Hamiltonian³. In the present paper, we shall derive as well analytically the behavior for all coupling strengths of the pseudogap as well as the thermodynamics of the pseudogapped phase in both two and three dimensions.

³ For Monte-Carlo study of the normal state just above T_c see Ref. [30]

The weak- to strong-coupling crossover of the Kosterlitz-Thouless in two dimensions was studied in Refs. [26], [24]. In Ref. [24] it was investigated within the same model as ours at a fixed carrier density, but only numerically⁴. The different properties of size and phase fluctuations was also exploited in Refs. [54] and [55].

The plan of the paper is the following:

We begin by reproducing the results of Refs. [11], [12] for the gap $\Delta(0)$ and chemical potential at zero temperature. These are subsequently extended by equations for the temperature behavior of gap and pseudogap, as well as of thermodynamic functions.

In Section 2 we calculate the crossover from BCS to strong-coupling superconductivity at small but finite temperatures and present solutions for the gap and thermodynamic functions of the superconductive state.

In Section 3 we study the crossover from BCS superconductivity near T_c to the onset of pseudogap behavior near T^* .

In Sections 4 and 5 we go beyond mean-field approximation and study crossover of superconductive transition in both two and three dimensions. In the strong-coupling limit of superconductors, we set up an equivalent XY -model in two as well as three dimensions with the help of a gradient expansion for the phase of the order parameter. This allows us to find the onset of phase coherence in a non-perturbative way. Even though in two dimensions, this was done before via Kosterlitz-Thouless arguments we argue that merging of temperatures of pair formation and condensation was missed. In three dimensions, our discussion is new. Previous results were derived from a study of the condensation of the gas of composite bosons via retaining corrections to the number equation.

In the weak-coupling limit of both two- and three-dimensional superconductors we show how the XY -model transition converges with the transition in the BCS theory.

⁴We shall see in Section 4 that these numerical results do not cover the entire crossover region, in particular, the merging of T_{KT} and T^* in the weak-coupling region is missing - the effect that we show analytically in our paper.

2 Crossover from BCS to Strong-Coupling Superconductivity Near Zero Temperature

The Hamiltonian of our model is the typical BCS Hamiltonian in D dimensions ($\hbar = 1$)

$$H = \sum_{\sigma} \int d^D x \psi_{\sigma}^{\dagger}(\mathbf{x}) \left(-\frac{\nabla^2}{2m} - \mu \right) \psi_{\sigma}(\mathbf{x}) + g \int d^D x \psi_{\uparrow}^{\dagger}(\mathbf{x}) \psi_{\downarrow}^{\dagger}(\mathbf{x}) \psi_{\downarrow}(\mathbf{x}) \psi_{\uparrow}(\mathbf{x}), \quad (1)$$

where $\psi_{\sigma}(\mathbf{x})$ is the Fermi field operator, $\sigma = \uparrow, \downarrow$ denotes the spin components, m is the effective fermionic mass, and $g < 0$ the strength of an attractive potential $g\delta(\mathbf{x} - \mathbf{x}')$.

The mean-field equations for the gap parameter Δ and the chemical potential μ are obtained in the standard way from the equations (see for example [53], [24] and Appendix A):

$$-\frac{1}{g} = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}} \tanh \frac{E_{\mathbf{k}}}{2T}, \quad (2)$$

$$n = \frac{1}{V} \sum_{\mathbf{k}} \left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \tanh \frac{E_{\mathbf{k}}}{2T} \right), \quad (3)$$

where the sum runs over all wave vectors \mathbf{k} , N is the total number of fermions, V the volume of the system, and

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2} \quad \text{with} \quad \xi_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - \mu \quad (4)$$

are the energies of single-particle excitations.

Changing the sum over \mathbf{k} to an integral over ξ and over the directions of \mathbf{k} , on which the integrand does not depend, we arrive in three dimensions at the gap equation:

$$\frac{1}{g} = \kappa_3 \int_{-\mu}^{\infty} d\xi \frac{\sqrt{\xi + \mu}}{2\sqrt{\xi^2 + \Delta^2}} \tanh \frac{\sqrt{\xi^2 + \Delta^2}}{2T}, \quad (5)$$

where the constant $\kappa_3 = m^{3/2}/\sqrt{2\pi^2}$ has dimension energy^{-3/2}/volume. In two-dimensions, the density of states is constant, and the gap equation becomes

$$\frac{1}{g} = \kappa_2 \int_{-\mu}^{\infty} d\xi \frac{1}{2\sqrt{\xi^2 + \Delta^2}} \tanh \frac{\sqrt{\xi^2 + \Delta^2}}{2T}, \quad (6)$$

with a constant $\kappa_2 = m/2\pi$ of dimension energy⁻¹/two-volume. In two dimensions, the particle number in Eq. (3) can be integrated with the result:

$$n = \frac{m}{2\pi} \left\{ \sqrt{\mu^2 + \Delta^2} + \mu + 2T \log \left[1 + \exp \left(-\frac{\sqrt{\mu^2 + \Delta^2}}{T} \right) \right] \right\}, \quad (7)$$

the right-hand side being a function $n(\mu, T, \Delta)$.

The δ -function potential produces an artificial divergence and requires regularization. A BCS superconductor possesses a natural cutoff supplied by the Debye frequency ω_D . For the crossover problem to be treated here this is no longer a useful quantity, since in the strong-coupling limit all fermions participate in the interaction, not only those in a thin shell of width ω_D around the Fermi surface. To be applicable in this regime, we renormalize the gap equation in three dimensions with the help of the experimentally observable s -wave scattering length a_s , for which the low-energy limit of the two-body scattering process gives an equally divergent expression [9]–[13]:

$$\frac{m}{4\pi a_s} = \frac{1}{g} + \frac{1}{V} \sum_{\mathbf{k}} \frac{m}{\mathbf{k}^2}. \quad (8)$$

Eliminating g from (8) and (2) we obtain a renormalized gap equation

$$-\frac{m}{4\pi a_s} = \frac{1}{V} \sum_{\mathbf{k}} \left[\frac{1}{2E_{\mathbf{k}}} \tanh \frac{E_{\mathbf{k}}}{2T} - \frac{m}{\mathbf{k}^2} \right], \quad (9)$$

in which $1/k_F a_s$ plays the role of a dimensionless coupling constant which monotonically increases from $-\infty$ to ∞ as the bare coupling constant g runs from small (BCS limit) to large values (BE limit). This equation is to be solved simultaneously with (3). These mean-field equations were first analyzed at a fixed carrier density in Refs. [5] and [7]. Here we shall first reproduce the obtained estimates for T^* and μ .

In the BCS limit, the chemical potential μ does not differ much from the Fermi energy ϵ_F , whereas with increasing interaction strength, the distribution function $n_{\mathbf{k}}$ broadens and μ decreases, and in the BE limit, on the other hand we have tightly bound pairs and nondegenerate fermions with a large negative chemical potential $|\mu| \gg T$. Analyzing Eqns. (3) and (9) we have from (3) for the critical temperature in the BCS limit ($\mu \gg T_c$) $T_c^{\text{BCS}} = 8e^{-2}e^{\gamma}\pi^{-1}\epsilon_F \exp(-\pi/2k_F|a_s|)$ where $\gamma = -\Gamma'(1)/\Gamma(1) = 0.577\dots$, from (9) we

have that chemical potential in this case is $\mu = \epsilon_F$. In the strong Eq. (9) determines T^* , whereas Eq. (3) determines μ . From Eq. (3) we obtain that in the BE limit $\mu = -E_b/2$, where $E_b = 1/ma_s^2$ is the binding energy of the bound pairs. In the BE limit, the pseudogap sets in at $T^* \simeq E_b/2 \log(E_b/\epsilon_F)^{3/2}$. A simple ‘‘chemical’’ equilibrium estimate ($\mu_b = 2\mu_f$) yields for the temperature of pair dissociation: $T_{\text{dissoc}} \simeq E_b/\log(E_b/\epsilon_F)^{3/2}$ which shows at strong couplings T^* is indeed related to pair formation [5], [6] (which in the strong-coupling regime lies above the temperature of phase coherence [2]-[32]).

The gap in the spectrum of single-particle excitations has a special feature [3], [1], [7] when the chemical potential changes its sign. The sign change occurs at the minimum of the Bogoliubov quasiparticle energy $E_{\mathbf{k}}$ where this energy defines the gap energy in the quasiparticle spectrum:

$$E_{\text{gap}} = \min \left(\xi_{\mathbf{k}}^2 + \Delta^2 \right)^{1/2}. \quad (10)$$

Thus, for positive chemical potential, the gap energy is given directly by the gap function Δ , whereas for negative chemical potential, it is larger than that:

$$E_{\text{gap}} = \begin{cases} \Delta & \text{for } \mu > 0, \\ (\mu^2 + \Delta^2)^{1/2} & \text{for } \mu < 0. \end{cases} \quad (11)$$

In three dimensions at $T = 0$, equations (9), (3) were solved analytically in entire crossover region in [11] to obtain Δ and μ as functions of crossover parameter $1/k_F a_s$. The results are

$$\frac{\Delta}{\epsilon_F} = \frac{1}{[x_0 I_1(x_0) + I_2(x_0)]^{2/3}}, \quad (12)$$

$$\frac{\mu}{\epsilon_F} = \frac{\mu \Delta}{\Delta \epsilon_F} = \frac{x_0}{[x_0 I_1(x_0) + I_2(x_0)]^{2/3}}, \quad (13)$$

$$\frac{1}{k_F a_s} = -\frac{4}{\pi} \frac{x_0 I_2(x_0) - I_1(x_0)}{[x_0 I_1(x_0) + I_2(x_0)]^{1/3}}, \quad (14)$$

with the functions

$$\begin{aligned} I_1(x_0) &= \int_0^\infty dx \frac{x^2}{(x^4 - 2x_0 x^2 + x_0^2 + 1)^{3/2}} \\ &= (1 + x_0^2)^{1/4} E\left(\frac{\pi}{2}, \kappa\right) - \frac{1}{4x_0^2(1 + x_0^2)^{1/4}} F\left(\frac{\pi}{2}, \kappa\right), \end{aligned}$$

(15)

$$I_2(x_0) = \frac{1}{2} \int_0^\infty dx \frac{1}{(x^4 - 2x_0x^2 + x_0^2 + 1)^{1/2}}$$

$$= \frac{1}{2(1+x_0^2)^{1/4}} F\left(\frac{\pi}{2}, \kappa\right),$$
(16)

$$\kappa^2 = \frac{x_1^2}{(1+x_0^2)^{1/2}},$$
(17)

$$x^2 = \frac{k^2}{2m} \frac{1}{\Delta}, \quad x_0 = \frac{\mu}{\Delta}, \quad x_1 = \frac{\sqrt{1+x_0^2} + x_0}{2},$$
(18)

and $E(\frac{\pi}{2}, \kappa)$ and $F(\frac{\pi}{2}, \kappa)$ are the usual elliptic integrals. The quantities (12) and (13) are plotted as functions of the crossover parameter x_0 in Fig. 1.

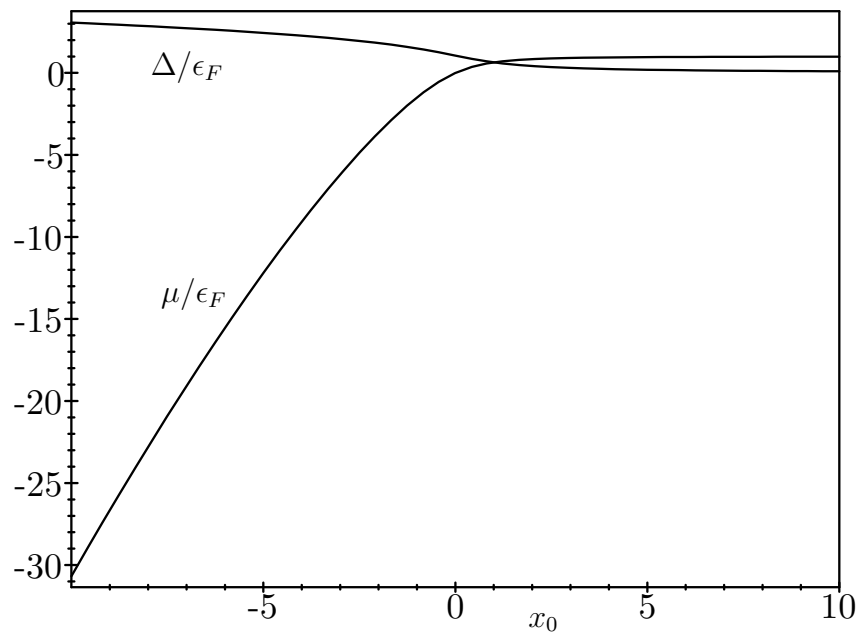


Figure 1: Gap function Δ and chemical potential μ at zero temperature as functions of x_0 in three dimensions.

In two dimensions, a nonzero bound state energy ϵ_0 exists for any coupling strength. The cutoff can therefore be eliminated by subtracting from the two-dimensional zero-

temperature gap equation

$$-\frac{1}{g} = \frac{1}{2V} \sum_{\mathbf{k}} \frac{1}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}} = \frac{m}{4\pi} \int_{-x_0}^{\infty} dz \frac{1}{\sqrt{1+z^2}}, \quad (19)$$

where $z = k^2/2m\Delta - x_0$, the bound-state equation

$$-\frac{1}{g} = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{\mathbf{k}^2/m + \epsilon_0} = \frac{m}{2\pi} \int_{-x_0}^{\infty} dz \frac{1}{2z + \epsilon_0/\Delta + 2x_0}. \quad (20)$$

After performing the elementary integrals, we find:

$$\frac{\epsilon_0}{\Delta} = \sqrt{1+x_0^2} - x_0. \quad (21)$$

From Eq. (7) we see that at zero temperature, gap and chemical potential are related to x_0 by

$$\frac{\Delta}{\epsilon_F} = \frac{2}{x_0 + \sqrt{1+x_0^2}}, \quad (22)$$

$$\frac{\mu}{\epsilon_F} = \frac{2x_0}{x_0 + \sqrt{1+x_0^2}}. \quad (23)$$

The two relations are plotted in Fig. 2. Combining (21) with (22) we find the dependence of the ratio ϵ_0/ϵ_F on the crossover parameter x_0 :

$$\frac{\epsilon_0}{\epsilon_F} = 2 \frac{\sqrt{1+x_0^2} - x_0}{\sqrt{1+x_0^2} + x_0} \quad (24)$$

We have extended all these relations to non-zero temperature. For this purpose, we do not fix the carrier density but assume the presence of a reservoir which provides us with a temperature-independent chemical potential $\mu = \mu(1/k_F a_s; T = 0)$. Such a fixed μ will be most convenient for deriving simple analytic results for the finite-temperature behavior of the system.⁵

⁵In Ref. [24], the temperature dependence of the chemical potential was calculated numerically within a "fixed carrier density model", where it turned out to be very small in comparison with the dependence on the coupling strength.

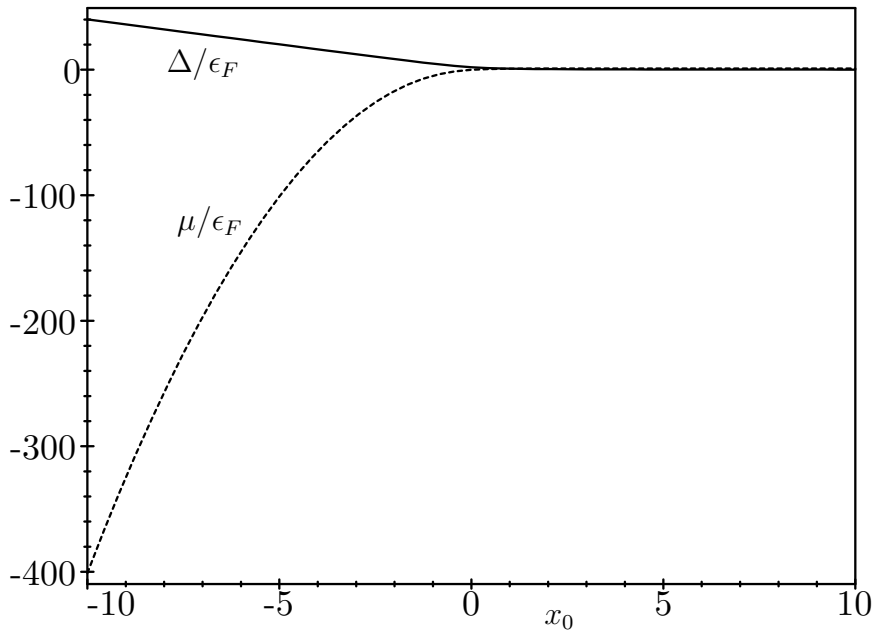


Figure 2: Gap function Δ and chemical potential μ at zero temperature as functions of x_0 in two dimensions.

In our calculation we use x_0 as the most convenient crossover parameter, since it depends via the simple relation (18) on the chemical potential which can be measured rather directly experimentally [56]. The parameter x_0 ranges from $-\infty$ in the strong-coupling (Bose-Einstein) limit to ∞ in the weak-coupling (BCS) limit. The relation between x_0 and the inverse reduced coupling strength between the electrons $1/k_F a_s$ is plotted for three-dimensional system in Fig. 3. The corresponding relation (24) in two dimensions between x_0 and the bound state energy ϵ_0 of the electron pairs is plotted on Fig. 4.

In Fig. 5 shows the temperature behavior of Δ near $T = 0$ for different coupling strengths in three dimensions. In Fig. 6 does the same thing in two dimensions. Figures 7 and 8 display dependence of the temperature T^* where the gap vanishes on the coupling strength parameter x_0 .

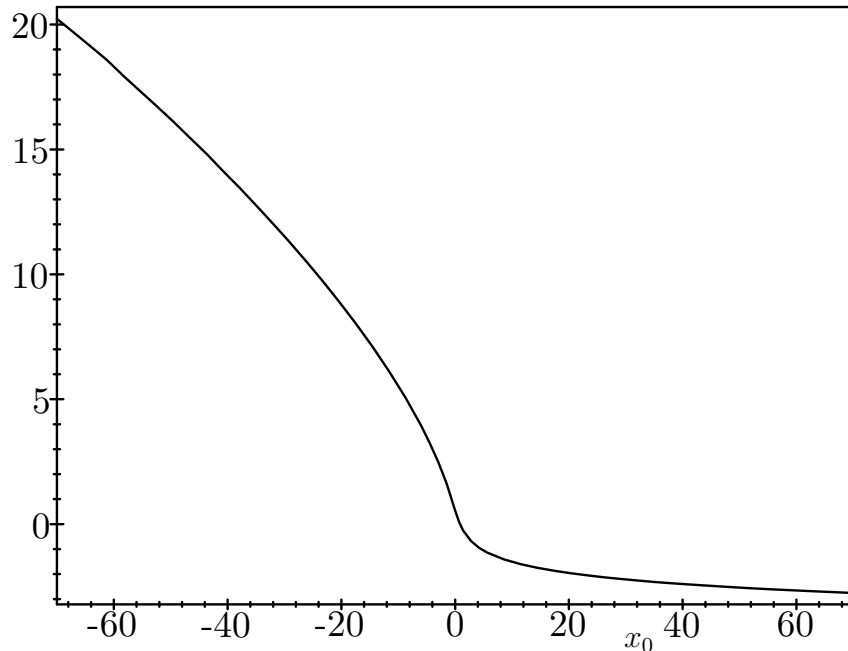


Figure 3: Dependence of $1/k_F a_s$ on the crossover parameter x_0 in two dimensions.

As was noted above, recent experiments on the underdoped cuprates showed that an ordinary superconductive gap develops smoothly to a pseudogap above T_c . Within the mean-field approximation we obtain the analytic results shown in Fig. 5 and 6.

Let us now turn to the region near zero temperature, where we can derive exact results for gap. From (12) we extract the asymptotic behavior in the three-dimensional case for $x_0 > 1$. In this region one can assume density of states to be roughly constant, since the integrand of (5) is peaked in the narrow region near $\xi = 0$. The small- T behavior is

$$\Delta(T) = \Delta(0) - \Delta(0) \sqrt{\frac{\pi}{2}} \sqrt{\frac{T}{\Delta(0)}} \exp\left[-\frac{\Delta(0)}{T}\right] \left[1 + \operatorname{erf}\left(\sqrt{\frac{\sqrt{x_0^2 + 1} - 1}{T/\Delta(0)}}\right)\right], \quad (25)$$

where $\operatorname{erf}(x)$ is the error function. Since the density of states is nearly constant in this limit, the same equation holds in two-dimensions—apart from a modified gap $\Delta(0)$ given by (22).

In the weak-coupling limit, $x_0 = \mu/\Delta(0)$ tends to infinity, and the expression above

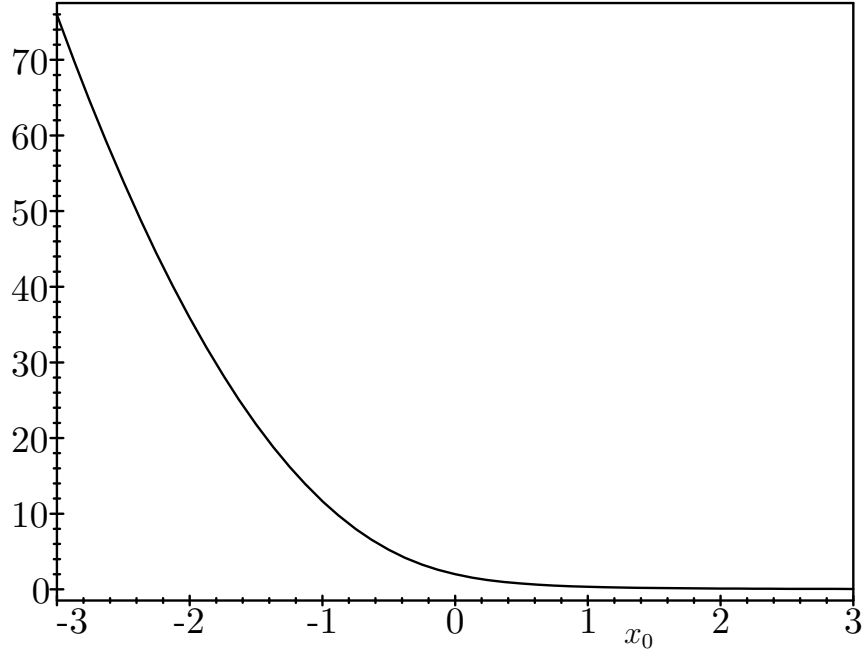


Figure 4: Dependence of ϵ_0/ϵ_F on the crossover parameter x_0 .

approaches exponentially fast the well-known BCS result:

$$\Delta(T) = \Delta(0) - [2\pi\Delta(0)T]^{1/2} \exp\left[-\frac{\Delta(0)}{T}\right] \quad (26)$$

For strong couplings with $x_0 < -1$, the three-dimensional integrands are no longer peaked in the narrow region so that the density of states can no longer be taken to be constant. Taking this into account, we find:

$$\Delta(T) = \Delta(0) - \frac{8}{\sqrt{\pi}} \sqrt{-x_0} \left(\frac{\Delta(0)}{T}\right)^{3/2} \exp\left[-\frac{\sqrt{\mu^2 + \Delta^2(0)}}{T}\right]. \quad (27)$$

From Eq. (27) we see that near $T = 0$ the gap $\Delta(T)$ tends in the strong-coupling limit exponentially to $\Delta(0)$, forming plateau near $T = 0$.

In two dimensions we arrive at similar result: an exponentially growing plateau near $T = 0$ in the strong coupling limit:

$$\Delta(T) = \Delta(0) - \frac{\Delta(0)}{2} E_1\left(\frac{\sqrt{\Delta(0)^2 + \mu^2}}{T}\right), \quad (28)$$

where E_1 is the exponential integral $E_1(z) = \int_z^\infty e^{-t}/t dt$.

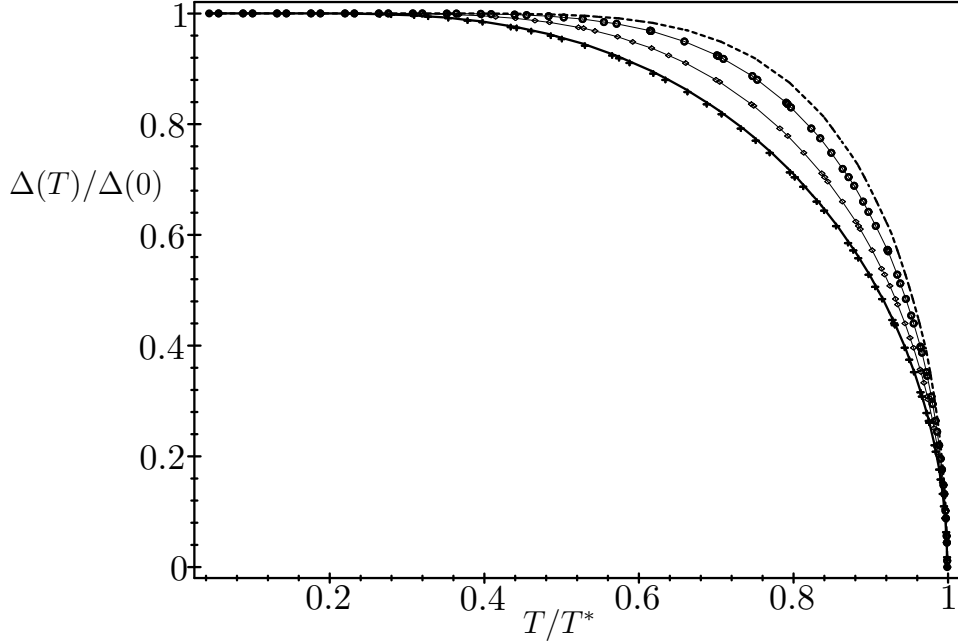


Figure 5: Temperature dependence of gap function in three dimensions. Solid line corresponds to crossover parameter $x_0 = 10$ (i.e., in the BCS regime), the crosses to $x_0 = 0$ (i.e., in the intermediate regime), lines with boxes and circles represent $x_0 = -2$ and $x_0 = -5$ cases correspondingly and the dashed line corresponds to $x_0 = -10$ (i.e., in strong-coupling regime).

For very strong couplings, Eq. (28) becomes:

$$\Delta(T) = \Delta(0) - \frac{\Delta(0)}{2} \frac{T}{\sqrt{\mu^2 + \Delta^2(0)}} \exp\left(-\frac{\sqrt{\mu^2 + \Delta(0)^2}}{T}\right) \quad (29)$$

Let us also calculate thermodynamical quantities near $T = 0$. For the thermodynamic Gibbs potential $\Omega(T, \mu, V)$ we calculate

$$\Omega = \sum_{\mathbf{k}} \left\{ \frac{\Delta^2}{2\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}} \tanh \frac{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}{2T} - 2T \log \left[2 \cosh \frac{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}{2T} \right] + \xi_{\mathbf{k}} \right\}. \quad (30)$$

Here and in the sequel in this section, $\Delta(0)$ will be replaced by Δ . In three dimensions, Eq. (30) turns into the

$$\frac{\Omega}{V} = \kappa_3 \int_{-\mu}^{\infty} d\xi \sqrt{\xi + \mu} \left[\frac{\Delta^2}{2\sqrt{\xi^2 + \Delta^2}} \tanh \frac{\sqrt{\xi^2 + \Delta^2}}{2T} - 2T \log \left(2 \cosh \frac{\sqrt{\xi^2 + \Delta^2}}{2T} \right) + \xi \right], \quad (31)$$

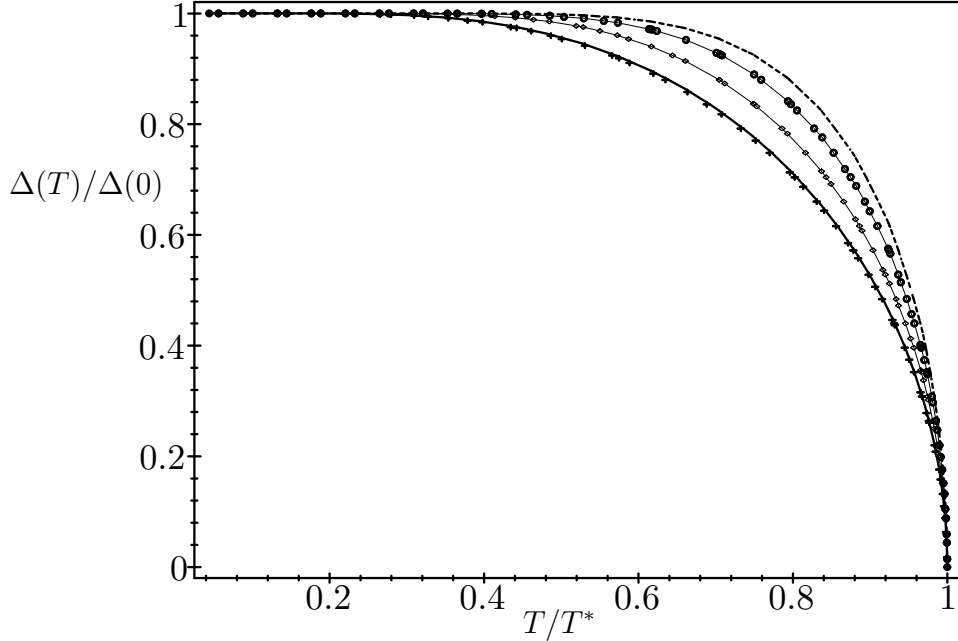


Figure 6: Temperature dependence of gap function in two dimensions. Solid line corresponds to crossover parameter $x_0 = 10$ (i.e., in the BCS regime), the crosses to $x_0 = 0$ (i.e., in the intermediate regime), lines with boxes and circles represent $x_0 = -2$ and $x_0 = -5$ cases correspondingly and the dashed line corresponds to $x_0 = -10$ (i.e., in strong-coupling regime).

In two dimensions, we obtain instead:

$$\frac{\Omega}{V} = \kappa_2 \int_{-\mu}^{\infty} d\xi \left[\frac{\Delta^2}{2\sqrt{\xi^2 + \Delta^2}} \tanh \frac{\sqrt{\xi^2 + \Delta^2}}{2T} - 2T \log \left(2 \cosh \frac{\sqrt{\xi^2 + \Delta^2}}{2T} \right) + \xi \right], \quad (32)$$

We regularize the thermodynamic potential Ω_s of the condensate subtracting $\Omega_n = \Omega(\Delta = 0)$. At $T = 0$ and for weak couplings this is found to depend on temperature as follows:

$$\frac{\Omega_s}{V} \equiv \frac{\Omega - \Omega_n}{V} = \kappa_3 \sqrt{\mu} \left[-\frac{\Delta^2}{4} + \frac{1}{2} \mu |\mu| - \frac{1}{2} \mu \sqrt{\mu^2 + \Delta^2} \right] \quad (33)$$

In the BCS limit ($x_0 \rightarrow \infty$) this reduces to the well-known result

$$\frac{\Omega_s}{V} = \kappa_3 \sqrt{\mu} \left[-\frac{\Delta^2}{2} \right]. \quad (34)$$

In two dimensions, we have a formula valid for any strength of coupling:

$$\frac{\Omega_s}{V} = \kappa_2 \left[-\frac{\Delta^2}{4} + \frac{1}{2} \mu |\mu| - \frac{1}{2} \mu \sqrt{\mu^2 + \Delta^2} \right], \quad (35)$$

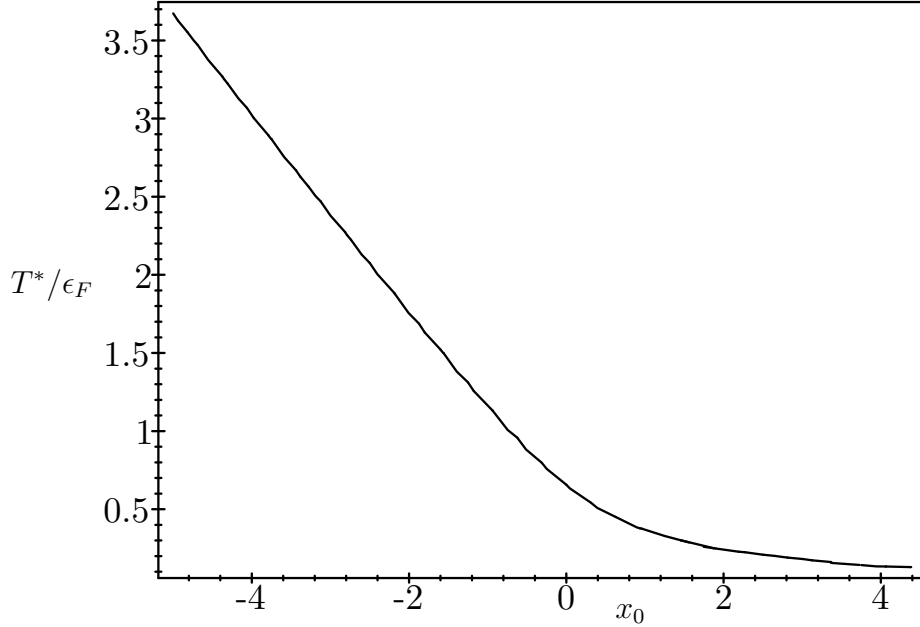


Figure 7: Dependence of T^* on crossover parameter in three dimensions.

with the BCS limit

$$\frac{\Omega_s}{V} = \kappa_2 \left[-\frac{\Delta^2}{2} \right]. \quad (36)$$

In both three- and two-dimensional cases in the BCS limit we can write a small temperature correction $\pi T^2/3$ to the thermodynamic potential. In the opposite limit of strong couplings, we find in three dimensions the strong-coupling limit:

$$\frac{\Omega}{V} = -\frac{\pi}{64} \kappa_3 \Delta^{5/2} (-x_0)^{-3/2}. \quad (37)$$

The gap $\Delta(0)$ has by Eq. (11) the strong-coupling limit $\Delta(0) \approx \epsilon_F [16/3\pi]^{3/2} |x_0|^{1/3}$ yielding the large- x_0 behavior

$$\frac{\Omega}{V} \sim -\kappa_3 \epsilon_F^{5/2} \frac{\pi}{64} \left(\frac{16}{3\pi} \right)^{15/4} |x_0|^{-2/3}. \quad (38)$$

In two dimensions, we substitute the gap function Δ of Eq. (22), into the thermodynamic potential (35), and obtain for strong couplings where $\mu < 0$:

$$\frac{\Omega}{V} \equiv 0 \quad (39)$$

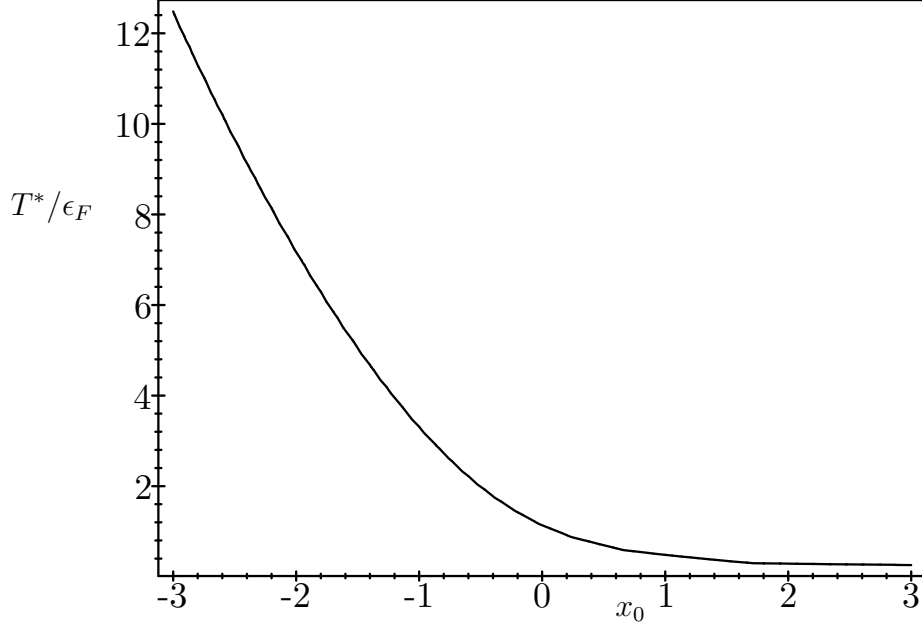


Figure 8: Dependence of T^* on crossover parameter in two dimensions.

Let us now turn to the entropy. In three dimensions near $T = 0$ it is given for weak couplings by:

$$\frac{S}{V} = \kappa_3 \sqrt{\mu} \left\{ \sqrt{\frac{2\pi\Delta^3}{T}} \exp\left(-\frac{\Delta}{T}\right) \left[1 + \operatorname{erf}\left(\sqrt{\frac{\sqrt{x_0^2 + 1} - 1}{T/\Delta}}\right) \right] + 2\mu \exp\left(-\frac{\sqrt{\mu^2 + \Delta^2}}{T}\right) \right\}, \quad (40)$$

For $\mu/\Delta \rightarrow \infty$, this reduces correctly to the BCS result:

$$\frac{S}{V} = \kappa_3 \sqrt{\mu} \sqrt{\frac{8\pi\Delta^3}{T}} \exp\left(-\frac{\Delta}{T}\right) \quad (41)$$

In two dimensions, the result is similar [with $\Delta = \Delta(0)$ given by Eq. (22)]:

$$\frac{S}{V} = \kappa_2 \left\{ \sqrt{\frac{2\pi\Delta^3}{T}} \exp\left(-\frac{\Delta}{T}\right) \left[1 + \operatorname{erf}\left(\sqrt{\frac{\sqrt{x_0^2 + 1} - 1}{T/\Delta}}\right) \right] + 2\mu \exp\left(-\frac{\sqrt{\mu^2 + \Delta^2}}{T}\right) \right\}, \quad (42)$$

The BCS limit of this is

$$\frac{S}{V} = \kappa_2 \sqrt{\frac{8\pi\Delta^3}{T}} \exp\left(-\frac{\Delta}{T}\right) \quad (43)$$

In the strong-coupling limit where $\mu/\Delta \ll -1$, we have for the entropy in three dimensions

$$\frac{S}{V} = \kappa_3 \frac{\sqrt{\pi}}{4} T^{1/2} \sqrt{\mu^2 + \Delta^2} \exp\left(-\frac{\sqrt{\mu^2 + \Delta^2}}{T}\right), \quad (44)$$

and in two dimensions:

$$\frac{S}{V} = -2\kappa_2 \mu \exp\left(-\frac{\sqrt{\mu^2 + \Delta^2}}{T}\right) \quad (45)$$

From the entropy, we easily derive the heat capacity at a constant volume c_V . In three dimensions it is given near $T = 0$ for weak couplings by

$$c_V = \kappa_3 \sqrt{\mu} \sqrt{2\pi} \Delta^3 \left\{ \frac{\Delta}{T^{3/2}} \exp\left(-\frac{\Delta}{T}\right) \left[1 + \operatorname{erf}\left(\sqrt{\frac{\sqrt{x_0^2 + 1} - 1}{T/\Delta}}\right) \right] \right\} \quad (46)$$

reducing in the limit $x_0 \rightarrow \infty$ to the BCS result

$$c_V = \kappa_3 \sqrt{\mu} \sqrt{2\pi} \Delta^3 \frac{2\Delta}{T^{3/2}} \exp\left(-\frac{\Delta}{T}\right). \quad (47)$$

In two dimensions, the weak-coupling behavior is

$$c_V = \kappa_2 \sqrt{2\pi} \Delta^3 \left\{ \frac{\Delta}{T^{3/2}} \exp\left(-\frac{\Delta}{T}\right) \left[1 + \operatorname{erf}\left(\sqrt{\frac{\sqrt{x_0^2 + 1} - 1}{T/\Delta}}\right) \right] \right\}, \quad (48)$$

while the strong-coupling behavior in three dimensions is

$$c_V = \kappa_3 \frac{\sqrt{\pi}}{4} T^{-1/2} (\mu^2 + \Delta^2) \exp\left(-\frac{\sqrt{\mu^2 + \Delta^2}}{T}\right) \quad (49)$$

and in two dimensions

$$c_V = 2\kappa_2 \frac{\mu^2}{T} \exp\left(-\frac{\sqrt{\mu^2 + \Delta^2}}{T}\right). \quad (50)$$

3 Crossover from BCS Superconductivity Near T_c to Onset of Pseudogap Behavior

We now turn to the region near T^* , for which we derive asymptotic behavior of the ratios $\Delta(T)/T^*$ and $\Delta(T)/\Delta(0)$ as well as thermodynamic quantities. In doing so, we shall

consider $\Delta(T)/T$ as a small parameter of the problem. In the calculations near T^* it is convenient to use $\mu/2T^*$ (this ratio tend to ∞ in the weak-coupling limit and to the $-\infty$ in the strong coupling one) as a crossover parameter rather than x_0 . In three dimensions, we find for weak couplings

$$\left[\frac{\Delta(T)}{2T^*}\right]^2 = \frac{\left(1 - \frac{T}{T^*}\right) \left(1 + \tanh \frac{\mu}{2T^*}\right)}{\frac{1}{4} \left[\frac{1}{\mu/2T^*} - \frac{1}{(\mu/2T^*)^2} \tanh \frac{\mu}{2T^*}\right] + \left(\frac{2}{\pi}\right)^2 \left(1 + \frac{2}{\pi} \arctan \frac{\mu}{\pi T^*}\right)}. \quad (51)$$

In the limit $\mu/2T^* \rightarrow \infty$ this tends to the BCS result

$$\frac{\Delta(T)}{T_c} \simeq 3\sqrt{1 - \frac{T}{T_c}}. \quad (52)$$

In the opposite limit of strong couplings, T^* and $\Delta(0)$ tend to infinity. The ratio $\Delta(T)/T$ near T^* tends to zero exponentially as a function of the crossover parameter $\mu/2T^*$:

$$\left[\frac{\Delta(T)}{2T^*}\right]^2 = \frac{16}{\sqrt{2\pi}} \left(1 - \frac{T}{T^*}\right) \left(-\frac{\mu}{2T^*}\right)^{3/2} e^{\mu/T^*}. \quad (53)$$

In two-dimensions the near- T^* formula (51) holds over the crossover region: In the weak-coupling limit, this formula reproduces the BCS result (52). In the strong-coupling limit, we find as in three dimensions a ratio $\Delta(T)/T$ which tends to zero exponentially as a function of the crossover parameter μ/T^* :

$$\left[\frac{\Delta(T)}{2T^*}\right]^2 = 2 \left[\frac{1}{4} - \left(\frac{2}{\pi}\right)^2\right]^{-1} \left(\frac{\mu}{2T^*}\right) \left(1 - \frac{T}{T^*}\right) e^{\mu/T}. \quad (54)$$

Let us calculate the dependence of T^* on the crossover parameter $\mu/2T^*$ in the strong-coupling limit. In three dimensions, we obtain from Eq. (5) the relation

$$\frac{T^*}{\epsilon_F} = \left(\frac{1}{3}\right)^{2/3} \exp\left(-\frac{2}{3} \frac{\mu}{T^*}\right). \quad (55)$$

This is solved for T^* (up to a logarithm) by

$$T^* \simeq -\frac{2}{3}\mu \log^{-1}\left(-\frac{\mu}{\epsilon_F}\right) \quad (56)$$

[see also the discussion after the formula (9)]. As a function of the crossover parameter x_0 we obtain

$$\frac{T^*}{\epsilon_F} \simeq \frac{1}{2} \left(\frac{16}{3\pi} \right)^{2/3} |x_0|^{4/3} \log^{-1} \left(\sqrt{16/\pi} |x_0| \right). \quad (57)$$

In two dimensions we find from (6)

$$\frac{T^*}{\epsilon_F} = \frac{1}{2} \exp \left(-\frac{\mu}{T^*} \right). \quad (58)$$

and thus

$$T^* \simeq -\mu \log^{-1} \left(-\frac{\mu}{\epsilon_F} \right) \quad (59)$$

As a function of x_0 , this implies

$$\frac{T^*}{\epsilon_F} = 2x_0^2 \log^{-1} \left(2\sqrt{2}|x_0| \right). \quad (60)$$

Let us also derive in the strong-coupling region the dependence of the ratio $\Delta(0)/T^*$ on the crossover parameter which in three dimensions reads

$$\frac{\Delta(0)}{T^*} = \frac{4}{\sqrt{\pi}} \left(-\frac{\mu}{T^*} \right)^{1/4} \exp \left(\frac{\mu}{2T^*} \right), \quad (61)$$

and in two dimensions:

$$\frac{\Delta(0)}{T^*} = 4 \left(-\frac{\mu}{2T^*} \right)^{1/2} \exp \left(\frac{\mu}{2T^*} \right). \quad (62)$$

In the weak-coupling regime, both three- and two- dimensional cases yield the result

$$\frac{\Delta(0)}{T^*} = \frac{\pi}{e^\gamma} \left(1 - \frac{\Delta(0)^2}{4\mu^2} \right)^{-1/2} = \frac{\pi}{e^\gamma} \left(1 - \frac{1}{4x_0^2} \right)^{-1/2} \simeq \frac{\pi}{e^\gamma} \left(1 + \frac{1}{8x_0^2} \right). \quad (63)$$

The temperature T^* is in the weak-coupling regime of three- and two dimensional systems the following function of x_0 :

$$\frac{T^*}{\epsilon_F} \simeq \frac{e^\gamma}{\pi} \left(\frac{1}{x_0} - \frac{3}{8x_0^3} \right) \quad (64)$$

Using these results, we can also calculate the asymptotic behavior of the ratio $\Delta(T)/\Delta(0)$ near T^* . In three dimensions, the strong-coupling limit yields

$$\left[\frac{\Delta(T)}{\Delta(0)} \right]^2 = \frac{\sqrt{\pi}}{2} \left(-\frac{\mu}{2T^*} \right) \left(1 - \frac{T}{T^*} \right). \quad (65)$$

In two dimensions we have that in the strong-coupling limit this ratio tends to

$$\left[\frac{\Delta(T)}{\Delta(0)}\right]^2 = \frac{1}{8} \left(\frac{4}{\pi^2} - \frac{1}{4}\right)^{-1} \left(1 - \frac{T}{T^*}\right). \quad (66)$$

At weak couplings, both three- and two-dimensional gap functions are given by

$$\left[\frac{\Delta(T)}{\Delta(0)}\right]^2 = \frac{4\pi^2}{e^{2\gamma}} \frac{\left(1 - \frac{T}{T^*}\right) \left[1 + \tanh \frac{\mu}{2T^*}\right]}{\frac{1}{4} \left[\frac{1}{\mu/2T^*} - \frac{1}{(\mu/2T^*)^2} \tanh \frac{\mu}{2T^*}\right] + \left(\frac{2}{\pi}\right)^2 \left(1 + \frac{2}{\pi} \arctan \frac{\mu}{\pi T^*}\right)} \quad (67)$$

In order to calculate thermodynamic potential near T^* we expand the general expression by $\Delta(T)/\Delta(0)$ and keeping the terms of the same order we get :

$$\begin{aligned} \frac{\Omega_s}{V} \simeq & -\frac{(T^* - T)\Delta^2}{4T^*} \int \frac{d^D \mathbf{p}}{(2\pi)^D} \cosh^{-2} \frac{\xi}{2T^*} \\ & - \frac{\Delta^4}{8} \int \frac{d^D \mathbf{p}}{(2\pi)^D} \frac{1}{\xi^2} \left(\frac{1}{2T^*} \cosh^{-2} \frac{\xi}{2T^*} - \frac{1}{\xi} \tanh \frac{\xi}{2T^*} \right) \end{aligned} \quad (68)$$

where D is the space dimension. Note once more that we consider here the temperature evolution of the system at a fixed chemical potential ($\mu(T, 1/k_F a_s) = \mu(0, 1/k_F a_s)$) and regularize Ω by a subtraction of $\Omega_n = \Omega(\Delta = 0)$.

The thermodynamic potential in three dimensions in the weak-coupling regime near T^* is given by

$$\begin{aligned} \frac{\Omega_s}{V} = & -\kappa_3 \sqrt{\mu} \left\{ \frac{(T^* - T)\Delta^2}{2T^*} \left[1 + \tanh \frac{\mu}{2T^*}\right] + \right. \\ & \left. + \frac{\Delta^4}{4} \frac{1}{(2T^*)^2} \left[\frac{1}{4} \left(\frac{1}{\mu/2T^*} - \frac{1}{(\mu/2T^*)^2} \tanh \frac{\mu}{2T^*} \right) + \left(\frac{2}{\pi}\right)^2 \left(1 + \frac{2}{\pi} \arctan \frac{\mu}{\pi T^*}\right) \right] \right\}. \end{aligned} \quad (69)$$

In the BCS limit, this reduces to the well-known formula:

$$\frac{\Omega_s}{V} = -\kappa_3 \sqrt{\mu} \Delta^2 \left(1 - \frac{T}{T_c} - \frac{1}{2\pi^2} \frac{\Delta^2}{T_c^2}\right) \quad (70)$$

In the strong-coupling limit we have:

$$\frac{\Omega_s}{V} = -\kappa_3 \left\{ \frac{\pi}{64} \Delta^4 (2T^*)^{-3/2} \left(-\frac{\mu}{2T^*}\right)^{-3/2} + \left(1 - \frac{T}{T^*}\right) \Delta^2 \frac{\sqrt{\pi}}{2} \sqrt{T^*} \exp\left(\frac{\mu}{T^*}\right) \right\} \quad (71)$$

Using the asymptotic estimates derived above for the strong-coupling limit, and the fact that in this limit

$$\frac{\mu}{T^*} \simeq -\log\left(-\frac{\mu}{\varepsilon_F}\right) \simeq -\text{const} \times \log(|x_0|), \quad (72)$$

we find near T^* the difference between the thermodynamic potential of the gapless and pseudogapped normal states:

$$\frac{\Omega_s}{V} \simeq -\text{const} \left(1 - \frac{T}{T^*}\right)^2 |x_0|^{-3/2}. \quad (73)$$

In two dimensions near $T = T^*$, the thermodynamic potential of the gas of pairs is given by the formula holding for the crossover region

$$\begin{aligned} \frac{\Omega_s}{V} = & -\kappa_2 \left\{ \frac{(T^* - T)\Delta^2}{2T^*} \left[1 + \tanh \frac{\mu}{2T^*}\right] + \right. \\ & \left. + \frac{\Delta^4}{4} \frac{1}{(2T^*)^2} \left[\frac{1}{4} \left(\frac{1}{\mu/2T^*} - \frac{1}{(\mu/2T^*)^2} \tanh \frac{\mu}{2T^*} \right) + \left(\frac{2}{\pi}\right)^2 \left(1 + \frac{2}{\pi} \arctan \frac{\mu}{\pi T^*}\right) \right] \right\}. \end{aligned} \quad (74)$$

In the BCS limit, this yields the familiar result

$$\frac{\Omega_s}{V} = -\kappa_2 \Delta^2 \left(1 - \frac{T}{T^*} - \frac{1}{2\pi^2} \frac{\Delta^2}{T^{*2}}\right), \quad (75)$$

and in the strong-coupling limit:

$$\frac{\Omega_s}{V} = -\kappa_2 \left\{ \left(1 - \frac{T}{T^*}\right) \Delta^2 \exp\left(\frac{\mu}{T^*}\right) + \frac{\Delta^4}{4} \frac{1}{(2T^*)^2} \left[\left(\frac{1}{4} - \frac{4}{\pi^2}\right) \frac{1}{\mu/2T^*} \right] \right\}. \quad (76)$$

Using the earlier-derived asymptotic behavior plus the limiting equation (72) which also holds for two-dimensional case, we derive for the thermodynamic potential the x_0 -behavior

$$\frac{\Omega_s}{V} \simeq -\text{const} \times \left(1 - \frac{T}{T^*}\right)^2 \log |x_0|. \quad (77)$$

The entropy behaves near T^* in three dimensions in the weak-coupling regime like

$$\frac{S_s}{V} \equiv \frac{S - S_n}{V} = -\kappa_3 \sqrt{\mu} \frac{\Delta^2}{2T^*} \left[1 + \tanh\left(\frac{\mu}{2T^*}\right)\right] \quad (78)$$

with the BCS limit

$$\frac{S_s}{V} = -\kappa_3 \sqrt{\mu} \frac{\Delta^2}{T_c}. \quad (79)$$

The opposite strong-coupling limit is in three dimensions:

$$\frac{S_s}{V} = -\kappa_3 \frac{\sqrt{\pi}}{2} \Delta^2 T^{*-1/2} \exp\left(\frac{\mu}{T^*}\right). \quad (80)$$

Inserting the above asymptotic formulas for Δ , μ , T^* , we find

$$\frac{S_s}{V} \simeq -\text{const} \times \left(1 - \frac{T}{T^*}\right) |x_0|^{-5/3}. \quad (81)$$

In two dimensions, the entropy is given in the entire crossover region by

$$\frac{S_s}{V} = -\kappa_2 \frac{\Delta^2}{2T^*} \left[1 + \tanh \frac{\mu}{2T^*}\right], \quad (82)$$

and has the BCS limit

$$\frac{S_s}{V} = -\kappa_2 \frac{\Delta^2}{T^*}, \quad (83)$$

while the strong-coupling limit becomes:

$$\frac{S_s}{V} = -\kappa_2 \frac{\Delta^2}{T^*} e^{\mu/T^*}. \quad (84)$$

Using corresponding asymptotic formulas for Δ , μ , T^* in two dimensions, this depends on x_0 as

$$\frac{S_s}{V} = -\text{const} \times \left(1 - \frac{T}{T^*}\right) x_0^{-2}. \quad (85)$$

In order to derive the specific heat we must take into account the temperature dependence of the gap.

In three dimensions, we find in the weak-coupling region near T^* :

$$\frac{C_s}{V} = 2T\kappa_3\sqrt{\mu} \frac{\left(1 + \tanh \frac{\mu}{2T^*}\right)^2}{\frac{1}{4} \left[\frac{1}{\mu/2T^*} - \frac{1}{(\mu/2T^*)^2} \tanh \frac{\mu}{2T^*} \right] + \left(\frac{2}{\pi}\right)^2 \left(1 + \frac{2}{\pi} \arctan \frac{\mu}{\pi T^*}\right)}, \quad (86)$$

which has the well-known BCS limit:

$$\frac{C_s}{V} \simeq \kappa_3 \sqrt{\mu} \pi^2 T_c. \quad (87)$$

In the strong-coupling limit, we find in three dimensions

$$\frac{C_s}{V} = \kappa_3 16\sqrt{2} T^{*3/2} \left(-\frac{\mu}{2T^*}\right)^{3/2} e^{2\mu/T^*} \quad (88)$$

Inserting earlier derived asymptotic formulas we see that C_s tends in the strong-coupling limit to zero like

$$\frac{C_s}{V} \sim \text{const} \times |x_0|^{-2} \quad (89)$$

In two dimensions, the result for the entire crossover region reads

$$\frac{C_s}{V} = 2T^* \kappa_2 \frac{\left(1 + \tanh \frac{\mu}{2T^*}\right)^2}{\frac{1}{4} \left[\frac{1}{\mu/2T^*} - \frac{1}{(\mu/2T^*)^2} \tanh \frac{\mu}{2T^*} \right] + \left(\frac{2}{\pi}\right)^2 \left(1 + \frac{2}{\pi} \arctan \frac{\mu}{\pi T^*}\right)}. \quad (90)$$

This becomes in the BCS limit

$$\frac{C_s}{V} \simeq \kappa_2 \pi^2 T^*, \quad (91)$$

and in the strong-coupling limit

$$\frac{C_s}{V} = 4\kappa_2 \mu \left(\frac{1}{4} - \frac{4}{\pi^2}\right)^{-1} \exp\left(\frac{2\mu}{T^*}\right). \quad (92)$$

As a function of x_0 , the result is

$$\frac{C_s}{V} \sim \text{const} \times x_0^{-2}. \quad (93)$$

From the above calculation near T^* we see that both quantities S_s and C_s tend quickly to zero with growing coupling strength in the pseudogapped regime (like a power of the crossover parameter $|x_0|$ or exponentially as a function of crossover parameter $\mu/2T^*$). So, at very strong couplings T^* is getting less and less pronounced with increasing coupling strength.

Note that in the strong-coupling regime, the modified gap function $\sqrt{\mu^2 + \Delta^2}$ [see Eq. 10] enters the expressions for thermodynamical quantities below T^* the same way as an ordinary gap in BCS limit [see Eqs. (44), (45), (49), (50)].

4 Phase fluctuations in Two Dimensions and Kosterlitz-Thouless Transition

In the previous sections we have calculated the properties of the model in the mean-field approximation.

Now we are ready to go beyond the mean-field approximation, which as discussed in the introduction, supply us with modulus of the gap function. In this chapter we make use of derivative expansion which determines the crucial stiffness parameter for the study of phase fluctuations, that in two dimensions leads to the Kosterlitz-Thouless transition, at which the expectation of the complex order field $\Delta(\mathbf{x}) = |\Delta|(\mathbf{x})e^{i\theta(\mathbf{x})}$ vanishes in the pseudogap state. In these calculations we assume with other authors ([24], [54]) that the phase fluctuations do not significantly affect modulus of Δ . We shall first study the two-dimensional system, where the mean-field solution receives the strongest modifications from the violent phase fluctuations, as articulated by the Coleman-Mermin-Wagner-Hohenberg theorem [57] which forbids the existence of a strict long-range order, leading to a power behavior of correlation functions for all temperatures below T_{KT} .

The crossover of the Kosterlitz-Thouless transition from weak to strong coupling was first considered in [14, 26], and studied recently by means of an XY-model in [24], with the stiffness derived from a fixed nonvanishing modulus of the order parameter Δ . In Appendix A, we outline the derivation of the effective Hamiltonian in [24]. Under the same assumptions, we shall analyze the weak- to strong-coupling crossover of the Kosterlitz-Thouless transition in the present work.

Writing the spacetime-dependent order parameter as $\Delta(x)e^{i\theta(x)}$, where x denotes the four-vector $x = (\tau, \mathbf{x})$ formed from imaginary time and position vector, the partition function may be written as a functional integral [53, 60, 58, 59]

$$Z(\mu, T) = \int \Delta \mathcal{D}\Delta \mathcal{D}\theta \exp[-\beta\Omega(\mu, T, \Delta(x), \partial\theta(x))], \quad (94)$$

where

$$\beta\Omega(\mu, T, \Delta(x), \partial\theta(x)) = \frac{1}{g} \int_0^\beta d\tau \int d\mathbf{x} \Delta^2(x) - \text{Tr} \log G^{-1} + \text{Tr} \log G_0^{-1} \quad (95)$$

is the one-loop effective action, containing the inverse Green function of the fermions in the collective pair field

$$\begin{aligned} G^{-1} &= -\hat{I}\partial_\tau + \tau_3 \left(\frac{\nabla^2}{2m} + \mu \right) + \tau_1 \Delta(\tau, \mathbf{x}) \\ &- \tau_3 \left[\frac{i\partial_\tau \theta(\tau, \mathbf{x})}{2} + \frac{(\nabla\theta(\tau, \mathbf{x}))^2}{8m} \right] + \hat{I} \left[\frac{i\nabla^2\theta(\tau, \mathbf{x})}{4m} + \frac{i\nabla\theta(\tau, \mathbf{x})\nabla}{2m} \right]. \end{aligned} \quad (96)$$

Here τ_1, τ_3 are the usual Pauli matrices, and $G_0 = G|_{\mu, \Delta, \theta=0}$ is added for regularization.

Let us now assume this smoothness implying that phase gradients are small. Then $\Omega(\mu, T, \Delta(x), \partial\theta(x))$ can be approximated as follows:

$$\Omega(\mu, \Delta(x), \partial\theta(x)) \simeq \Omega_{\text{kin}}(\mu, T, \Delta, \partial\theta(x)) + \Omega_{\text{pot}}(\mu, T, \Delta), \quad (97)$$

with the “kinetic” term (see [24], [59])

$$\Omega_{\text{kin}}(\mu, T, \Delta, \partial\theta(x)) = T \text{Tr} \sum_{n=1}^{\infty} \frac{1}{n} (\mathcal{G}\Sigma)^n \Big|_{\Delta=\text{const}} \quad (98)$$

and the “potential” term

$$\Omega_{\text{pot}}(\mu, T, \Delta) = \left(\frac{1}{g} \int d^D x \Delta^2 - T \text{Tr} \log \mathcal{G}^{-1} + T \text{Tr} \log G_0^{-1} \right) \Big|_{\Delta=\text{const}}. \quad (99)$$

The latter coincides with our earlier mean-field energy (see also Appendix A), determining the modulus of $\Delta(\mu, T)$ and thus the stiffness of phase fluctuations. The kinetic part Ω_{kin} contains gradient terms whose size is determined by the modulus of $\Delta(\mu, T)$. Given the stiffness, one may immediately set up an equivalent XY-model. Both Ω_{kin} and Ω_{pot} are expressed in terms of the Green function of the fermions, which solves the equation

$$\left[-\hat{I} \partial_\tau + \tau_3 \left(\frac{\nabla^2}{2m} + \mu \right) + \tau_1 \Delta \right] \mathcal{G}(\tau, \mathbf{x}) = \delta(\tau) \delta(\mathbf{x}) \quad (100)$$

and

$$\Sigma(\partial\theta) \equiv \tau_3 \left[\frac{i \partial_\tau \theta}{2} + \frac{(\nabla\theta)^2}{8m} \right] - \hat{I} \left[\frac{i \nabla^2 \theta}{4m} + \frac{i \nabla\theta(\tau, \mathbf{x}) \nabla}{2m} \right]. \quad (101)$$

The gradient expansion that we use to determine stiffness was first made in Ref. [60] at zero temperature. In Ref. [24], the kinetic term Ω_{kin} was calculated at finite temperature for arbitrary chemical potential retaining terms with $n = 1, 2$ in the expansion (98).

The result is (see Appendix A)

$$\Omega_{\text{kin}} = \frac{T}{2} \int_0^\beta d\tau \int d^D x \left[n(\mu, T, \Delta) i \partial_\tau \theta + J(\mu, T, \Delta(\mu, T)) (\nabla\theta)^2 + K(\mu, T, \Delta(\mu, T)) (\partial_\tau \theta)^2 \right], \quad (102)$$

where $J(\mu, T, \Delta)$ is the stiffness coefficient

$$J(\mu, T, \Delta) = \frac{1}{4m} n(\mu, T, \Delta) - \frac{T}{4\pi} \int_{-\mu/2T}^{\infty} dx \frac{x + \mu/2T}{\cosh^2 \sqrt{x^2 + \Delta^2/4T^2}}, \quad (103)$$

$$K(\mu, T, \Delta) = \frac{m}{8\pi} \left(1 + \frac{\mu}{\sqrt{\mu^2 + \Delta^2}} \tanh \frac{\sqrt{\mu^2 + \Delta^2}}{2T} \right), \quad (104)$$

and $n(\mu, T, \Delta)$ is the density of fermions (7) which varies with temperature in our model. At the temperature T^* where the modulus of Δ vanishes, also the stiffness disappears. The kinetic term corresponds to an XY-model with a Hamiltonian [61], [8]:

$$H = \frac{J}{2} \int d\mathbf{x} [\nabla\theta(\mathbf{x})]^2, \quad (105)$$

the only difference with the standard XY-model lying in the dependence of the stiffness constant J on the temperature, which is determined from the solutions of gap and number equations (2) and (3). Clearly, in this model the Kosterlitz-Thouless transition always take place below T^* . In the XY-model with vortices of a high fugacity, the temperature of the phase transition is determined by a simple formula [62]:

$$T_{\text{KT}} = \frac{\pi}{2} J \quad (106)$$

which follows from the divergence of the average square size of a vortex-antivortex pair. Since these attract each other by a Coulomb potential $v(r) = 2\pi J \log(r/r_0)$, the average square distance is

$$\langle r^2 \rangle \propto \int_{r_0}^{\infty} dr r r^2 e^{-(2\pi J/T) \log(r/r_0)} \propto \frac{1}{4 - 2\pi J/T}, \quad (107)$$

which diverges indeed at the temperature (106). In our case T_{KT} should be determined self-consistently:

$$T_{\text{KT}} = \frac{\pi}{2} J(\mu, T_{\text{KT}}, \Delta(\mu, T_{\text{KT}})). \quad (108)$$

From (103), (7) and (106) it is easily seen that T_{KT} indeed tends to zero when the pair attraction vanishes in which case $\Delta(T = 0) = 0$. In general, the behavior of T_{KT} for strong and weak couplings is found by the following considerations. We observe from the above-derived limiting formulas for $\Delta(T, \mu)$ and T^* , that the particle number n does not vary appreciably in these limits with temperature in the range $0 < T < T^*$, so that weak-coupling estimates for T_{KT} derived within the model with temperature-independent chemical potential (i.e. when the system is coupled to a large reservoir, see discussion after

the formula (23)) practically coincide with those derived from a fixed fermion density. Further it is immediately realized from the equations (108), (7), (103) and (6) that in the weak-coupling limit $\Delta(T_{\text{KT}}, \mu)/T_{\text{KT}}$ is a small parameter. At zero coupling, the stiffness $J(\mu, T_{\text{KT}}, \Delta(\mu, T_{\text{KT}}))$ vanishes identically, such that an estimate of J at weak couplings requires calculating a lowest-order correction to the second term of eq.(103) proportional to $\Delta(T_{\text{KT}}, \mu)/T_{\text{KT}}$. Thus weak-coupling expression for stiffness reads [24]:

$$J(T) \simeq \frac{7\zeta(3)}{16\pi^3} \epsilon_F \frac{\Delta(T)^2}{T^{*2}}. \quad (109)$$

Using the limiting behavior (51) of $\Delta(T)$ [see also discussion after (53)] we find after some algebra the weak-coupling equation for T_{KT} :

$$T_{\text{KT}} \simeq \frac{\epsilon_F}{4} \left(1 - \frac{T_{\text{KT}}}{T^*} \right). \quad (110)$$

where $\epsilon_F = (\pi/m)n$ is the Fermi energy of free fermions. It is useful to introduce reduced dimensionless temperatures $\tilde{T}_{KT} \equiv T_{KT}/\epsilon_F$ and $\tilde{T}^* = T^*/\epsilon_F$ which are small in the weak-coupling limit. Then we rewrite Eq. (110) as

$$\tilde{T}_{KT} \simeq \frac{1}{4} \frac{1}{1 + 1/4\tilde{T}^*}. \quad (111)$$

For small \tilde{T}^* we may expand

$$\tilde{T}_{KT} \approx \tilde{T}^* - \frac{\tilde{T}^{*2}}{4}. \quad (112)$$

This equation shows nicely how for decreasing coupling strength T_{KT} merges with T^* .

As a function of the crossover parameter x_0 , the temperature T_{KT} behaves like

$$\tilde{T}_{\text{KT}} \approx \frac{e^\gamma}{\pi} \frac{1}{x_0}. \quad (113)$$

The merging of the two temperatures in the weak-coupling regime is displayed in Fig. 9.

Consider now the opposite limit of strong couplings. There Eqs. (108), (7), (103), and (6) for T_{KT} , $n(T, \mu)$, and $\Delta(T, \mu)$ show that T_{KT} tends to a constant value. From Eqs. (103), (6), (7), and (108), as well as the limiting expressions (28) and (29) it follows that in the strong-coupling limit $\Delta(T_{\text{KT}})$ is always situated close to the zero-temperature

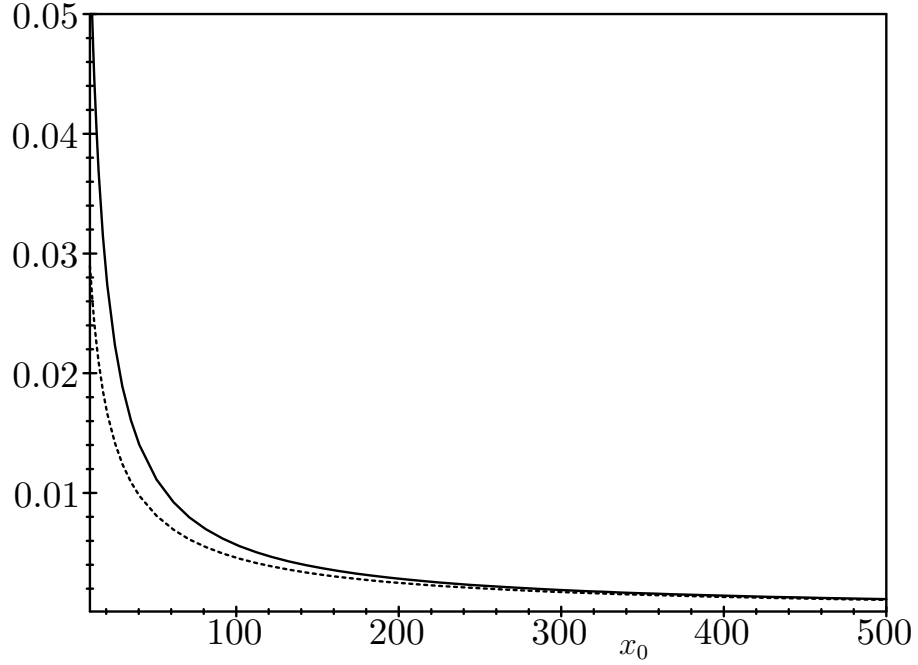


Figure 9: Weak-coupling behavior of $T_{KT}(x_0)$. The solid line is T^* , the dashed line represent T_{KT} .

value of $\Delta(T_{KT}, \mu) \approx \Delta(T = 0, \mu)$. Taking this into the account we derive an estimate for the second term in (103), thus obtaining the strong-coupling equation for T_{KT} :

$$T_{KT} \simeq \frac{\pi}{8} \left\{ \frac{1}{m} n - \frac{T_{KT}}{\pi} \exp \left[\frac{\sqrt{\mu^2 + \Delta^2(T_{KT}, \mu)}}{T_{KT}} \right] \right\} \quad (114)$$

which may further be expanded as

$$T_{KT} \simeq \frac{\pi}{8} \left\{ \frac{1}{m} n - \frac{T_{KT}}{\pi} \exp \left[\frac{\mu}{T_{KT}} + \frac{\Delta^2(T_{KT}, \mu)}{2\mu T_{KT}} \right] \right\}. \quad (115)$$

With the approximation $\Delta(T_{KT}, \mu) \approx \Delta(T = 0, \mu)$ and the limiting behavior (22) we find that the first term in the exponent tends in the strong-coupling limit to a constant, $\Delta^2(T_{KT}, \mu)/2\mu T_{KT} \rightarrow -4$, whereas the first term in the brackets tends to $-\infty$, so that Eq. (115) has the limiting form

$$T_{KT} \simeq \frac{\pi}{8} \frac{n}{m} \left\{ 1 - \frac{1}{8} \exp \left[\frac{2\mu}{\epsilon_F} - 4 \right] \right\}. \quad (116)$$

As a function of x_0 crossover parameter, this reads

$$T_{KT} \simeq \frac{\pi}{8} \frac{n}{m} \left\{ 1 - \frac{1}{8} \exp \left[8x_0^2 - 6 \right] \right\}. \quad (117)$$

Thus for increasing coupling strength, i.e., decreasing crossover parameter $x_0 \ll -1$, the phase-decoherence temperature T_{KT} tends very quickly towards a constant:

$$T_{KT} \simeq \frac{\pi}{8} \frac{n}{m}. \quad (118)$$

In this limit we know from Eq. (7) that the difference in the carrier density at zero temperature, $n(T=0)$, becomes equal to $n(T=T_{KT})$, so that our limiting result coincides with that obtained in the "fixed carrier density model":

$$T_{KT} = \frac{\epsilon_F(n_0)}{8} = \frac{\pi}{8m} n_0, \quad (119)$$

where we have inserted again $\epsilon_F(n) = (\pi/m)n$ for the Fermi energy of free fermions at the carrier density $n_0 = n(T=0)$.

Note that this strong-coupling behavior of T_{KT} coincides roughly with the estimate of T_c for three-dimensional superconductors in Refs. [2], [5], and [9] via the onset of Bose condensation of tightly bound, almost free composite bosons. In the first two of these references which include only quadratic fluctuations around the mean field (corresponding to ladder diagrams), T_c was shown to tend to a constant value which does not depend on the internal structure of composite boson and is simply equal to the condensation temperature of a gas of free bosons of mass $2m$ and density $n/2$, implying that the interactions between the composite bosons is irrelevant in this approximation.

We find the same situation in two dimensions, where T_{KT} tends to a constant depending only on the mass $2m$ and the density $n/2$ of the pairs. No dependence on the coupling strength is left. The difference with respect to the three-dimensional case is that here the transition temperature $T_c = T_{KT}$ is linear in the carrier density n , while growing like $n^{2/3}$ in three dimensions. Our result (119) agrees with Ref. [26] and [24].

If interactions between condensed and noncondensed composite are taken into account, as done in Ref. [9] in three dimensions, then T_c turns out to grow slowly with the coupling strength in the strong coupling regime.

Equation (119) determines the critical temperature in the strong-coupling limit completely in terms of the carrier density n_0 . There exists a corresponding equation for the

temperature T^* in the strong-coupling limit $\epsilon_0 \gg \epsilon_F$:

$$T^* \simeq \frac{\epsilon_0}{2} \frac{1}{\log \epsilon_0/\epsilon_F}. \quad (120)$$

For experimental purposes, the dependence of the ratio $2\Delta(0)/T_{KT}$ on the coupling strength is of interest. It is plotted in Fig. 10. Analytically, we have in the weak-coupling limit

$$\frac{2\Delta(0)}{T_{KT}} = \frac{2\pi}{e^\gamma} \left\{ 1 + \frac{e^\gamma}{\pi} \frac{4}{x_0} + \left[\frac{1}{8} + \left(\frac{4e^\gamma}{\pi} \right)^2 \right] \frac{1}{x_0^2} \right\} + \mathcal{O}(x_0^{-3}), \quad (121)$$

and for strong-couplings:

$$\frac{2\Delta(0)}{T_{KT}} \simeq \frac{32}{\sqrt{x_0^2 + 1} + x_0} \simeq -64x_0. \quad (122)$$

The two curves can easily be interpolated graphically to all coupling strengths, as seen in the figure.

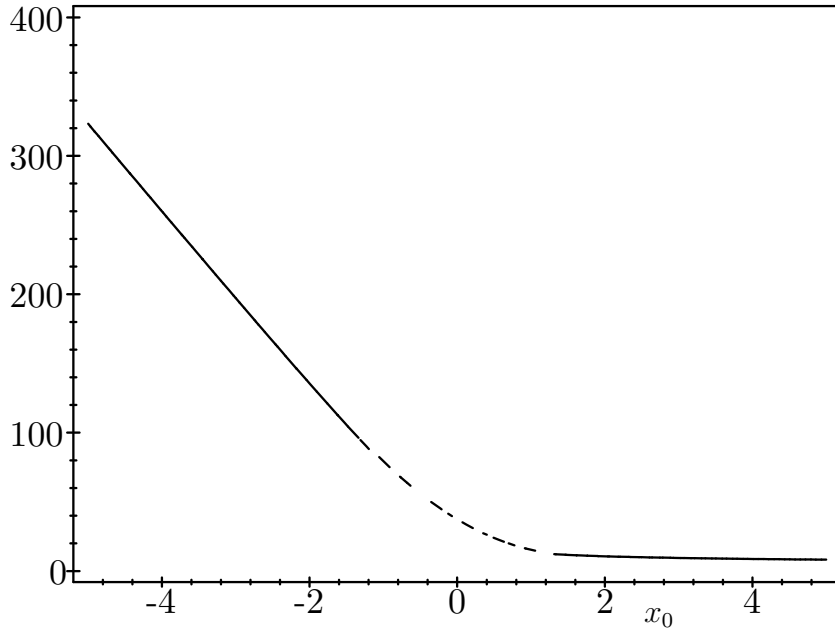


Figure 10: Weak-coupling and strong-coupling estimates for the ratio $2\Delta(0)/T_{KT}$ (solid curves). The dashed line is a graphical interpolation.

5 Phase Fluctuations in Three Dimensions and Superconductive Transition in 3D XY-Model

In this chapter we discuss effects of fluctuations in three-dimensional systems. In three dimensions we have the expression for stiffness

$$J_{3D}(\mu, T, \Delta) = \frac{1}{4m}n(\mu, T, \Delta) - \frac{\sqrt{2m}}{16\pi^2} \frac{1}{T} \int_{-\mu}^{\infty} d\xi \frac{(\xi + \mu)^{3/2}}{\cosh^2(\sqrt{\xi^2 + \Delta^2}/2T)} \quad (123)$$

With it we can immediately set up 3D XY model, whose Hamiltonian reads:

$$H = \frac{J_{3D}}{2} \int d^3\mathbf{x} [\nabla\theta(\mathbf{x})]^2. \quad (124)$$

Temperature of the phase transition of this model can be estimated using mean-field methods for the lattice 3D XY-Model [8]:

$$T_{3D}^{MF} \simeq 3J_{3D}a, \quad (125)$$

$a = 1/n_b^{1/3}$ is the lattice spacing of the theory [8] where n_b is number of pairs.

In the weak-coupling limit, the stiffness can be expanded near T^* as follows:

$$J = \frac{7}{32\pi^4} \zeta(3) \frac{p_F^3}{m} \frac{\Delta^2}{T^{*2}}, \quad (126)$$

This is similar to the coefficient of the gradient term in the Ginzburg-Landau expansion except that there it is obtained from a small- Δ expansion, whereas here the background gap has a nonzero modulus.

Obviously we have no separation of the two temperatures in the BCS limit. For moderately strong coupling, the two temperatures are related by

$$\tilde{T}_c = \tilde{T}^* - \alpha \tilde{T}^{*5/2}, \quad (127)$$

where $\alpha = (2\pi^2)^{2/3}/3$.

In the strong-coupling limit of the theory where we have tightly bound composite bosons, the phase stiffness tends asymptotically to:

$$J = \frac{n}{4m} - \frac{3\sqrt{2\pi m}}{16\pi^2} T^{3/2} \exp\left[-\frac{\sqrt{\mu^2 + \Delta^2}}{T}\right], \quad (128)$$

which may be expanded

$$J = \frac{n}{4m} - \frac{3\sqrt{2\pi m}}{16\pi^2} T^{3/2} \exp\left[\frac{\mu}{T}\right]. \quad (129)$$

It obviously tends in this limit quickly to

$$J = \frac{n}{4m}. \quad (130)$$

An estimate for the critical temperature, obtained via the mean-field treatment of the 3D XY-model on the lattice reads in this limit:

$$T_c = \frac{3}{2m} \left[\left(\frac{n}{2}\right)^{2/3} - \frac{1}{n^{1/3}} \frac{2^{7/6}}{\pi^{3/2}} T_c^{3/2} m^{3/2} \exp\left(-\frac{\sqrt{\mu^2 + \Delta^2}}{T_c}\right) \right] \quad (131)$$

This quickly tends from below to the value:

$$T_c^{\text{3DXY}} = \frac{3n^{2/3}}{2^{5/3}m} = \epsilon_F \frac{3}{(6\pi^2)^{2/3}} \simeq 0.198\epsilon_F. \quad (132)$$

This result is very close to the temperature of the condensation of bosons of mass $2m$ and density $n/2$, which, as it was discussed in the introduction was obtained including the effect of Gaussian fluctuations into the mean-field equation for the particle number [2, 5] yielding ⁶

$$T_c^{\text{Bosons}} = [n/2\zeta(3/2)]^{2/3}\pi/m = 0.218\epsilon_F. \quad (133)$$

The XY-model nature of the phase transition at T_c has been demonstrated in recent experiments [48] on $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ near the region of optimal doping. The phase transition of this model was discussed in great detail on a lattice in the textbook [8], since it describes the critical properties of the superfluid transition of Helium.

6 Conclusion

We have discussed the two different transitions taking place in superconductors at strong couplings, the formation of pairs and onset of phase coherence. For theory it was a

⁶When critical temperature is studied via retaining gaussian corrections to the number equation the crossover of the critical temperature has an artificial maximum in the region of intermediate couplings [2, 5], so in this case it's limiting value is approached from above in the strong-coupling limit, this is not the case in our approach.

fortunate fact of history that the early-discovered metallic superconductors had such a weak coupling that there was only one transition which, moreover, can be understood by mean-field methods. In high- T_c superconductors, the existence of a pseudogap brings in complications which we have tried to illuminate in the framework of the simple fermion model with δ -function interaction. The interpretation of the experimental data is still complicated due to the complex chemical structure of these compounds. Little is known up to now on the real forces causing the pairing.

We have studied the crossover from BCS-type to Bose-type superconductivity and the behavior of the pseudogap state. Our crossover parameter is x_0 , a quantity closely related to the chemical potential. For this purpose we have used the gradient expansion of the effective energy functional to set up an equivalent XY-model which allows us to investigate the onset of long-range order in the phase fluctuations. In two dimensions, we have given a simple analytic expression which shows how the resulting Kosterlitz-Thouless temperature T_{KT} at which quasi-long-range order sets in moves towards the pair-binding temperature T^* , and merges with it in the weak-coupling limit. We have found similar results in three dimensions, setting up a three dimensional XY-model for the description of the onset of phase coherence in the superconductive transition, and how this transition evolves to the ordinary BCS transition in the weak-coupling regime.

We have also studied the weak- to strong-coupling crossover of thermodynamic functions of the superconductive state near zero temperature as well in the pseudogap phase near the critical temperature.

Certainly, our mean-field estimates for T^* are quite crude, and we expect significant modifications due to fluctuations, in particular of the character of the transition which experimentally does not seem to be of second order, and may not be a phase transition after all. All our formulas for thermodynamic quantities will have to be smeared out in temperature near T^* , before any possible comparison with experiments.

Let us finally remark that the separation of T^* and T_c has an analogy in the ferroelectrics and magnets which also contain two separate characteristic temperatures, for example in the later case— the Stoner- and the Curie-temperature.

7 Acknowledgments

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A Action functional of Collective Pair Field

In this appendix we briefly outline derivation of the effective action (95). As shown in Ref. [53], a pair field Δ is introduced to eliminate the quartic interaction term in the functional integral involving the action of the Hamiltonian (1):

$$\mathcal{A} = \int dt \left[\sum_{\sigma} \int d^D x \psi_{\sigma}^{\dagger}(\mathbf{x}) i\hbar \partial_t \psi_{\sigma}(\mathbf{x}) - H(t) \right], \quad (134)$$

After that, the fermions can be integrated out. At a constant pair field, we find the potential part (99) of the collective-field action

$$\begin{aligned} \Omega_{\text{pot}}(\mu, T, \Delta, \Delta^*) = V \left\{ \frac{|\Delta|^2}{g} - T \sum_{n=-\infty}^{+\infty} \int \frac{d^D k}{(2\pi)^D} \text{tr}[\ln G^{-1}(i\omega_n, \mathbf{k}) e^{i\delta\omega_n\tau_3}] \right. \\ \left. + T \sum_{n=-\infty}^{+\infty} \int \frac{d^D k}{(2\pi)^D} \text{tr}[\log G_0^{-1}(i\omega_n, \mathbf{k}) e^{i\delta\omega_n\tau_3}] \right\}, \quad \delta \rightarrow +0, \end{aligned} \quad (135)$$

where $\Delta = \Delta e^{i\theta}$ and

$$G^{-1}(i\omega_n, \mathbf{k}) = i\omega_n \hat{I} - \tau_3 \xi(\mathbf{k}) + \tau_+ \Delta + \tau_- \Delta^* = \begin{pmatrix} i\omega_n - \xi(\mathbf{k}) & \Delta \\ \Delta^* & i\omega_n + \xi(\mathbf{k}) \end{pmatrix}. \quad (136)$$

Using the identity $\text{tr} \log \hat{A} = \log \det \hat{A}$, equation (135) takes the form

$$\begin{aligned} \Omega_{\text{pot}}(\mu, T, \Delta, \Delta^*) = V \left\{ \frac{|\Delta|^2}{g} - T \sum_{n=-\infty}^{+\infty} \int \frac{d^D k}{(2\pi)^D} \log \frac{\omega_n^2 + \xi^2(\mathbf{k}) + |\Delta|^2}{\omega_n^2 + \varepsilon^2(\mathbf{k})} \right. \\ \left. - \int \frac{d^D k}{(2\pi)^D} [-\xi(\mathbf{k}) + \varepsilon(\mathbf{k})] \right\}, \end{aligned} \quad (137)$$

After performing the sum over the Matsubara frequencies in (137), we obtain the well-known mean-field expression for Ω_{pot} [53]:

$$\begin{aligned} \Omega_{\text{pot}}(\mu, T, \Delta, \Delta^*) = V \left\{ \frac{|\Delta|^2}{g} - \int \frac{d^D k}{(2\pi)^D} \left[2T \log 2 \cosh \frac{\sqrt{\xi^2(\mathbf{k}) + |\Delta|^2}}{2T} - \xi(\mathbf{k}) \right] \right. \\ \left. + \int \frac{d^D k}{(2\pi)^D} \left[2T \log 2 \cosh \frac{\varepsilon(\mathbf{k})}{2T} - \varepsilon(\mathbf{k}) \right] \right\}. \end{aligned} \quad (138)$$

In order to derive the kinetic part Ω_{kin} of the mean-field energy, we must calculate the first two terms of the series (98), the first being [24]

$$\Omega_{\text{kin}}^{(1)} = T \int_0^\beta d\tau \int d^D x \frac{T}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int d^D k \text{tr}[\mathcal{G}(i\omega_n, \mathbf{k})\tau_3] \left[\frac{i\partial_\tau \theta}{2} + \frac{(\nabla\theta)^2}{8m} \right], \quad (139)$$

with

$$\mathcal{G}(i\omega_n, \mathbf{k}) = -\frac{i\omega_n \hat{I} + \tau_3 \xi(\mathbf{k}) - \tau_1 \Delta}{\omega_n^2 + \xi^2(\mathbf{k}) + \Delta^2}. \quad (140)$$

After summing over the Matsubara frequencies and integration over \mathbf{k} , we obtain

$$\Omega_{\text{kin}}^{(1)} = T \int_0^\beta d\tau \int d^D x n(\mu, T, \Delta) \left[\frac{i\partial_\tau \theta}{2} + \frac{(\nabla\theta)^2}{8m} \right], \quad (141)$$

with $n(\mu, T, \Delta)$ given in by (3). After an ansatz $\Sigma = \tau_3 O_1 + \hat{I} O_2$, where O_1 and O_2 are the two gradient terms in Eq. (101), we derive for the second term $\Omega_{\text{kin}}^{(2)}$ the two contributions from O_1 and O_2 :

$$\Omega_{\text{kin}}^{(2)}(O_1) = -\frac{T}{2} \int_0^\beta d\tau \int d^D x K(\mu, T, \Delta) \left[\frac{i\partial_\tau \theta}{2} + \frac{(\nabla\theta)^2}{8m} \right]^2, \quad (142)$$

where $K(\mu, T, \Delta)$ was given in (104). This is the second term in (102). The second term in (103) is obtained from the second contribution to $\Omega_{\text{kin}}^{(2)}$:

$$\Omega_{\text{kin}}^{(2)}(O_2) = -\int_0^\beta d\tau \int d^D x \frac{1}{32\pi^2 m^2} \int d^D k \frac{\mathbf{k}^2}{\cosh^2[\sqrt{\xi^2(\mathbf{k}) + \Delta^2/2T}]} (\nabla\theta)^2. \quad (143)$$

Combining (143), (142) and (141) we obtain (102).

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