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Universality Principle for Orbital Angular Momentum and Spin in Gravity with Torsion

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We argue that compatibility with elementary particle physics requires gravitational theories with torsion to be unable to distinguish between orbital angular momentum and spin. An important consequence of this principle is that spinless particles must move along autoparallel trajectories, not along geodesics.

KEY WORDS : Spin and orbital angular momentum

1. INTRODUCTION

Universality principles provide us with important guidelines for constructing candidates of fundamental theories which have a chance of being true. For example, an essential property of Maxwell's theory is that electromagnetic interactions depend only on the charge of a particle, not on the various physical origins of this charge. The charge of an ion is composed of electronic and nuclear charges, the latter of proton charges, these in turn of quark charges, or of any further charged substructures to be discovered in the future. The motion of a charged particle in an electromagnetic field does not depend on these details. An atom moves like a neutral point particle, irrespective of the completely different origins of electron and proton charges, the exact neutrality of an atom being the very basis for the electrostatic stability of large gravitational bodies (and thus for the existence

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of theoretical physics).

The irrelevance of the physical origin of the "charge" of gravitational interactions, the mass, led Einstein to the discovery of a geometric theory of these interactions. Just like the electric charge, the mass of a particle also has a multitude of origins, arising from the masses of constituents and the various field energies binding these together. Gravitational interactions depend only on the total mass, and this property makes all particles run along the same trajectories, which can therefore be used to define a geometry of spacetime. In Einstein's equations, the independence of the physical origin of the mass manifests itself in the coupling of the Einstein curvature tensor to the total energy-momentum tensor of all fields in the universe. Its precise composition depends on the actual status of elementary particle physics, but the motion is invariant with respect to this composition, and thus to future discoveries about the internal structure of the particles.

More recently, the universality of weak and color charges was an important principle in the construction of unified theories of electromagnetic and weak, as well as of strong interactions.

2. SPIN PRECESSION IN SPECIAL TORSION FIELD

For a number of years, theoreticians have enjoyed the idea that the geometry of spacetime may not only be curved but also carry torsion. The line of argument leading to this idea was that that gravitational equations may be rederived from a gauge theory of local translations. These local translations generalize the global translations under which all local theories are invariant in Minkowski spacetime. But the latter theories are also invariant under the larger Poincaré group, the group of translations *and* Lorentz transformations. It therefore seemed natural to postulate the existence of a second gauge field which ensures the invariance under local Lorentz transformations [1], and to seek for experimentally observable effects. One basic feature of such a second field is an interaction between torsion and spin which is common to all field theories of gravitation in a four-dimensional spacetime with general coordinates q^μ:[2]

$$\mathcal{A} = -\frac{1}{2}\int d^4q\sqrt{-g}\, K_{\mu\nu\lambda}\Sigma^{\nu\lambda,\,\mu}, \tag{1}$$

where $K_{\mu\nu\lambda}$ is the contortion tensor, containing the torsion in the combination $K_{\mu\nu\lambda} = S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu}$. The tensor $\Sigma^{\nu\lambda,\,\mu}$ is the local spin current density of all matter fields.

[2] Our notation of field theoretic and geometric quantities is the same as in the textbook [3].

Consider now a particle at rest in a Riemann-flat space with euclidean coordinates $\mathbf{x} = (x^1, x^2, x^3)$ with $x^i = q^i$ for $(i = 1, 2, 3)$, and a time $t = q^0$. Suppose the space has a specific torsion, whose only nonzero components are

$$S_{ij0} = \tfrac{1}{2}\epsilon_{ijk}b_k\,, \qquad \partial_i b_i = 0\,. \tag{2}$$

For the argument to come, the divergenceless vector $\mathbf{b} = (b_1, b_2, b_3)$ may be assumed to be constant, for simplicity. Then (1) specifies an interaction energy

$$H_{\rm int} = -\frac{1}{2}\, b_k\, \frac{1}{2}\int d^3x\, \epsilon_{kij}\Sigma_{ij,0}\,. \tag{3}$$

For a particle at rest, the factor to the right of b_k is the spin vector S_k of a particle, so that the interaction energy becomes

$$H_{\rm int} = -\tfrac{1}{2}\,\mathbf{b}\cdot\mathbf{S}. \tag{4}$$

This looks just like the interaction energy of a unit magnetic moment with a constant magnetic field. For this reason we shall refer to a torsion of the type (2) as a *magneto-torsion*, and call the field $\mathbf{b}$ a *torsion-magnetic field*. From the Heisenberg equation $\dot{\mathbf{S}} = i[H, \mathbf{S}]$, we obtain the equation of motion for the spin,

$$\frac{d}{dt}\,\mathbf{S} = -\frac{1}{2}\,\mathbf{S}\times\mathbf{b}, \tag{5}$$

describing a precession with frequency $\omega = |\mathbf{b}|/2$.

The microscopic origin of the spin of the particle is completely irrelevant to obtain this result. The spin, being the total angular momentum in the particle's rest frame, is composed of the orbital angular momenta of all constituents and their spins. The details of this composition depend on the actual quantum field theoretic description of the particle. A ρ-meson, for instance, has a unit spin. From the hadronic strong-interaction viewpoint of bootstrap physics, the unit spin is explained by ρ being a bound state of a pair of spinless pions with unit orbital angular momentum. In quark physics, on the other hand, ρ is a bound state of a quark and an antiquark with zero orbital angular momentum in a spin triplet state.

Thus, in the quark description, the spin of a ρ-meson in a torsion field (2) will precess. Clearly, a theory of gravity with torsion can only be consistent with particle physics, if the same precession frequency is found for the hadronic description of ρ as a bound state of two spinless pions, which contains only orbital angular momentum. The equations of motion for the motion of spinless particles must therefore lead to the same precession.

For present-day theories of gravity with torsion [1], this postulate presents a serious problem. In these theories, the energy momentum tensor $T^{\mu\nu}(q)$ of a spinless point particle satisfies the local conservation law

$$\bar{D}_\nu T^{\mu\nu}(q) = D^*_\nu T^{\mu\nu}(q) + 2S_{\kappa}{}^{\mu}{}_{\lambda}(q)T^{\kappa\lambda}(q) = 0, \qquad D^*_\nu \equiv D_\nu + 2S_\nu\,, \quad (6)$$

where $\bar{D}_\mu$ is the covariant derivative involving the Riemann connection $\bar{\Gamma}_{\mu\nu\lambda}$, whereas D_μ is the covariant derivative involving the full affine connection $\Gamma_{\mu\nu\lambda} = \bar{\Gamma}_{\mu\nu\lambda} + K_{\mu\nu\lambda}$. It is obvious from the torsionless left-hand part of eq. (6), and was proved in [4] (see also Section VII in Ref. 5), that such a conservation law leads directly to *geodesic* particle trajectories for point-like spinless particles, governed by the equation of motion

$$\ddot{q}^\nu + \bar{\Gamma}_{\lambda\kappa}{}^{\nu}\dot{q}^\lambda\dot{q}^\kappa = 0, \qquad (7)$$

where $q^\mu(\tau)$ is the orbit parametrized in terms of the proper time τ. This motion is not influenced by torsion. As a consequence, the motion of the two pions in a ρ meson would be independent of torsion, so that the spin of ρ at rest would *not* precess in the two-pion description, in contradiction with the quark-antiquark description. Since both field-theoretic descriptions of ρ are equally good, the true ρ-meson fluctuating between the two states, we conclude that geodesics cannot be the correct trajectories of spinless particles.

3. PRECESSION OF ORBITAL ANGULAR MOMENTUM FOR AUTOPARALLEL MOTION

The discrepancy can be avoided by another option for the trajectories of spinless particles in this geometry. These are the *autoparallels*, which obey an equation of motion like (7), but with the full affine connection:

$$\ddot{q}^\nu + \Gamma_{\lambda\kappa}{}^{\nu}\dot{q}^\lambda\dot{q}^\kappa = 0. \qquad (8)$$

The conservation law for the energy momentum tensor of a spinless point particle leading to autoparallel motion is [5]

$$D^*_\nu T^{\mu\nu}(q) = 0\,. \qquad (9)$$

In a flat space with torsion, eq. (8) becomes

$$\ddot{q}^\nu + 2S^{\nu}{}_{\lambda\kappa}\dot{q}^\lambda\dot{q}^\kappa = 0\,. \qquad (10)$$

Specializing further to a constant magneto-torsion (2), we obtain $\dot{q}^0 =$ const, and find for the spatial motion in euclidean coordinates the equation

$$\frac{d^2}{dt^2}\mathbf{x} = -\dot{\mathbf{x}} \times \mathbf{b}. \tag{11}$$

Thus the constant torsion (2) acts on the orbital motion of the spinless point particle just like a Lorentz force. It is well known from electrodynamics, that this Lorentz force causes a precession of the orbital angular momentum of an electron. Its frequency is determined by the magnetic moment of the *orbital* motion, whose size for a certain orbital angular momentum $\mathbf{L}$ is half as big as that of a spin $\mathbf{S}$ of equal size. The precession frequency following from (11) is therefore $\omega = |\mathbf{b}|/2$. To show this explicitly we simply observe that (11) follows from a Lagrangian $L = \dot{\mathbf{x}}^2/2 + \mathbf{a}\cdot\dot{\mathbf{x}}$, describing a particle of unit mass moving in a *torsion-magnetic* vector potential $\mathbf{a} = \mathbf{b} \times \mathbf{x}$. The associated Hamiltonian depending on $\mathbf{x}$ and the momentum $\mathbf{p}$ reads

$$H = \tfrac{1}{2}(\mathbf{p} - \mathbf{A})^2 = \tfrac{1}{2}\,\mathbf{p}^2 - \tfrac{1}{2}\,\mathbf{b}\cdot(\mathbf{x} \times \mathbf{p}) + \tfrac{1}{8}(\mathbf{b} \times \mathbf{x})^2. \tag{12}$$

The smallness of the gravitational coupling makes torsion small enough to ignore the last term. From the second term written as $-\frac{1}{2}\mathbf{b}\cdot\mathbf{L}$ we calculate via the Heisenberg equation $\dot{\mathbf{L}} = i[H, \mathbf{L}]$ the equation of motion for the orbital angular momentum:

$$\frac{d}{dt}\mathbf{L} = -\frac{1}{2}\,\mathbf{L} \times \mathbf{b}, \tag{13}$$

which is the same as eq. (5) for the spin, leading to the same precession frequency $\omega = |\mathbf{b}|/2$.

A similar study can of course by performed for an *electro-torsion* field ${S_{i0}}^0 = e^i/2$ with $e^i = \partial_i a^0$, in which case the autoparallel differential equation (10) can be rewritten as

$$\frac{d^2}{dt^2}\mathbf{x} = -\mathbf{e} - \dot{\mathbf{x}} \times \mathbf{b}, \tag{14}$$

thus extending (11) to an analog of the full Lorentz equation. The Hamiltonian (12) contains then an extra electro-torsion contribution equal to a^0.

This Hamiltonian may be quantized as usual to obtain the quantum mechanics of a spinless point particle in the presence of *electromagneto-torsion* fields. The eikonal approximation of the Schrödinger wave function will describe autoparallel trajectories.

Although the discussion up to this point has assumed constant electro-magneto-torsion fields $\mathbf{e}$ and $\mathbf{b}$, it is easy to convince ourselves that the final theory is also valid for space-dependent fields.

Let us compare this with the couplings in proper magnetism, where in analogy to the universal coupling of an electric field to the charge of a particle, a magnetic field $\mathbf{B}$ couples universally to the magnetic moments. For orbital angular momenta and spin, however, the magnetic coupling is nonuniversal. Consider atomic electrons. They have a gyromagnetic ratio $g = 2$ caused by the Thomas precession, so that the magnetic interaction Hamiltonian is (ignoring the anomalous magnetic moment)

$$H_{\rm int} = -\mu_B \, \mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S}), \qquad \mu_{\rm B} \equiv \frac{e}{2Mc}\,, \tag{15}$$

where $\mu_{\rm B}$ is the Bohr magnetic moment (using $\hbar = 1$). In weak magnetic fields, an atom has an interaction energy $-g\mu_{\rm B}BM$, with the gyromagnetic ratio $g = 1 + [J(J+1) + S(S+1) - L(L+1)]/2J(J+1)$, where J is the quantum number of the toral spin vector $\mathbf{J} = \mathbf{L} + \mathbf{S}$. This ratio g causes the characteristic level splitting of the Zeeman effect.

4. DIRAC PARTICLE IN SPECIAL TORSION FIELD

It is useful to attempt to set up a modified action for a Dirac field which is compatible with the proposed universality principle in the special electro-magneto torsion field. So far, we have been able to find such an action only for slowly moving particles with spin, and it is not clear whether the construction can be extended properly to the fully relativistic situation.

In a first step, consider a Riemann-flat spacetime with Minkowski coordiantes $x^\alpha = (x^0, \mathbf{x})$ and an action

$$\mathcal{A} = \int d^4x \, \bar\psi(x) \left[\gamma^\alpha \left(i f_\alpha{}^\beta \partial_\beta - eA_\alpha - \frac{1}{2}\, K_{\alpha\beta\gamma} \Sigma^{\beta\gamma} \right) - M \right] \psi(x), \tag{16}$$

where $f_\alpha{}^\beta = 1 - a_\alpha{}^\beta$, with $a^{\alpha 0}$ being the electromagneto-torsion field $(a^0, \mathbf{a})$, and the other components $a_\alpha{}^i = 0$ vanishing. It is a gauge field whose curl yields the torsion, $S_{ij}{}^0 = (\partial_i a_j{}^0 - \partial_j a_i{}^0)/2$. This action is not covariant since it is only supposed to illustrate how the torsion forces can be made compatible. The action is gauge-invariant under $a_i{}^0(x) \to a_i{}^0(x) + \partial_i \Lambda^0(\mathbf{x})$ with a simultaneous transformation $\psi(x) \to e^{-i\Lambda^0(\mathbf{x})\partial_0}\psi(x)$. The 4×4-matrices $\Sigma_{\beta\gamma} \equiv (i/4)[\gamma_\beta, \gamma_\gamma]_-$ are the generators of Lorentz transformations, so that the spin current density in (1) is

$\Sigma_{\beta\gamma,\alpha} = -(i/2)\bar{\psi}[\gamma_\alpha, \Sigma_{\beta\gamma}]_+\psi$. Here $[\,.\,,\,.\,]_\mp$ denotes commutator and anticommutator, respectively, and all quantities have standard Dirac notation. Now we use the Gordon formula

$$\bar{u}(\mathbf{p}', s_3')\gamma^\alpha u(\mathbf{p}, s_3) = \bar{u}(\mathbf{p}', s_3')\left[\frac{1}{2M}\,(p'^\alpha + p^\alpha) + \frac{i}{2M}\,\sigma^{\alpha\beta}q_\beta\right]u(\mathbf{p}, s_3) \quad (17)$$

to calculate the interaction energy for slow electrons between single-electron states of small momenta $\mathbf{p}'$ and $\mathbf{p}$ with momentum transfer $q = p' - p$,

$$H_{\rm int} = \int d^3x \left[\frac{e}{M}\,\mathbf{A}(x)\cdot(\mathbf{p}+\mathbf{q}-i\mathbf{q}\times\mathbf{\Sigma}) + \mathbf{a}(x)\cdot(\mathbf{p}+\mathbf{q}-i\mathbf{q}\times\mathbf{\Sigma}) - Ma^0(x) - \frac{1}{2}\,\mathbf{b}\cdot\mathbf{\Sigma}\right]e^{-i\mathbf{qx}}, \quad (18)$$

where $\Sigma_i = \frac{1}{2}\epsilon_{ijk}\Sigma_{jk}$ are the Dirac spin matrices. We have omitted the external spinors $\bar{u}(\mathbf{p}', s_3')$ and $u(\mathbf{p}, s_3)$, for brevity, since we shall immediately take the limit $\mathbf{p}' \to \mathbf{p}$ where $\bar{u}(\mathbf{0}, s_3')u(\mathbf{0}, s_3) = \delta_{s_3's_3}$, $\bar{u}(\mathbf{0}, s_3')\Sigma_{ij}u(\mathbf{0}, s_3) = \epsilon_{ijk}(S_k)_{s_3's_3}$, and $S_k = \sigma_k/2$, with Pauli spin matrices σ_k. Before going to this limit, we convert $\mathbf{q}$ into a derivative of $e^{-i\mathbf{qx}}$, then via an integration by parts into a derivative of $\mathbf{A}(x)$, and using the vector potentials $\mathbf{A} = \frac{1}{2}\,\mathbf{B}\times\mathbf{x}$ and $\mathbf{a} = \frac{1}{2}\,\mathbf{b}\times\mathbf{x}$, we obtain in the limit $\mathbf{p}' \to \mathbf{p}$ for a slow electron

$$H_{\rm int} = \int d^3x \left[\alpha_B\,\mathbf{B}\cdot(\mathbf{L}+2\mathbf{S}) - \frac{1}{2}\,\mathbf{b}\cdot(\mathbf{L}+\mathbf{S}) - Ma^0\right]. \quad (19)$$

Observe that the last spin term in (18) has removed precisely half of the spin term coming from the coupling of torsion to γ^i, thus leading to the universal coupling $\mathbf{b}\cdot(\mathbf{L}+\mathbf{S}) = \mathbf{b}\cdot\mathbf{J}$. The Hamiltonian (19) ensures that nonrelativistic electrons follow the equation of motion eq. (14), thus running along autoparallels (10).

To complete the analogy with magnetism, we make the dimension of the magnetotorsion field equal to that of the magnetic field by defining

$$\mathbf{b} \equiv \alpha_K\,\mathbf{B}^K, \quad (20)$$

with the *torsionmagneton*

$$\alpha_K \equiv \sqrt{G}\,\hbar/2c, \quad (21)$$

where $G = \hbar c/M_{\rm P}^2$, and $M_{\rm P}$ is the Planck mass $M_{\rm P} = 2.38962\times10^{22}\,M$. The torsionmagneton is the same factor smaller than the Bohr magneton.

Note that in present-day gravity with torsion [1,3], the term $\frac{1}{2}\mathbf{b}\cdot\mathbf{L}$ is absent in (19), while $\frac{1}{2}\mathbf{b}\cdot\mathbf{S}$ is present, in violation of our universality principle.

5. UNIVERSALITY OF ORBITAL ANGULAR MOMENTUM AND SPIN IN EINSTEIN'S THEORY

Note that in a torsionless spacetime, the universality of orbital angular momentum and spin is satisfied. Then $A_{\mu\alpha\beta} = K^h_{\mu\alpha\beta}$ and $f_\mu{}^\sigma = \delta_\mu{}^\sigma$, and for a nearly flat $h_i{}^0 = \tilde{a}^i$ the gradient term in (16) gives a coupling $\tilde{\mathbf{b}} \cdot (\mathbf{L} + 2\mathbf{S})$, with $\tilde{b}_i = \epsilon_{ijk}\partial_j h_i{}^0$, while the spin term in (16) removes half the spin coupling just as in (19), thus leading once more to the universal form $\tilde{\mathbf{b}} \cdot \mathbf{J}$.

This universality is an experimentally observed result of Einstein's theory of gravitation, which predicts that a spinning star exerts a rotational drag upon a distant point particle (Lense–Thirring effect). The deviation of the metric from the Minkowski form is at large distance from a star of mass M at the origin,

$$\begin{aligned} \phi^{00}(\mathbf{x}) &= -4\frac{GM}{c^2 r} + \dots\,, \qquad \phi^{ji}(\mathbf{x}) = 0, \\ \phi^{0i}(\mathbf{x}) &= \phi^{i0}(\mathbf{x}) = 2\frac{G}{c^3 r^3}\,(\mathbf{x} \times \mathbf{J}) + \dots\,, \end{aligned} \tag{22}$$

where G is the gravitational constant and $\mathbf{J}$ the total angular momentum of the star at the origin, obtained from the spatial integral over the stellar volume V:

$$J^k = \frac{1}{2}\,\epsilon_{ijk} J^{ij} = \frac{1}{2}\,\epsilon_{ijk} \int_V d^3x [x^i T^{j0}(\mathbf{x}, t) - x^j T^{i0}(\mathbf{x}, t)]. \tag{23}$$

The energy-momentum tensor on the right-hand side receives contributions from both orbital as well as spin angular momentum. Thus, a nonrotating polarized neutron star with total spin $\mathbf{S}$ gives rise to the *same* Lense–Thirring effect as a rotating star composed of spinless dust with purely orbital angular momentum $\mathbf{L}$, if this is equal to the spin $\mathbf{S}$ of the neutron star.

Going from the rotating source to a rotating test particle, it is coupled to an external gravitational field $g_{\mu\nu}(q)$ only via its total energy-momentum tensor $T^{\mu\nu}(q)$. In the rest-frame of the test particle, the off-diagonal matrix elements of $T^{\mu\nu}(q)$ receive equal contributions from orbital and spin angular momenta.

6. CONCLUSION

In conclusion we see that only autoparallel trajectories comply with the universality principle of orbital and spin angular momentum. Only for these trajectories can we calculate the gravitational behavior of particles in an electromagneto torsion field without a complete knowledge of the source of their spin in terms of its constituents.

Let us end by remarking that autoparallel equations of motion can be derived from the standard action of a classical point particle action via a modified variational procedure [7–9] which follows from geometric considerations (closure failure of parallelograms in the presence of torsion). The geometric basis for these developments was deduced from an analogy of these spaces with a crystal with defects, which in crystal physics play the same geometric role as curvature and torsion in gravity [3]. A *nonholonomic mapping principle* was found [5,9] to transform equations of motion from flat space to spaces with curvature and torsion. This principle was of great help in solving another fundamental problem, the path integral of the hydrogen atom [9].

Autoparallel trajectories are also the most natural trajectories obtained from an embedding of spaces with torsion in a Riemannian space [10]. Recently it was pointed out by Hammond [2] that string theories contain a coupling parameter which could accommodate autoparallel trajectories.

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