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Theory and satellite experiment for critical exponent α of λ-transition in superfluid helium

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Abstract

On the basis of recent seven-loop perturbation expansion for $\nu^{-1} = 3/(2-\alpha)$ we perform a careful reinvestigation of the critical exponent α governing the power behavior $|T_c - T|^{-\alpha}$ of the specific heat of superfluid helium near the phase transition. With the help of variational strong-coupling theory, we find $\alpha = -0.01126 \pm 0.0010$, in very good agreement with the space shuttle experimental value $\alpha = -0.01056 \pm 0.00038$.

1. The critical exponent α characterizing the power behavior $|T_c - T|^{-\alpha}$ of the specific heat of superfluid helium near the transition temperature T_c is presently the best-measured critical exponent of all. A microgravity experiment in the Space Shuttle in October 1992 rendered a value with amazing precision [1]:

$$\alpha^{\mathrm{SS}} = -0.01056 \pm 0.00038. \tag{1}$$

This represents a considerable change and improvement of the experimental number found a long time ago on earth by Ahlers [2]:

$$\alpha = -0.026 \pm 0.004, \tag{2}$$

in which the sharp peak of the specific heat was broadened to 10^{-6} K by the tiny pressure difference between top and bottom of the sample. In space, the temperature could be brought to within 10^{-8} K close to T_c without seeing this broadening.

The exponent α is extremely sensitive to the precise value of the critical exponent ν which determines the growth of the coherence length when approaching the critical temperature, $\xi \propto |T - T_c|^{-\nu}$. Since ν lies very close to 2/3, and α is related to ν by the scaling relation $\alpha = 2 - 3\nu$, a tiny change of ν produces a large relative change of α. Ahlers' value was for many years an embarrassment to quantum field theorists who never could find α quite as negative — the field theoretic ν-value came usually out smaller than $\nu_{\mathrm{Ahl}} = 0.6753 \pm 0.0013$. The space shuttle measurement was therefore extremely welcome, since it comes much closer to previous theoretical values. In fact, it turned out to agree extremely well with the most recent theoretical determination of α by strong-coupling perturbation theory [3] based on the recent seven-loop power series expansions of ν [4], which gave [5]

$$\alpha^{\mathrm{sc}} = -0.0129 \pm 0.0006. \tag{3}$$

The purpose of this note is to present yet another resummation of the perturbation expansion for ν^{-1} and for $\alpha = 2 - 3\nu$ by variational perturbation theory applied in a different way than in [5]. Since it is

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a priori unclear which of the two results should be more accurate, we combine them to the slightly less negative average value with a larger error

$$\alpha^{\mathrm{sc}} = -0.01126 \pm 0.0010. \tag{4}$$

Before entering the more technical part of the paper, a few comments are necessary on the reliability of error estimates for any theoretical result of this kind. They can certainly be trusted no more than the experimental numbers. Great care went into the analysis of Ahlers' data [2]. Still, his final result (2) does not accommodate the space shuttle value (1). The same surprise may happen to theoretical results and their error limits in papers on resummation of divergent perturbation expansions, since there exists so far no safe way of determining the errors. The expansions in powers of the coupling constant g are strongly divergent, and one knows accurately only the first seven coefficients, plus the leading growth behavior for large orders k like $\gamma(-a)^k k! k\Gamma(k+b)$. The parameter b is determined by the number of zero modes in a solution to a classical field equation, a is the inverse energy of this solution, and γ the entropy of its small oscillations.

The shortness of the available expansions and their divergence make estimates of the error range of the result a rather subjective procedure. All publications resumming critical exponents such as α calculate some sequences of Nth-order resummed approximations α_N, and estimate an error range from the way these tend to their limiting value. While these estimates may be statistically significant, there are unknown systematic errors. Otherwise one should be able to take the expansion for any function $\tilde{f}(g) \equiv f(\alpha(g))$ and find a limiting number $f(\alpha)$ which lies in the corresponding range of values. This is unfortunately not true in general. Such reexpansions can approach their limiting values in many different ways, and it is not clear which yields the most reliable result. One must therefore seek as much additional information on the series as possible.

One such additional information becomes available by resumming the expansions in powers of the bare coupling constant g_0 rather than the renormalized one g. The reason is that any function of the bare coupling constant $f(g_0)$ which has a finite critical limit approaches this limit with a nonleading inverse power of g_0^{ω}, where ω is called the *critical exponent of approach to scaling*, whose size is known to be about 0.8 for superfluid helium. Any resummation method which naturally incorporates his power behavior should converge faster than those which ignore it. This incorporation is precisely the virtue of variational perturbation theory, which we have therefore chosen for the resummation of α.

For a second additional information we take advantage of our theoretical knowledge on the general form of the large-order behavior of the expansion coefficients:

$$\gamma(-a)^k k! k\Gamma(k+b)\left(1 + \frac{c^{(1)}}{k} + \frac{c^{(2)}}{k^2} + \cdots\right). \tag{5}$$

In the previous paper [5] we have done so by choosing the nonleading parameters c^i to reproduce exactly the first seven known expansion coefficients of α. The resulting expression (5) determines all expansion coefficients. The so-determined expression (5) predicts approximately *all* expansion coefficients, with increasing precision for increasing orders. The extended power series has then been resummed for increasing orders N, and from the N-behavior we have found the α-value (3) with quite a small error range.

As a third additional information we use the fact that we know from theory [3] in which way the infinite-order result is approached. Thus we may fit the approximate values α_N by an appropriate expansion in $1/N$ and achieve in this way a more accurate estimate of the limiting value than without such an extrapolation. The error can thus be made much smaller than the distance between the last two approximations, as has been verified in many model studies of divergent series [6].

The strategy of this Letter goes as follows: We want to use all the additional informations on the expansion of the critical exponent α as above, but apply the variational resummation method in two more alternative ways. First, we reexpand the series $\alpha(g_0)$ in powers of a variable h whose critical limit is no longer infinity but $h = 1$. The closer distance to the expansion point $h = 0$ leads us to expect a faster convergence. Second, we resum two different expansions, one for α, and one for $f(\alpha) = \nu^{-1} \equiv 3/(2-\alpha)$. From the difference in the resulting α-values and a comparison with the earlier result (3) we obtain an estimate of the systematic errors specified in Eq. (4).

2. The seven-loop power series expansion for ν in powers of the unrenormalized coupling constant of O(2)-invariant ϕ^4-theory which lies in the universality class of superfluid helium reads [4,7,8]

$$\nu^{-1} = 2 - 0.4g_0 + 0.4681481481482289{g_0}^2 - 0.66739{g_0}^3 + 1.079261838589703{g_0}^4 - 1.91274{g_0}^5 + 3.644347291527398{g_0}^6 - 7.37808{g_0}^7 + \cdots. \tag{6}$$

By fitting the expansion coefficients with the theoretical large-order behavior (5), this series has been extended to higher orders as follows [5]:

$$\Delta\nu^{-1} = 15.75313406543747{g_0}^8 - 35.2944{g_0}^9 + 82.6900901520064{g_0}^{10} - 202.094{g_0}^{11} + 514.3394395526179{g_0}^{12} - 1361.42{g_0}^{13} + 3744.242656157152{g_0}^{14} - 10691.7{g_0}^{15} + \cdots. \tag{7}$$

The renormalized coupling constant is related to the unrenormalized one by an expansion $g = \sum_{k=1}^{7} a_k g_0^k$. Its power behavior for large g_0 is determined by a series

$$s = \frac{d\log g(g_0)}{d\log g_0} = 1 - g_0 + \frac{947{g_0}^2}{675} - 2.322324349407407{g_0}^3 + 4.276203609026057{g_0}^4 - 8.51611440473227{g_0}^5 + 18.05897631325589{g_0}^6 + \cdots. \tag{8}$$

A similar best fit of these by the theoretical large-order behavior extends this series by

$$\Delta s = 40.38657228730114{g_0}^7 + 94.6453399123477{g_0}^8 - 231.3922442162566{g_0}^9 + 588.3206172579102{g_0}^{10} - 1552.116358404217{g_0}^{11} + 4242.372685080157{g_0}^{12} - 12001.18866491822{g_0}^{13} + 35115.23006646194{g_0}^{14} - 106234.4643086436{g_0}^{15} + 332239.2175082959{g_0}^{16} + \cdots. \tag{9}$$

Scaling implies that $g(g_0)$ becomes a constant for $g_0 \to \infty$, implying that the power s goes to zero in this limit. By inverting the expansion for s, we obtain an expansion for ν^{-1} in powers of $h \equiv 1 - s$ as follows:

$$\nu^{-1}(h) = 2 - 0.4h - 0.093037h^2 + 0.000485012h^3 - 0.0139286h^4 + 0.007349h^5 - 0.0140478h^6 + 0.0159545h^7 - 0.029175h^8 + 0.0521537h^9 - 0.102226h^{10} + 0.224026h^{11} - 0.491045h^{12} + 1.22506h^{13} - 3.00608h^{14} + 8.29528h^{15} - 22.5967h^{16}. \tag{10}$$

This series has to be evaluated at $h = 1$. For estimating the systematic errors of our resummation, we also calculate from (10) a series for $\alpha = 2 - 3\nu$:

$$\alpha(h) = 0.5 - 0.3h - 0.129778h^2 - 0.0395474h^3 - 0.0243203h^4 - 0.0032498h^5 - 0.0121091h^6 + 0.00749308h^7 - 0.0194876h^8 + 0.0320172h^9 - 0.0651726h^{10} + 0.14422h^{11} - 0.315055h^{12} + 0.802395h^{13} - 1.95455h^{14} + 5.49143h^{15} - 14.8771h^{16} + \cdots. \tag{11}$$

3. In order to get a rough idea about the behavior of the reexpansions in powers of h, we plot their partial sums at $h = 1$ in the upper row of Fig. 1. After an initial apparent convergence, these show the typical divergence of perturbation expansions.

A rough resummation is possible using Padé approximants. The results are shown in Table 1. The highest Padé approximants yield

$$\alpha^{\mathrm{Pad}} = -0.0123 \pm 0.0050. \tag{12}$$

The error is estimated by the distance to the next lower approximation.

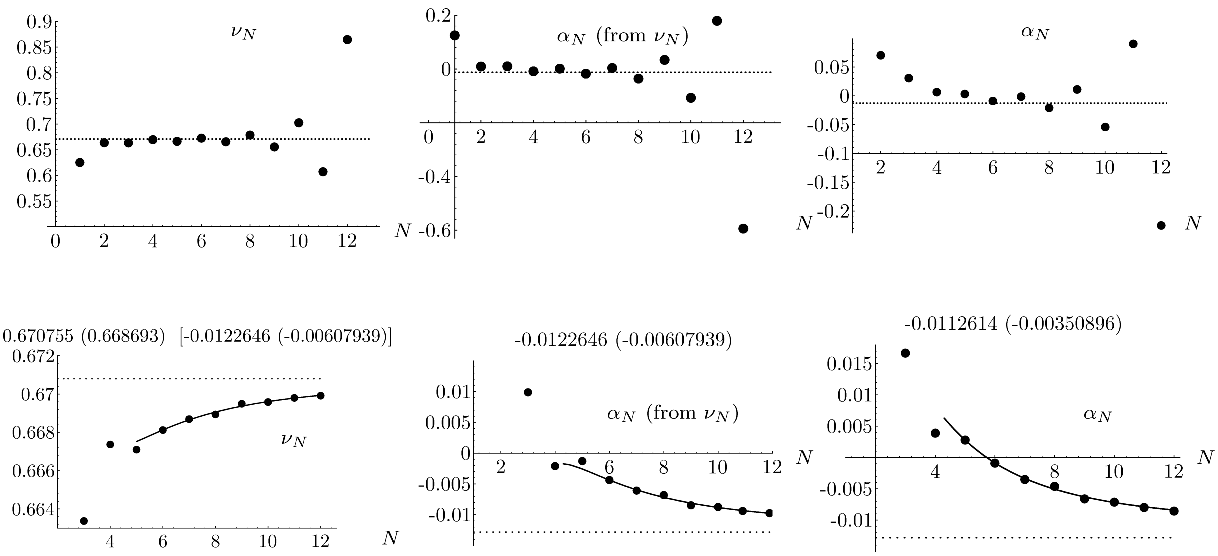

Fig. 1. Upper plots: Results of partial sums of series (10) for ν^{-1} up to order N, once plotted as $\nu_N = 1/\nu_N^{-1}$, and once as $\alpha_N = 2 - 3\nu_N$. The third plot shows the corresponding partial sums of the series for α. The dotted line is the experimental space shuttle value α^{ss} of Eq. (1). Lower plots: The corresponding resummed values and a fit of them by $c_0 + c_1/N^2 + c_2/N^4$. The constant c_0 is written on top, together with the seventh-order approximation (in parentheses). The square brackets on top of the left-hand plot for ν shows the corresponding α-values.

Table 1
Results of the Padé approximations $P_{MN}(h)$ at $h = 1$ to the power series $\nu^{-1}(h)$ and $\alpha(h)$. The parentheses show the associated values of α and ν

M	N	ν	(α)	(ν)	α
4	4	0.678793	(−0.0363802)	(0.678793)	−0.0363802
5	4	0.671104	(−0.0133107)	(0.670965)	−0.0128940
4	5	0.670965	(−0.0128940)	(0.670901)	−0.0127031
5	5	0.670756	(−0.0122678)	(0.670756)	−0.0122678

4. We now resum the expansions $\nu^{-1}(h)$ and $\alpha(h)$ by variational perturbation theory. This is applicable to divergent perturbation expansions

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{13}$$

which behave for large x like

$$f(x) = x^{p/q} \sum_{m=0}^{\infty} b_m x^{-2m/q}. \tag{14}$$

It is easy to adapt our function to this general behavior. Plotting the successive truncated power series for $\nu^{-1}(h)$ against h in Fig. 2, we see that this function will have a zero somewhere above $h = h_0 = 3$.

We therefore go over to the variable x defined by $h = h(x) \equiv h_0 x/(h_0 - 1 + x)$, in terms of which $f(x) = \nu^{-1}(h(x))$ behaves like (14) with $p = 0$ and $q = 2$, and has to be evaluated at $x = 1$. The large-x behavior is imposed upon the function with expansion (13) as follows. We insert an auxiliary scale parameter κ and define the truncated functions

$$f_N(x) \equiv \kappa^p \sum_{n=0}^{N} a_n \left(\frac{x}{\kappa^q}\right)^n. \tag{15}$$

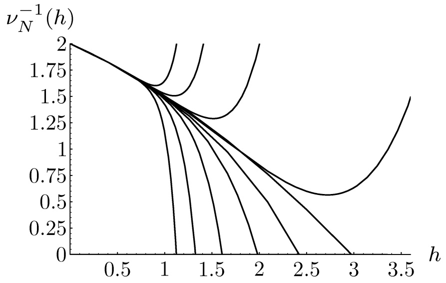

Fig. 2. Successive truncated expansions of $\nu^{-1}(h)$ of orders $N = 2, \ldots, 12$.

The parameter κ will be set equal to 1 at the end. Then we introduce a variational parameter K by the replacement

$$\kappa \to \sqrt{K^2 + \kappa^2 - K^2}. \tag{16}$$

The functions $f_N(x)$ are so far independent of K. This is changed by expanding the square root in (16) in powers of $\kappa^2 - K^2$, thereby treating this difference as a quantity of order x. This transforms the terms $\kappa^p x^n/\kappa^{qn}$ in (15) into polynomials of $r \equiv (\kappa^2 - K^2)/K^2$:

$$\kappa^p \frac{x^n}{\kappa^{qn}} \to K^p \frac{x^n}{K^{qn}} \left[1 + \binom{(p-qn)/2}{1} r + \binom{(p-qn)/2}{2} r^2 + \cdots + \binom{(p-qn)/2}{N-n} r^{N-n} \right]. \tag{17}$$

Setting now $\kappa = 1$, and replacing the variational parameter K by v defined by $K^2 \equiv x/v$, we obtain from (15) at $x = 1$ the variational expansions

$$f_N(v) = \sum_{n=0}^{N} a_n v^{qn-p/2} \big[1 + (v-1)\big]_{N-n}^{(p-qn)/2}, \tag{18}$$

where the symbol $[1 + A]_{N-n}^{(p-qn)/2}$ is a short notation for the binomial expansion of $(1 + A)^{(p-qn)/2}$ in powers of A up to the order A^{N-n}.

The variational expansions are optimized in v by minima for odd, and by turning points for even N, as shown in Fig. 3. The extrema are plotted as a function of the order N in the lower row of Fig. 1. The left-hand plot shows directly the extremal values of $\nu_N^{-1}(v)$, the middle plot shows the α-values $\alpha_N = 2 - 3\nu_N$ corresponding to these. The right-hand plot, finally, shows the extremal values of $\alpha_N(v)$. All three sequences of approximations are fitted very well by a large N expansion $c_0 + c_1/N^2 + c_2/N^4$, if we omit the lowest five data points which are not yet very regular. The inverse powers 2 and 4 of N in this fit are determined by starting from a more general ansatz $c_0 + c_1/N^{p_1} + c_2/N^{p_2}$ and varying p_1, p_2 until the sum of the square deviations of the fit from the points is minimal.

The highest-order data point is taken to be the one with $N = 12$ since, up to this order, the successive asymptotic values c_0 change monotonously by decreasing amounts. Starting with $N = 13$, the changes increase and reverse direction. In addition, the mean square deviations of the fits increasing drastically, indicating a decreasing usefulness of the extrapolated expansion coefficients in (7) and (9) for the extrapolation $N \to \infty$. From the parameter c_0 of the best fit for α which is indicated on top of the lower right-

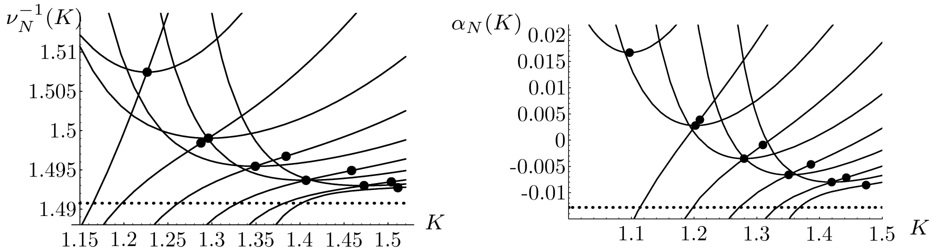

Fig. 3. Successive variational functions $\nu_N^{-1}(h)$ and $\alpha_N(h)$ with $N = 3, \ldots, 12$ of Table 2 plotted for $h = x = 1$ against the variational parameter $K = \sqrt{x/v}$, together with their minima for odd N, or turning points for even N. These points are plotted against N in the lower row of Fig. 1, where they are extrapolated to $N \to \infty$, yielding the critical exponents.

Table 2
Variational reexpansions of $\nu_N^{-1}(h)$ and $\alpha_N(h)$ for $N = 2, \ldots, 9$ at $h = x = 1$ which are plotted in Fig. 3 and whose minima and turning points are extrapolated to $N = \infty$ in the lower left- and right-hand plots of Fig. 1. The lists are carried only to $N = 9$, to save space, whereas the plots are for $N = 3, \ldots, 12$

$$\nu_2^{-1} = 2 - 1.2v + 0.69067v^2$$
$$\nu_3^{-1} = 2 - 1.8v + 2.07200v^2 - 0.72036v^3$$
$$\nu_4^{-1} = 2 - 2.4v + 4.14400v^2 - 2.88145v^3 + 0.53412v^4$$
$$\nu_5^{-1} = 2 - 3.0v + 6.90667v^2 - 7.20363v^3 + 2.67060v^4 + 0.28949v^5$$
$$\nu_6^{-1} = 2 - 3.6v + 10.3600v^2 - 14.4073v^3 + 8.01180v^4 + 1.73692v^5 - 2.96286v^6$$
$$\nu_7^{-1} = 2 - 4.2v + 14.5040v^2 - 25.2127v^3 + 18.6942v^4 + 6.07922v^5 - 20.7401v^6 + 11.1835v^7$$
$$\nu_8^{-1} = 2 - 4.8v + 19.3387v^2 - 40.3403v^3 + 37.3884v^4 + 16.2113v^5 - 82.9602v^6 + 89.4683v^7 - 36.9575v^8$$
$$\nu_9^{-1} = 2 - 5.4v + 24.8640v^2 - 60.5105v^3 + 67.2992v^4 + 36.4753v^5 - 248.881v^6 + 402.607v^7 - 332.617v^8 + 121.914v^9$$

$$\alpha_2 = 0.5 - 0.90v + 0.3830v^2$$
$$\alpha_3 = 0.5 - 1.35v + 1.1490v^2 - 0.26997v^3$$
$$\alpha_4 = 0.5 - 1.80v + 2.2980v^2 - 1.07989v^3 + 0.025254v^4$$
$$\alpha_5 = 0.5 - 2.25v + 3.8300v^2 - 2.69972v^3 + 0.126271v^4 + 0.57604v^5$$
$$\alpha_6 = 0.5 - 2.70v + 5.7450v^2 - 5.39945v^3 + 0.378812v^4 + 3.45629v^5 - 2.19244v^6$$
$$\alpha_7 = 0.5 - 3.15v + 8.0430v^2 - 9.44903v^3 + 0.883895v^4 + 12.0970v^5 - 15.3471v^6 + 6.89011v^7$$
$$\alpha_8 = 0.5 - 3.60v + 10.724v^2 - 15.1184v^3 + 1.767790v^4 + 32.2587v^5 - 61.3884v^6 + 55.1208v^7 - 21.5704v^8$$
$$\alpha_9 = 0.5 - 4.05v + 13.788v^2 - 22.6777v^3 + 3.182020v^4 + 72.5821v^5 - 184.165v^6 + 248.044v^7 - 194.134v^8 + 70.781v^9$$

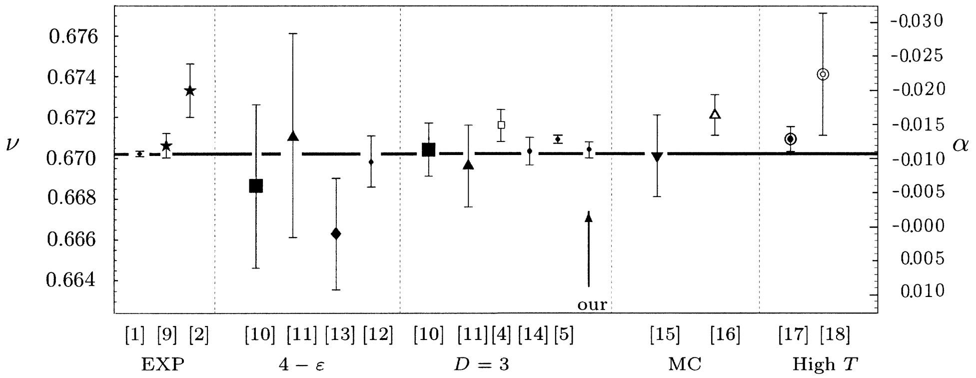

Fig. 4. Survey of experimental and theoretical values for α. The latter come from resummed perturbation expansions of ϕ^4-theory in $4 - \varepsilon$ dimensions, in three dimensions, and from high-temperature expansions of XY-models on a lattice. The sources (Refs. [1,2,4,5,9–18]) are indicated below.

hand plot in Fig. 1, we find the critical exponent $\alpha = -0.01126$ stated in Eq. (4), where the error estimate takes into account the basic systematic errors indicated by the difference between the resummation of $\alpha = 2 - 3\nu$, and of ν^{-1}, which by the lower middle plot in Fig. 1 yields $\alpha = -0.01226$. It also accommodates our earlier seven-loop strong-coupling result (3) of Ref. [5]. The dependence on the choice of h_0 is negligible as long as the resummed series $\nu^{-1}(x)$ and $\alpha(x)$ do not change their Borel character. Thus

$h_0 = 2.2$ leads to results well within the error limits in (4).

Our number as well as many earlier results are displayed in Fig. 4.

The entire subject is discussed in detail in Ref. [19].

Note added in proof

A recent calculation of α by an improved high-temperature expansion yields the exponent $\alpha = -0.0150(17)$ [20].

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