

# Fast-Convergent Resummation Algorithm and Critical Exponents of $\phi^4$ -Theory in Three Dimensions

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## Abstract

We develop an efficient algorithm for evaluating divergent perturbation expansions of field theories in the bare coupling constant  $g_B$  for which we possess a finite number  $L$  of expansion coefficients plus two more informations: The knowledge of the large-order behavior proportional to  $(-\alpha)^k k! k^\beta g_B^k$ , with a known growth parameter  $\alpha$ , and the knowledge of the approach to scaling being of the type  $c + c'/g_B^\omega$ , with constants  $c, c'$  and a critical exponent of approach  $\omega$ . The latter information leads to an increase in the speed of convergence and a high accuracy of the results. The algorithm is applied to the six- and seven-loop expansions for the critical exponents of  $O(N)$ -symmetric  $\phi^4$ -theories, and the result for the critical exponent  $\alpha$  is compared with the recent satellite experiment.

## 1 Introduction

The field-theoretic approach to critical phenomena provides us with power series expansions for the critical exponents of a wide variety of universality classes. For  $\phi^4$ -theories with  $O(N)$  symmetry in three dimensions, these expansions have been calculated numerically as power series in the renormalized coupling constant up to seven loops for the critical exponents  $\nu$  and  $\eta$  [1] and up to six loops for the exponent  $\omega$  governing the approach to scaling. In  $4 - \epsilon$  dimensions, exact  $\epsilon$ -expansions are available up to five loops for all critical exponents with  $O(N)$  symmetry [2], cubic symmetry, and mixtures of these [3]. When inserted into the renormalization group equations, these expansions are supposed to determine the critical exponents via their values at an infrared-stable fixed point  $g = g^*$ . The latter step is nontrivial since the expansions are divergent and require resummation, for which sophisticated methods have been developed, summarized and applied most recently in [4]. The resummation methods use the information from the known large-order behavior  $(-\alpha)^k k! k^\beta g_B^k$  of the expansions and analytic mapping techniques to obtain quite accurate results.

A completely different resummation procedure was developed recently on the basis of variational perturbation theory [5] to the expansions in powers of the *bare coupling constant*, which goes to infinity at the critical point. The resulting *strong-coupling theory* [6] was successfully applied in three [8, 9] and  $4 - \epsilon$  dimensions [10], and rendered for the first time an interpolation between expansions of  $4 - \epsilon$  and  $2 + \epsilon$ -dimensional theories. This method converges as fast as the previous

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ones, even though it does not take into account the information on the large-order behavior of the expansions. Instead, it uses the fact that the power series for the critical exponents approach their constant critical value in the form  $c + c'/g_B^\omega$ , where  $c, c'$  are constants, and  $\omega$  is the critical exponent of the approach to scaling. The results showed that the latter information is just as efficient in increasing the speed of convergence as the information on the large-order behavior.

We may therefore expect that a resummation method which incorporates both informations should lead to results with an even higher accuracy, and it is the purpose of this paper to present such a method in the form of a simple algorithm.

## 2 The Problem

The development of our resummation algorithm is based on an improvement of the problem formulated in [6, 7] and solved via variational perturbation theory [5]. Mathematically, the problem we want to solve is the following: Let

$$f_L(g_B) = \sum_{k=0}^L f_k g_B^k \quad (1)$$

be the first  $L$  terms of a divergent asymptotic expansion

$$f(g_B) = \sum_{k=0}^{\infty} f_k g_B^k \quad (2)$$

of a function  $f(g_B)$ , which possesses a strong-coupling expansion of the type

$$f(g_B) = g_B^s \sum_{k=0}^{\infty} b_k g_B^{-k\omega}, \quad (3)$$

which is assumed to have some finite convergence radius  $|g_B| \geq g_B^{\text{conv}}$ . Suppose that the function is analytic in the complex  $g_B$ -plane with a cut along the negative real axis, with a discontinuity known from instanton calculations [11, 12] to have near the tip of the cut the generic form

$$\text{disc}f(-g_B) \equiv 2\pi i \gamma (\alpha|g_B|)^{-\beta-1} e^{-1/\alpha|g_B|}. \quad (4)$$

Via a dispersion relation,

$$f(g_B) = \frac{1}{2\pi i} \int_0^\infty dg'_B \frac{\text{disc}f(-g'_B)}{g'_B + g_B}, \quad (5)$$

or a sufficiently subtracted version of it, this discontinuity corresponds to the large-order behavior of the expansion coefficients  $f_k$

$$f_k \stackrel{k \rightarrow \infty}{\equiv} \gamma k! (-\alpha)^k k^\beta [1 + \mathcal{O}(1/k)]. \quad (6)$$

The constant  $\alpha$  is given by the inverse action of the radial symmetric solution to the classical field equations. The parameter  $\beta$  counts the number of zero modes in the fluctuation determinant around this solution. The absolute normalization  $\gamma$  of the large-order behavior requires the calculation of the fluctuation determinant [11].

As far as the leading strong-coupling coefficient  $b_0$  is concerned, this problem has been attacked before by Parisi [13] using a resummation method based on Borel transformations in combination with analytic mapping techniques. However, when applied to the asymptotic expansions of the ground state energy of the anharmonic oscillator, his method converged very slowly, too slow to lead to reliable critical exponents, where only five to seven expansion coefficients  $f_k$  are known. The reason is that in Parisi's approach, the corrections to the leading power behavior are failing to match the true fractional powers of the strong-coupling expansion (3).

This deficiency was cured by the strong-coupling theory of one of the authors (HK) in Ref. [6], and the subsequent application to critical exponents in Refs. [8, 9, 10], which showed a surprisingly rapid convergence. However, that theory did not take advantage of the knowledge of the large-order behavior (6), which can lead to an increase in the speed of convergence and thus of the accuracy of theoretical values for the critical exponents. This will be done in the present improved resummation method.

### 3 Borel Methods

Basis for this method is the development of a more general Borel-like transformation which will automatically guarantee the form of the strong-coupling expansion (3) for each approximant  $f_L(g_B)$ . Let us first recall briefly the important properties of the ordinary Borel transformations: It is a function  $B(t)$  associated with  $f(g_B)$  which is defined by the Taylor series

$$B(t) = \sum_{k=0}^{\infty} B_k t^k \equiv \sum_{k=0}^{\infty} \frac{f_k}{k!} t^k. \quad (7)$$

By dividing the expansion coefficients  $f_k$  by  $k!$ , the factorial growth of  $f_k$  is reduced to a power growth, thus giving  $B(t)$  a finite convergence radius.

An alternative definition of the Borel transform is given by the contour integral

$$B(t) \equiv \frac{1}{2\pi i} \oint_C \frac{dz}{z} e^z f(t/z), \quad (8)$$

where the contour  $C$  encloses anticlockwise the negative real axis. Indeed, inserting here (1) and performing the integral we obtain (7).

If  $f(g_B)$  is an analytic function in the sector

$$S_{\pi/2+\delta}^R \equiv \{g_B \mid |g_B| < R, |\arg(g_B)| < \pi/2 + \delta\} \quad (9)$$

of a circle, and satisfies the so-called strong asymptotic condition

$$|f(g_B) - \sum_{k=0}^L f_k g_B^k| < A g_B^{L+1} \alpha^{L+1} (L+1)!, \quad \text{with } \alpha, A > 0, \quad (10)$$

then  $B(t)$  is analytic in  $S_\delta^\infty$ , with a finite radius of convergence  $t < 1/\alpha$ . The original function  $f(g_B)$  can be recovered from  $B(t)$  by the inverse Borel transformation

$$f(g_B) = \int_0^\infty e^{-t} B(t g_B) dt. \quad (11)$$

Obviously, the inverse transformation can only be performed if  $B(t)$  is known on the entire positive real  $t$ -axis. The Taylor series (7) for  $B(t)$ , however, converges only inside the circle of radius  $1/\alpha$ . Before we can do the integral in (11), we must therefore perform a suitable analytic continuation of (7) [14]. This can be done by reexpanding  $B(t)$  in powers of the function  $\kappa(t)$  defined implicitly by

$$t = \frac{1}{\sigma} \frac{\kappa(t)}{[1 - \kappa(t)]^p}. \quad (12)$$

This function maps the interval  $[0, \infty]$  of the  $t$ -axis to the interval  $[0, 1]$  of the  $\kappa$ -plane. By a proper choice of  $\sigma$  it is possible to make the unit circle free of singularities. Then we may use the reexpansion  $B(t)$  in powers of  $\kappa(t)$  truncated after  $\kappa^L$ ,

$$B_L(t) \equiv \sum_{k=0}^L v_k \kappa^k(t), \quad (13)$$

as an approximation to  $B(t)$  on the entire positive real  $t$  axis. Inserting this into the inverse transformation formula (11), we obtain an approximation  $f_L^a(g_B)$  for  $f(g_B)$ , which has the same first  $L$  expansion coefficients as  $f_L(g_B)$  and, in addition, the correct large-order behavior (6).

How can we incorporate the strong-coupling expansion (3) of  $f(g_B)$  into the approximation  $f_L^a(g_B)$ ? In the Borel transform  $B(t)$ , the strong-coupling expansion (3) amounts to a large- $t$  expansion

$$B(t) = t^s \sum_{k=0}^{\infty} \frac{\sin \pi(k\omega - s)}{\pi} \Gamma(k\omega - s) b_k t^{-k\omega}. \quad (14)$$

This follows directly by inserting (3) into (8) and integrating each term. Here and in the sequel,  $C$  denotes a path of integration which encloses anticlockwise the negative real axis in the complex plane.

If the series (3) has a finite radius of convergence, the large- $t$  expansion of  $B(t)$  is a divergent asymptotic one, because of the factor  $\Gamma(k\omega - s)$  in the  $k$ -th expansion coefficient.

It should be stressed, that the relation between the coefficients of the strong-coupling expansion (3) and the coefficients of expansion (14) is not generally invertible, because of the factor  $\sin \pi(k\omega - s)$  which causes the coefficients of negative integer powers of  $t$  to vanish.

Note that, in general, an expansion in the Borel-plane with a power sequence in  $t$  as in (14) is not sufficient to ensure an expansion in the same powers in the  $g_B$ -plane as in (3), because of the appearance of extra integer powers in  $g_B$ . This is illustrated by the simple function  $B(t) = (1+t)^s$ , which possesses a strong-coupling expansion in the powers  $t^{s-k}$ . If  $s$  is non-integer the expansion of the corresponding function  $f(g_B)$  reads

$$f(g_B) = \int_0^\infty dt e^{-t}(1+g_B t)^s = e^{1/g_B} \Gamma(s+1) g_B^s + e^{1/g_B} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+s+1)k!} g_B^{-k-1}, \quad (15)$$

and expanding the exponential we see that the sum contains integer powers which are not contained in the strong-coupling expansion of  $B(t)$ .

It is advantageous to perform a further analytic continuation of the reexpansion (13) which enforces automatically the leading power behavior  $t^s$  of  $B(t)$ . For this we change (13) to

$$B_L(t) \equiv [1 - \kappa(t)]^{-ps} \sum_{k=0}^L h_k \kappa^k(t). \quad (16)$$

The coefficients  $h_k$  are determined by using (12) to expand  $\kappa(t)$  in powers of  $t$ , inserting this into (16), reexpanding in powers of  $t$ , and comparing the final coefficients with those in (7). When the approximation (16) is inserted into (11), we obtain  $f_L^a(g_B)$  with the correct leading power behavior  $g_B^s$  for large  $g_B$ .

Unfortunately, the simple prefactor does not produce the correct subleading powers  $(g_B)^{s-k\omega}$  of the strong-coupling expansion (3), and we have not been able to find another simple analytic continuation of  $B(t)$  which would achieve this.

## 4 Hyper-Borel Transformation

A solution of this problem is, however, possible with the help of a generalization of the Borel-Leroy transformation to what we shall call a *hyper-Borel transformation* [15]

$$\tilde{B}(y) = \sum_{k=0}^{\infty} \tilde{B}_k y^k, \quad (17)$$

with coefficients

$$\tilde{B}_k \equiv \omega \frac{\Gamma(k(1/\omega - 1) + \beta_0)}{\Gamma(k/\omega - s/\omega) \Gamma(\beta_0)} f_k. \quad (18)$$

## 4.1 General Properties

The inverse transformation is given by the double integral

$$f(g_B) = \frac{\Gamma(\beta_0)}{2\pi i} \oint_C dt e^t t^{-\beta_0} \int_0^\infty \frac{dy}{y} \left[ \frac{g_B}{yt^{(1-\omega)/\omega}} \right]^s \exp \left[ \frac{yt^{(1-\omega)/\omega}}{g_B} \right]^\omega \tilde{B}(y), \quad (19)$$

as can easily shown with the help of the integral representation of the inverse Gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C dt e^t t^{-z}. \quad (20)$$

The transformation possesses a free parameter  $\beta_0$  which will be used to optimize the approximation  $f_L(g_B)$  at each order  $L$ . The power  $\omega$  of the strong-coupling expansion is assumed to lie in the interval  $0 < \omega < 1$ , as it does in the upcoming physical applications.

The hyper-Borel transformation has the desired property of allowing for a resummation of  $f_L(g_B)$  with the full sequence of powers of  $g_B$  in the strong-coupling expansion (3). To show this we first observe that as in the ordinary Borel transform (7), the large-argument behavior of the Gamma function known from Stirling's formula

$$\Gamma(pk + q) \stackrel{k \rightarrow \infty}{\equiv} \sqrt{2\pi}^{1-p} p^{q-1/2} k^{-1/2+q-p/2} p^{pk} (k!)^p [1 + \mathcal{O}(1/k)], \quad (21)$$

removes the factorial growth (6) from the expansion coefficients  $f_k$ , and leads to a simple power behavior of the coefficients  $\tilde{B}_k$ :

$$\tilde{B}_k \stackrel{k \rightarrow \infty}{\equiv} \text{const.} \times \left[ \alpha \omega (1-\omega)^{1/\omega-1} \right]^k k^{\beta+\beta_0+1/2+s/\omega} [1 + \mathcal{O}(1/k)]. \quad (22)$$

Thus our transform  $\tilde{B}(y)$  shares with the ordinary Borel transform  $B(t)$  the property of being analytic at the origin. Its radius of convergence is determined by the singularity on the negative real axis at

$$y_s = -\frac{1}{\sigma} \equiv -\frac{1}{\alpha \omega (1-\omega)^{1/\omega-1}}. \quad (23)$$

## 4.2 Resummation Procedure

A resummation procedure can now be set up on the basis of the transform  $\tilde{B}(y)$  as before. The inverse transformation (19) contains an integral over the entire positive axis, requiring again an analytic continuation of the Taylor expansion of  $\tilde{B}(y)$  beyond the convergence radius.

The reason for introducing the transform  $\tilde{B}(y)$  was to allow us to reproduce the complete power sequence in the strong-coupling expansion (3), with a leading power  $g_B^s$  and a subleading sequence of powers  $g_B^{s-k\omega}$ ,  $k = 1, 2, 3, \dots$ . This is achieved by removing a factor  $e^{-\rho\sigma y}$  with  $\rho, \sigma > 0$  from the

truncated series (18) of our transform  $\tilde{B}(y)$ . Furthermore by removing a second simple prefactor of the form  $(1 + \sigma y)^{-\delta}$  we weaken the leading singularity in the hyper-Borel complex  $y$ -plane, which determines the large order behavior (6). The remaining series has still a finite radius of convergence. To achieve convergence on the entire positive  $y$  axis for which we must do the integral (19), we reexpand the remaining series of  $y$  in powers of  $\kappa(y)$  which is related to  $y$  by an equation like Eq. (12). For simplicity we choose the parameter  $p = 1$ , i.e.

$$y = \frac{1}{\sigma} \frac{\kappa(y)}{1 - \kappa(y)}, \quad (24)$$

which maps a shifted right half of the complex  $y$ -plane with  $\text{Re}[y] \geq -1/2\sigma$  onto the unit circle in the complex  $\kappa$ -plane. Thus we reexpand  $\tilde{B}(y)$  in the following way:

$$\tilde{B}(y) \equiv \sum_{k=0}^{\infty} \tilde{B}_k y^k = e^{-\rho\sigma y} [1 + \sigma y]^{-\delta} \sum_{k=0}^{\infty} h_k \kappa^k(y) = e^{-\rho\sigma y} \sum_{k=0}^{\infty} h_k \frac{(\sigma y)^k}{(1 + \sigma y)^{k+\delta}}. \quad (25)$$

The inverse hyper-Borel transform of  $\tilde{B}(y)$  is now found by forming the integrals of the expansion functions in (25)

$$I_n(g_B) = \frac{\Gamma(\beta_0)}{2\pi i} \oint_C dt e^t t^{-\beta_0} \int_0^\infty \frac{dy}{y} \left[ \frac{g_B}{y t^{1/\omega-1}} \right]^s \exp \left[ -\frac{y t^{1/\omega-1}}{g_B} \right]^\omega e^{-\rho\sigma y} \frac{(\sigma y)^n}{(1 + \sigma y)^{n+\delta}}, \quad (26)$$

so that the approximants  $f_L^a(g_B)$  may be written as

$$f_L^a(g_B) = \sum_{n=0}^L h_n I_n(g_B). \quad (27)$$

The same functions  $I_n(g_B)$  may be used as basis functions for a wide variety of divergent truncated perturbation expansions  $f_L(g_B)$ . The complete list of parameters on which they depend reads as follows

$$I_n(g_B) = I_n(g_B, \omega, s, \rho, \sigma, \delta, \beta_0) = I_n(\sigma g_B, \omega, s, \rho, 1, \delta, \beta_0), \quad (28)$$

but in the following we shall mostly use the shorter notation  $I_n(g_B)$ . The integral representation of  $I_n(g_B)$  breaks down at  $s = n$ , requiring an analytical continuation. For the upcoming applications in the large- $g_B$  regime it will be sufficient to perform this continuation only in the convergent strong-coupling expansion of  $I_n(g_B)$ . This is obtained by performing a Taylor series expansion of the exponential function in (26), which is an expansion in powers of  $1/g_B^\omega$ . After integrating over  $t$  and  $y$  using (20), we obtain an expansion

$$I_n(g_B) = g_B^s \sum_{k=0}^{\infty} b_k^{(n)} g_B^{-k\omega}, \quad (29)$$

which has indeed the same power sequence as the strong-coupling expansion (3) of the function  $f(g_B)$  to be resummed.

The expansion coefficients are

$$b_k^{(n)} = \frac{(-1)^k}{k!} \frac{\sigma^{s-k\omega} \Gamma(\beta_0)}{\Gamma[(\omega-1)k + \beta_0 + (1/\omega-1)s]} i_k^{(n)}, \quad (30)$$

where  $i_k^{(n)}$  denotes the integral

$$i_k^{(n)} = \int_0^\infty dy e^{-\rho y} (1+y)^{-\delta-n} y^{k\omega+n-s-1}. \quad (31)$$

This integral is seen to coincide with the Kummer function

$$U(\alpha, \gamma, z) \equiv \frac{1}{\Gamma(\alpha)} \int_0^\infty dy e^{-zy} y^{\alpha-1} (1+y)^{\gamma-\alpha-1}, \quad (32)$$

in terms of which we can write

$$i_k^{(n)} = \Gamma(k\omega + n - s) U(k\omega + n - s, k\omega - s - \delta + 1, \rho). \quad (33)$$

The latter expression is useful since in some applications the integral (31) may diverge, and requires an analytic continuation by deforming the contour of integration. Such deformations are automatically supplied by choosing other representations for the Kummer function, for instance

$$U(\alpha, \gamma, z) = \frac{\pi}{\sin \pi \gamma} \left[ \frac{M(\alpha, \gamma, z)}{\Gamma(1 + \alpha - \gamma) \Gamma(\gamma)} - z^{1-\gamma} \frac{M(1 + \alpha - \gamma, 2 - \gamma, z)}{\Gamma(\alpha) \Gamma(2 - \gamma)} \right], \quad (34)$$

where  $M(\alpha, \gamma, z)$  is the confluent hypergeometric function with a Taylor expansion

$M(\alpha, \gamma, z) = 1 + \frac{\alpha}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha-1)}{\gamma(\gamma-1)} \frac{z^2}{2!} + \dots$  (35) The alternative expression (33) for  $i_k^{(n)}$ , with (34) and (35), is useful for resumming various asymptotic expansions, for example that of the ground state energy of the anharmonic oscillator, in which case the leading strong-coupling power  $s$  has the value  $1/3$ . There the integral representation (31) would have to be evaluated for values  $n = 0$ ,  $k = 0$ , where the integral does not exist, whereas formula (33) with (34) and (35) is well-defined.

For large  $k$ , the integral on the right-hand side of (31) can be estimated with the help of the saddle point approximation. The saddle point lies at

$$y_s \approx \frac{k\omega}{\rho}, \quad (36)$$

leading to the asymptotic estimate

$$\begin{aligned} i_k^{(n)} &\stackrel{k \rightarrow \infty}{\approx} \left( \frac{k\omega}{\rho} \right)^{-\delta-n} \int_0^\infty dy e^{-\rho y} y^{\omega k+n-s-1} [1 + \mathcal{O}(1/k)] \\ &= \left( \frac{\omega k}{\rho} \right)^{-\delta-n} \rho^{-k\omega-n+s} \Gamma(k\omega + n - s) [1 + \mathcal{O}(1/k)]. \end{aligned} \quad (37)$$

The behavior of the strong-coupling coefficients  $b_k^{(n)}$  for large  $k$  is obtained with the help of the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad (38)$$

and Stirling's formula (21), yielding

$$b_k^{(n)} \stackrel{k \rightarrow \infty}{\equiv} \gamma \sin \pi [k(\omega - 1) + \beta_0 + (1/\omega - 1)s] \left[ -\frac{(1-\omega)^{(1-\omega)}}{(\sigma\rho)^\omega} \right]^k k^{\gamma_1} [1 + \mathcal{O}(1/k)]. \quad (39)$$

The values of the real constants  $\gamma, \gamma_1$  will not be needed in the upcoming discussions, and are therefore not calculated explicitly.

Equation (39) shows that the strong-coupling expansion (3) has a convergence radius

$$|g_B| \geq \frac{(\rho\sigma)^\omega}{(1-\omega)^{1-\omega}}, \quad (40)$$

which means that the basis functions  $I_n(g_B)$ , and certainly also  $f(g_B)$  itself, possess additional singularities beside  $g_B = 0$ . The parameter  $\rho$  will be optimally adjusted to match the positions of these singularities.

### 4.3 Taylor Series of Basis Functions

For reexpanding  $f_L(g_B)$  in terms of the basis functions  $I_n(g_B)$ , we must know their Taylor series. These are obtained by substituting into (26) the variable  $y$  by  $g_B y'$ , and expanding the integrand of (26) in powers of  $g_B$ . After performing the integrals over  $y'$  and  $t$ , we find

$$I_n(g_B) = \sum_{k=n}^{\infty} f_k^{(n)} g_B^k, \quad (41)$$

with the coefficients

$$f_k^{(n)} = \frac{1}{\omega} \frac{\Gamma(\beta_0)\Gamma(k/\omega - s/\omega)}{\Gamma(k(1/\omega - 1) + \beta_0)} \sum_{j=0}^{k-n} \binom{-\delta - n}{j} \frac{(-\rho)^{k-n-j}}{(k-n-j)!} \sigma^k. \quad (42)$$

The coefficients in the last sum arise from the  $t$ -integral:

$$\sum_{j=0}^{k-n} \binom{-n - \delta}{j}$$

$j(-\rho)^{k-n-j} \frac{1}{(k-n-j)!} \frac{(-1)^{k-n}}{\Gamma(k-n+1)\Gamma(n+\delta)} \int_0^\infty dt e^{-t\delta+n-1} (\rho+t)^{k-n}$ . For large  $k$ , the integral may be evaluated with the help of the saddle-point approximation. Using this and Stirling's formula (21), we find

$$\sum_{j=0}^{k-n} \binom{-n - \delta}{j} \frac{(-\rho)^{k-n-j}}{(k-n-j)!} \stackrel{k \rightarrow \infty}{\equiv} \frac{(-1)^{k-n} e^\rho}{\Gamma(\delta + n)} k^{\delta+n-1} [1 + \mathcal{O}(1/k)]. \quad (44)$$

Inserting this into (42) and using once more Stirling's formula, we obtain for the expansion coefficients  $f_k^{(n)}$  the following factorial growth

$$f_k^{(n)} \stackrel{k \rightarrow \infty}{\simeq} \frac{(-1)^n e^{\rho} \Gamma(\beta_0)}{\sqrt{2\pi} \Gamma(\delta + n)} (1 - \omega)^{1/2 - \beta_0} \omega^{\beta_0 - 1 + s/\omega} k^{\delta - \beta_0 + n - 3/2 - s/\omega} \left[ -\frac{\sigma}{\omega(1 - \omega)^{1/\omega - 1}} \right]^k k! \times [1 + \mathcal{O}(1/k)]. \quad (45)$$

For an optimal reexpansion (27), we shall choose the free parameters of the basis functions  $I_n(g_B, \omega, s, \rho, \sigma, \delta, \beta_0)$  to match the large-order behavior of the coefficients  $f_k$  in (6).

#### 4.4 Convergence Properties of Resummed Series

We shall now discuss the speed of convergence of the resummation procedure. For this it will be sufficient to estimate the convergence of the strong-coupling coefficients  $b_k^L$  of the approximations  $f_L(g_B)$  against the true strong-coupling coefficients  $b_k$  in (3). The convergence for arbitrary values of  $g_B$  will always be better than that. Such an estimate is possible by looking at the large- $n$  behavior of the expansion coefficients  $b_k^{(n)}$  in the strong-coupling expansion of  $I_n(g_B)$  in (29). This is determined by the saddle point approximation to the integral  $i_k^{(n)}$  in Eq. (31), which we rewrite as

$$i_k^{(n)} = \int_0^\infty dy e^{-\rho y - n \ln(1+1/y)} (1+y)^{-\delta} y^{k\omega - s - 1}. \quad (46)$$

The saddle point lies at

$$y_s = \sqrt{\frac{n}{\rho}} [1 + \mathcal{O}(1/\sqrt{n})]. \quad (47)$$

At this point, the total exponent in the integrand is

$$-\rho y_s - n \ln \left( 1 + \frac{1}{y_s} \right) = -2\sqrt{\rho n} [1 + \mathcal{O}(1/\sqrt{n})], \quad (48)$$

implying the large- $n$  behavior

$$b_k^{(n)} \stackrel{n \rightarrow \infty}{\simeq} \text{const.} \times n^{k\omega - s - 1 - \delta} e^{-2\sqrt{\rho n}} [1 + \mathcal{O}(1/\sqrt{n})]. \quad (49)$$

The strong-coupling coefficients  $b_k^L$  of the approximations  $f_L^a(g_B)$  are linear combinations of the coefficients  $b_k^{(n)}$  of the basis functions  $I_n(g_B)$ :

$$b_k^L = \sum_{n=0}^L b_k^{(n)} h_n. \quad (50)$$

The speed of convergence with which the  $b_k^L$ 's approach  $b_k$  as the number  $L$  goes to infinity is governed by the growth with  $n$  of the reexpansion coefficients  $h_n$  and of the coefficients  $b_k^{(n)}$  in Eq. (49). We shall see that for the series to be resummed, the reexpansion coefficients  $h_n$  will grow at most like

some power  $n^r$ , implying that the approximations  $b_k^L$  approach their  $L \rightarrow \infty$ -limit  $b_k$  with an error proportional to

$$b_k^L - b_k \sim L^{r+k\omega-s-\delta-1/2} \times e^{-2\sqrt{\rho L}}. \quad (51)$$

The leading exponential falloff of the error  $e^{-2\sqrt{\rho L}}$  is independent of the other parameters in the basis functions  $I_n(g_B, \omega, p, \rho, \sigma, \delta, \beta_0)$  which still need adjustment. This is the important advantage of the present resummation method with respect to variational perturbation theory [5, 8] where the error decreases merely like  $e^{-\text{const} \times L^{1-\omega}}$  with  $1 - \omega$  close to  $1/4$ .

The nonexponential prefactor in Eq. (51) depends on the parameters in  $I_n(g_B, \omega, p, \rho, \sigma, \delta, \beta_0)$ . Some of them are related to observables, others are free and may be chosen to optimize the convergence.

#### 4.4.1 Parameters $s$ and $\omega$

The perturbation expansions for the critical exponents are power series in the bare coupling constant  $g_B$  whose strong-coupling limit is a constant [8, 9]. The same is true for the series expressing the renormalized coupling constant  $g$  in powers of the bare coupling constant. This implies that the growth parameter  $s$  for the basis functions  $I_n(g_B)$  is equal to zero in all cases. The constant asymptotic values are approached with the subleading powers  $1/g_B^{k\omega-s}$ , where  $\omega$  is a universal experimentally measurable critical exponent.

#### 4.4.2 Parameter $\sigma$

In the ordinary Borel-transformation, the parameter  $\alpha$  in the large-order behavior of the expansion coefficients  $f_k$  in Eq. (6), which is determined directly by the inverse value of the reduced action of the classical solution to the field equations, specifies also the position of the singularity on the negative  $t$  axis in  $B(t)$ . In our transform  $\tilde{B}(y)$ , the singularity position of the singularity is proportional to  $\alpha$ , with an  $\omega$ -dependent prefactor. It lies at [see Eq. (23)]

$$\sigma = \alpha\omega(1 - \omega)^{1/\omega-1}. \quad (52)$$

This value of  $\sigma$  ensures that the expansion coefficients  $f_k^n$  of the basis functions  $I_n(g_B)$  in Eq. (45) grow for large  $k$  with the same factor  $(-\alpha)^k$  as the expansion coefficients for  $f(g_B)$  in Eq. (6).

The conformal mapping (24) maps the singularity at  $t = -1/\sigma$  to  $\kappa = \infty$ , and converts the cut along the negative into a cut in the  $\kappa$ -plane from 1 to  $\infty$ . The growth of the reexpansion coefficients  $h_n$  with  $n$  is therefore determined by the nature of the singularity of  $\tilde{B}(y)$  at  $\infty$ .

In the upcoming applications to critical exponents it will turn out that the value (52) following from the inverse action of the solution to the classical field equations and  $\omega$  will not yield the fastest convergence of the approximations  $f^L(g_B)$  towards  $f(g_B)$ , but that a slightly smaller value gives better results. This seems to be due to the fact that the classical solution gives only the nearest singularity in the hyper-Borel transform  $\tilde{B}(y)$  of  $f(g_B)$ . In reality, there are many additional cuts from other fluctuating field configurations which determine the size of the expansion coefficients  $f_k$  at pre-asymptotic orders  $k$ . Since the few known  $f_k$ 's are always pre-asymptotic, they are best accounted for by an effective shift of the position of the singularity into the direction of the additional cuts at larger negative  $y$ , corresponding to a smaller  $\sigma$ .

#### 4.4.3 Parameter $\rho$

According to Eq. (40), the parameter  $\rho$  determines the radius of convergence of the strong-coupling expansion of the basis functions  $I_n(g_B)$ . It should therefore be adjusted to fit optimally the corresponding radius of the original function  $f(g_B)$ . Since we do not know this radius, this adjustment will be done phenomenologically by varying  $\rho$  to optimize the speed of convergence. Specifically, we shall search at each order  $L$  for a vanishing highest reexpansion coefficient  $h_L$  or, if it does not vanish anywhere, for a vanishing derivative with respect to  $\rho$ :

$$h_L(\rho) = 0, \quad \text{or} \quad \frac{dh_L(\rho)}{d\rho} = 0. \quad (53)$$

#### 4.4.4 Parameter $\delta$

From Eq. (45)] we see that the parameter  $\delta$  influences the power  $k^\beta$  in the large-order behavior (6). By comparing the two equations, we identify the growth parameter  $\beta$  of  $I_n(g_B)$  as being

$$\beta = \delta - \beta_0 - 3/2 - s/\omega + n. \quad (54)$$

At first it appears to be impossible to give *all* basis functions  $I_n(g_B)$  the same growth power  $\beta$  in (45) by simply letting  $\delta$  depend on the order  $n$  as required by (54). If we were to do this, we would have to assign to  $\delta$  the value

$$\delta = \delta_n \equiv \beta + \beta_0 + 3/2 + s/\omega - n, \quad (55)$$

which depends on the index  $n$  of the function  $I_n(g_B)$ , and this means that we perform an analytical continuation of the power series expansion of  $\tilde{B}(y)$  by reexpanding it as follows:

$$\tilde{B}(y) = \sum_{k=0}^{\infty} \tilde{B}_k y^k = e^{-\rho\sigma y} (1 + \sigma y)^{-\delta} \sum_{k=0}^{\infty} h_k(\sigma y)^k. \quad (56)$$

But the series in this formula which is obtained from the series of  $\tilde{B}(y)$  by removing a simple factor still has the same finite radius of convergence and could not be used to estimate  $\tilde{B}(y)$  for large values of  $y$  needed to perform the back-transform (19). It is, however, possible to sidetrack this problem by letting  $\rho$  grow linearly with the order  $L$ . Then the exponential factor of (56) suppresses the integrals over  $y$  for large  $y$  sufficiently to make the divergence of the reexpanded series (56) at large  $y$  irrelevant. If we determine  $\rho$  from the condition (53), the growth of  $\rho$  with  $L$  turns out to emerge by itself.

#### 4.4.5 Parameter $\beta_0$

The parameter  $\beta_0$  has two effects. From Eq. (30) we see that for

$$k > k_c \equiv \frac{\beta_0 + (1/\omega - 1)s}{1 - \omega} \quad (57)$$

the signs of the strong-coupling expansion coefficients start to alternate irregularly. This irregularity weakens the convergence of the higher strong-coupling coefficients  $b_k^L$  with  $k > k_c$  against  $b_k$ . The convergence can therefore be improved by choosing a  $\beta_0$  which grows proportionally to the order  $L$  of the approximation.

In addition,  $\beta_0$  appears in the power of  $k$  in (45), which is a consequence of the fact that it determines the nature of the cut in  $\tilde{B}(y)$  in the complex  $y$ -plane starting at  $y = -1/\sigma$  [see Eq. (25)].

If we expand both sides of (25) in powers of  $\kappa = \sigma y/(1 + \sigma y)$  and compare the coefficients of powers of  $\kappa$ , it is easy to write down an explicit formula for the reexpansion coefficients  $h_n$  in terms of the coefficients  $\tilde{B}_j$  of  $\tilde{B}(y)$  by

$$h_n = \sum_{k=0}^n \sum_{j=0}^k \frac{\tilde{B}_j \sigma^{-j} \rho^{k-j}}{(k-j)!} \binom{\delta + n - 1}{n - k}, \quad (58)$$

where  $\tilde{B}_j$  are obtained from the original expansion coefficients  $f_k$  of  $f(g_B)$  by relation (18).

Before beginning with the resummation of the perturbation expansions for the critical exponents of  $\phi^4$ -field theories, it will be useful to obtain a feeling for the quality of the above-developed resummation procedure, in particular for the significance of the parameters upon the speed of convergence. We do this by resumming the often-used example of an asymptotic series, the perturbation expansion of the ground state energy of the anharmonic oscillator.

### 4.5 Resummation of Ground State Energy of Anharmonic Oscillator

Consider the one-dimensional anharmonic oscillator with the Hamiltonian

$$H = \frac{p^2}{2} + m^2 \frac{x^2}{2} + g_B x^4. \quad (59)$$

In this quantum mechanical system, there is no need to distinguish bare and renormalized coupling constants, but since the previous resummation formulas were all formulated in terms of  $g_B$  we shall keep this notation also here. The ground state energy has a perturbation expansion

$$E^{(0)}(g_B) = \sum_k^{\infty} f_k g_B^k, \quad (60)$$

whose coefficients can be calculated via the Bender-Wu recursion relation [16] to arbitrarily high orders, with a large-order behavior

$$f_k = -\sqrt{\frac{6}{\pi^3}} k! (-3)^k k^{-1/2} [1 + \mathcal{O}(1/k)]. \quad (61)$$

By comparison with (6) we identify the growth parameters

$$\alpha = 3, \quad \beta = -1/2. \quad (62)$$

A scale transformation  $x \rightarrow g^{1/6} x$  applied to the Hamiltonian (59) reveals the scaling property [17] for the energy as a function of  $g_B$  and  $m^2$ :

$$E(m^2, g_B) = g_B^{1/3} E(g_B^{-2/3} m^2, 1). \quad (63)$$

Combining this with the knowledge [17] that  $E(m^2, 1)$  is an analytic function at  $m^2 = 0$ , we see that  $E(1, g_B)$  possesses a power series expansion of the form (3), with the parameters

$$s = 1/3, \quad \omega = 2/3. \quad (64)$$

Inserting the latter number together with  $\alpha$  from Eq. (62) into (52), we identify

$$\sigma = \frac{2}{\sqrt{3}}. \quad (65)$$

The ground state energy  $E^{(0)}(g_B)$  obeys a once-subtracted dispersion relation [17]:

$$E^{(0)}(g_B) = \frac{1}{2} + \frac{g_B}{\pi} \int_0^{\infty} \frac{dg'_B}{g'_B} \frac{\text{Im} E^{(0)}(-g'_B)}{g'_B + g_B}. \quad (66)$$

The perturbation expansion (60) is obtained from this by expanding  $1/(g'_B + g_B)$  in powers of  $g_B$ , and performing the integral term by term. This shows explicitly that the large-order behavior (61) is caused by an imaginary part

$$\text{Im} E^{(0)}(-|g_B|) = \sqrt{\frac{6}{\pi}} \sqrt{\frac{1}{3|g_B|}} e^{-1/3|g_B|} [1 + \mathcal{O}(|g_B|)] \quad (67)$$

near the tip of the left-hand cut in the complex  $g_B$ -plane, in agreement with the general form (5) associated with the large-order behavior (6).

Let us now specify the parameter  $\delta$ . We shall do this here in an  $n$ -dependent way using Eq. (55), which now reads with (64):

$$\delta = \delta_n \equiv \beta_0 + 3/2 - n. \quad (68)$$

The corresponding basis functions

$$I_n(g_B, 2/3, 1/3, \rho, 2/\sqrt{3}, \beta_0 + 3/2 - n, \beta_0), \quad (69)$$

have then all the same large-order growth parameter  $\beta$  in (6).

The two parameters  $\rho$  and  $\beta_0$  are still arbitrary. The first is determined by an order-dependent optimization of the approximations via the conditions (53). The best choice of  $\beta_0$  will be made differently depending on the regions of  $g_B$ .

Let us test the convergence of our algorithm at small negative coupling constants  $g_B$ , i.e., near the tip of the left-hand cut in the complex  $g_B$ -plane. We do this by calculating the prefactor  $\gamma$  in the large-order behavior (6). In this case the convergence turns out to be fastest by giving the parameter  $\beta_0$  a small value, i.e.  $\beta_0 = 2$ . With the large-order behavior (45) of the basis functions  $I_n(g_B)$ , we find the resummed functions  $f_L(g_B)$  of  $L$ th order  $\sum_{n=0}^L h_n I_n(g_B)$  to have a large-order behavior (6) with a prefactor

$$\gamma_L = \frac{e^\rho \Gamma(\beta_0)}{\sqrt{2\pi} \Gamma(\delta)} \sum_{k=0}^L (-1)^k h_k. \quad (70)$$

The values of these sums for increasing  $L$  are shown in Fig. 1. They converge exponentially fast against the exact limiting value

$$\gamma = \sqrt{\frac{6}{\pi^3}}, \quad (71)$$

with superimposed oscillations. The oscillations are of the same kind as those observed in variational perturbation theory for the convergence of the approximations to the strong-coupling coefficients  $b_k$  (see Figs. 5.19 and 5.20 in Ref. [5]) Also here, the strong-coupling coefficients  $b_k^L$  converge exponentially fast towards  $b_k$ , but with a larger power of  $L$  in the exponent of the last term  $\approx e^{-\text{const} \times \sqrt{L}}$  [see Eq. (51)], rather than  $\approx e^{-\text{const} \times L^{1/3}}$  for variational perturbation theory [see Eq. (5.199) in Ref. [5]]. This is seen on the right-hand side of Fig. 1.

We have applied our resummation method to the first 10 strong-coupling coefficients using the expansion coefficients  $f_k$  up to order 70. The results are shown in Table 1. Comparison with a similar table in Refs. [18, 5] shows that the new resummation method yields in 70th order the same accuracy as variational perturbation theory did in 251st order. In all cases the optimal parameter  $\rho$  turns out to be a slowly growing function with  $L$ .

$n$	$b_n$
0	0.667 986 259 155 777 108 270 962 02
1	0.143 668 783 380 864 910 020 319
2	-0.008 627 565 680 802 279 127 963
3	0.000 818 208 905 756 349 542 41
4	-0.000 082 429 217 130 077 219 91
5	0.000 008 069 494 235 040 964 75
6	-0.000 000 727 977 005 945 772 63
7	0.000 000 056 145 997 222 351 17
8	-0.000 000 002 949 562 732 709 36
9	-0.000 000 000 064 215 331 956 97
10	0.000 000 000 048 214 263 789 07

Table 1: Strong-coupling coefficients  $b_n$  of the 70-th order approximants  $E_{70}^0(g) = \sum_{n=0}^{70} h_n I_n(g)$  to the ground state energy  $E^0(g)$  of the anharmonic oscillator. They have the same accuracy as the variational perturbation-theoretic calculations up to order 251 in Refs. [18, 5].

In the strong-coupling regime, the convergence is fastest by choosing for  $\beta_0$  an  $L$ -dependent value

$$\beta_0 = L. \quad (72)$$

Note that this choice of  $\beta_0$  ruins the convergence to the imaginary part for small negative  $g_B$  which was resummed best with  $\beta_0 = 2$ .

## 4.6 Resummation of Critical Exponents

Having convinced ourselves of the fast convergence of our new resummation method, let us now turn to the perturbation expansions of the  $O(N)$ -symmetric  $\phi^4$ -theories in powers of the bare coupling constant  $\bar{g}_B$ , defined by the euclidean action

$$\mathcal{A} = \int d^D x \left\{ \frac{1}{2} [\partial \phi_0(x)]^2 + \frac{1}{2} m_0^2 \phi_0^2(x) + 2\pi \bar{g}_B [\phi_0^2(x)]^2 \right\} \quad (73)$$

in  $D = 3$  dimensions. The field  $\phi_0$  is an  $N$ -component vector  $\phi_0 = (\phi_0^1, \phi_0^2, \dots, \phi_0^N)$ , and the action is  $O(N)$ -symmetric. We define renormalized mass  $m$  and field strength by parametrizing the behavior of the connected two point function  $G^{(2)}$  in momentum space near zero momentum as

$$G^{(2)}(p, \alpha; -p, \beta) = Z_\phi \frac{\delta_{\alpha\beta}}{m^2 + p^2 + \mathcal{O}(p^4)}. \quad (74)$$

The renormalized coupling constant  $g$  is defined by the value of the connected four-point function at zero momenta:

$$G^{(4)}(0, \alpha; 0, \beta; 0, \gamma; 0, \delta) = m^{-4-D} Z_\phi^2 g (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}). \quad (75)$$

If we introduce the dimensionless bare coupling constant  $g_B \equiv \bar{g}_B/m$ , the critical exponents are defined by

$$\begin{aligned}\eta(g_B) &= g_B \frac{d}{dg_B} \log Z_\phi, \\ 2 - \nu(g_B)^{-1} &= g_B \frac{d}{dg_B} \log \frac{m_0^2}{m^2}.\end{aligned}\quad (76)$$

The following expansions for the critical indices in the bare dimensionless coupling constant are available [19] in the literature for all  $O(N)$ :

$$\begin{aligned}\eta(g_B) &= (16/27 + 8N/27)g_B^2 + (-9.086537459 - 5.679085912N - 0.5679085912N^2)g_B^3 \\ &+ (127.4916153 + 94.77320534N + 17.1347755N^2 + 0.8105383221N^3)g_B^4 \\ &+ (-1843.49199 - 1576.46676N - 395.2678358N^2 - 36.00660242N^3 \\ &\quad - 1.026437849N^4)g_B^5 \\ &+ (28108.60398 + 26995.87962N + 8461.481806N^2 + 1116.246863N^3 \\ &\quad + 62.8879068N^4 + 1.218861532N^5)g_B^6,\end{aligned}\quad (77)$$

$$\begin{aligned}2 - \nu^{-1}(g_B) &= g_B(2 + N) + (523/27 + 316N/27 + N^2)g_B^2 \\ &+ (229.3744544 + 162.8474234N + 26.08009809N^2 + N^3)g_B^3 \\ &+ (-3090996037 - 2520.848751N - 572.3282893N^2 - 44.32646141N^3 - N^4)g_B^4 \\ &+ (45970.71839 + 42170.32707N + 12152.70675N^2 + 1408.064008N^3 \\ &\quad + 65.97630108N^4 + N^5)g_B^5 \\ &+ (-740843.1985 - 751333.064N - 258945.0037N^2 - 39575.57037N^3 \\ &\quad - 2842.8966N^4 - 90.7145582N^5 - N^6)g_B^6.\end{aligned}\quad (78)$$

In addition, seventh order coefficients have been calculated for  $N = 0, 1, 2, 3$ : [1]:

$$\eta^{(\tau)} = \left\{ \begin{array}{c} -45387.48927 \\ -114574.4876 \\ -241424.7646 \\ -454761.4731 \end{array} \right\} g_B^7, \quad \nu^{-1(\tau)} = \left\{ \begin{array}{c} -12792269.773 \\ -33711416.972 \\ -73780809.849 \\ -143831857.01 \end{array} \right\} g_B^7 \quad \text{for} \quad \left\{ \begin{array}{c} N = 0 \\ N = 1 \\ N = 2 \\ N = 3 \end{array} \right\}.\quad (79)$$

When approaching the critical point, the renormalized mass  $m$  tends to zero, so that the problem is to find the strong-coupling limit of these expansions. In order to have the critical exponents approach a constant value, the power  $s$  in Eq. (3) must be set equal to zero.

In contrast to the quantum-mechanical discussion in the last section, the exponent  $\omega$  governing the approach to the scaling limit is now unknown, and must also be determined from the available

perturbation expansions. As in Ref. [8, 9], we solve this problem by using the fact that the existence of a critical point implies the renormalized coupling constant  $g$  in powers of  $g_B$  to converge against a constant renormalized coupling  $g^*$  for  $m \rightarrow 0$ . The expansion of  $g(g_B)$  is known up to six loops [19] for all  $O(N)$ :

$$\begin{aligned}
g(g_B) = & g_B + (-8 - N)g_B^2 + (2108/27 + 514/27N + N^2)g_B^3 \\
& + (-878.7937193 - 312.63444671N - 32.54841303N^2 - N^3)g_B^4 \\
& + (11068.06183 + 5100.403285N + 786.3665699N^2 + 48.21386744N^3 + N^4)g_B^5 \\
& + (-153102.85023 - 85611.91996N - 17317.7025N^2 - 1585.114189N^3 \\
& \quad - 65.82036203N^4 - N^5)g_B^6 \\
& + (2297647.148 + 1495703.313N + 371103.0896N^2 + 44914.04818N^3 \\
& \quad + 2797.291579N^4 + 85.21310501N^5 + N^6)g_B^7 .
\end{aligned} \tag{80}$$

The convergence against a fixed coupling  $g^*$  occurs only for the correct value of  $\omega$  in the resummation functions  $I_n(g_B, \omega, s, \rho, \sigma, \delta, \beta_0)$ . At different values,  $g(g_B)$  has some strong-coupling power behavior  $g_B^s$  with  $s \neq 0$ . We may therefore determine  $\omega$  by forming from (80) a series for the power  $s$ ,

$$s = \frac{d \log g(g_B)}{d \log g_B} = \frac{g_B}{g} g'(g_B), \tag{81}$$

resumming this for various values of  $\omega$  in the basis functions, and finding the critical exponent  $\omega$  from the zero of  $s$ . Alternatively, since  $g(g_B) \rightarrow g^*$ , we can just as well resum the series for  $-gs$ , which coincides with the  $\beta$ -function of renormalization group theory [not to be confused with the growth parameter  $\beta$  in (6)]

$$\beta(g_B) \equiv -g_B \frac{dg(g_B)}{dg_B}. \tag{82}$$

If we denote its strong-coupling limit by  $\beta^*$ ,

$$\beta^* \equiv \beta(g_B)|_{g_B \rightarrow \infty}, \tag{83}$$

we resum the expansion for  $\beta(g_B)$  to form the approximations

$$\beta_L(g_B) = \sum_{n=0}^L h_n I_n(g_B, \omega), \tag{84}$$

and plot the strong-coupling limits of the  $L$ th approximations  $\beta_L^*$  for various values of  $\omega$ . This is shown in Fig. 2. From these plots we extract the critical exponent  $\omega$  by finding the  $\omega$ -value for which the approximations  $\beta_L^*$  extrapolate best to zero for  $L \rightarrow \infty$ , taking into account that the convergence is exponentially fast with superimposed oscillations. These  $\omega$ -values are called  $\omega_L$ .

For these resummations, we must of course specify the remaining parameters in the basis functions  $I_n(g_B, \omega, 0, \rho, \sigma, \delta, \beta_0)$ . This can, in principle, proceed as in the case of the anharmonic oscillator. The parameter  $\alpha$  is determined from the action of a classical instanton solution  $\phi_c(x)$  of the field equations, and has for all expansions the  $N$ -independent value [11]

$$\alpha = \frac{32\pi}{I_4} = 1.32997, \quad (85)$$

where  $I_p \equiv \int d^D x [\phi_c(x)]^p$  are integrals over powers of  $\phi_c(x)$ .

To determine the parameter  $\delta$ , we recall the remaining growth parameters  $\beta$  and  $\gamma$  of the large-order behavior (6) of the perturbative series for the critical exponents. The growth parameter  $\beta$  is given by the number of zero modes in the fluctuation spectrum around this classical solution:

$$\begin{Bmatrix} \beta_\eta \\ \beta_{\nu-1} \\ \beta_\beta \end{Bmatrix} = \begin{Bmatrix} 3 + N/2 \\ 4 + N/2 \\ 4 + N/2 \end{Bmatrix} \quad (86)$$

The prefactors  $\gamma$  in (6) requires the calculation of the fluctuation determinants around the classical solution, which yields in the case of the  $\beta$ -function

$\gamma_\beta = \frac{2^{N/2+2} 3^{-3/2} \pi^{-2}}{\Gamma(N/2+2)} \left(\frac{I_1^2}{I_4}\right)^2 \left(\frac{I_6}{I_4} - 1\right)^{3/2} D_L^{-1/2} D_T^{-(N-1)/2}$ . (87) where  $D_L$  and  $D_T$  are characteristic quantities of the longitudinal and transverse parts of the fluctuation determinant, respectively. Their numerical values are [11]

$D_L$	$D_T$	$I_1$	$I_4$	$I_6$	$H_3$
$10.544 \pm 0.004$	$1.4571 \pm 0.0001$	31.691522	75.589005	659.868352	13.563312

(88)

The parameters  $\gamma_\eta, \gamma_{\nu-1}$  are obtained from  $\gamma_\beta$  by:

$$\gamma_\eta = \gamma_\beta \frac{2H_3}{I_1 D(4-D)}, \quad \gamma_{\nu-1} = \gamma_\beta \frac{N+2}{N+8} (D-1) 4\pi \frac{I_2}{I_1^2}, \quad (89)$$

where  $I_2 = (1 - D/4)I_4$  and  $H_3$  is listed in (88). Note that the expansions in powers of the renormalized coupling constant  $g$  have the same parameters  $\alpha$  and  $\beta$ , but different parameters  $\gamma_R$ . These differ from the above  $\alpha$ 's by a common factor:

$$\gamma_R = \gamma e^{-(N+8)/\alpha}. \quad (90)$$

From Eq. (85), the parameter  $\sigma$  is found using relation (52). It turns out, however, that this value does not lead to an optimal convergence. This can be understood qualitatively by observing that the large-order behavior of the expansion coefficients of the critical exponents and of the  $\beta$ -function

in powers of the bare coupling constant  $g_B$  is not nearly as precocious in reaching the large-order form (6) as the corresponding expansions in powers of the renormalized coupling constant  $g$  (see Fig. 1 in Ref. [9]). The lack of precocity here is illustrated for the expansion coefficients  $\beta_k$  of the  $\beta$ -function in Table 2, which gives the ratios of  $\beta_k$  and their leading asymptotic estimates  $\beta_k^{\text{as}}$ :

$N$	0	1	2	3
$k$	$\beta_k/\beta_k^{\text{as}}$	$\beta_k/\beta_k^{\text{as}}$	$\beta_k/\beta_k^{\text{as}}$	$\beta_k/\beta_k^{\text{as}}$
2	0.57	0.45	0.35	0.27
3	0.61	0.45	0.32	0.22
4	0.73	0.51	0.34	0.22
5	0.89	0.61	0.40	0.25
6	1.07	0.73	0.47	0.29
7	1.26	0.88	0.56	0.34
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\gamma_\beta$	110.0	97.0	75.5	53.2

Table 2: First six perturbative coefficients in the expansions of the  $\beta$ -function in powers of the bare coupling constant  $g_B$ , divided by their asymptotic large-order estimates  $(-\alpha)^k k! k^{\beta}$ . The ratios increase quite slowly towards the theoretically predicted normalization constant  $\gamma_\beta$  in the asymptotic regime.

$$\beta_k/\beta_k^{\text{as}} \equiv \beta_k/k!(-\alpha)^k k^\beta. \quad (91)$$

The first six approach their large-order limits quite slowly. For this reason we prefer to adapt  $\sigma$  not from  $\alpha$  by Eq. (52), but by an optimization of the convergence. Since the reexpanded series converges for fixed values of  $\delta$  and  $\sigma$  it is reasonable to determine these parameters by searching for a point of least dependence in largest available order  $L$ . This is done by imposing the conditions

$$\frac{d\kappa_L}{d\sigma} = 0 \quad \text{and} \quad \frac{d^2\kappa_L}{d\sigma^2} = 0 \quad (92)$$

to determine *both* parameters  $\delta$ ,  $\sigma$ , where  $\kappa_L$  denotes the  $L$ th approximation to any exponent  $\gamma, \nu$  or  $\eta$ . In accordance with the discussions in section 2.1.2 this procedure provides a value of  $\sigma$  which is smaller than that given by (52).

After trying out a few choices, we have given the parameters  $\beta$  and  $\rho$  the fixed values 1 and 10, respectively, to accelerate the convergence.

The results for the critical exponents of all  $O(N)$ -symmetries are shown in Figs. 2–6 and Table 3.

The total error is indicated in the square brackets. It is deduced from the error of resummation of the critical exponent at a fixed value of  $\omega$  indicated in the parentheses, and from the error  $\Delta\omega$  of

$\omega$ , using the derivative of the exponent with respect to  $\omega$  given in curly brackets. Symbolically, the relation between these errors is

$$[\dots] = (\dots) + \Delta\omega\{\dots\}. \quad (93)$$

The accuracy of our results can be judged by comparison with the most accurately measured critical exponent  $\alpha$  parametrizing the divergence of the specific heat of superfluid helium at the  $\lambda$ -transition by  $|T_c - T|^{-\alpha}$ . By going into a vicinity of the critical temperature with  $\Delta T \approx 10^{-8}$  K, a recent satellite experiment has provided us with the value [20]

$$\alpha = -0.01056 \pm 0.00038. \quad (94)$$

Our value for  $\nu$  in Table 3 is

$$\nu = -0.6704 \pm 0.007 \quad (95)$$

and yields via the scaling relation  $\alpha = 2 - 3\nu$ :

$$\alpha = -0.0112 \pm 0.0021, \quad (96)$$

in excellent agreement with the experimental number (94). A comparison with other experiments and theories is shown in Fig. 7, showing that our result is among the more accurate ones.

A remark is necessary concerning the errors quoted in this paper. We do not know how to estimating precisely the errors which can appear in an involved numerical approximation scheme such as the one presented here. Our estimates are based on the range of critical exponents which can be reached by reasonably modifying the parameters in the calculations. What may be considered as reasonable is a somewhat subjective procedure. As such, our error estimates follow the rule of maximal optimism, and are probably underestimated. This is, however, not uncommon in resummations of divergent power series of critical exponents.

$n$	$\gamma$	$\eta$	$\nu$	$\omega$
0	1.1604[8] (4) {0.075}	0.0285[6] (4) {0.037}	0.5881[8] (4) {0.075}	0.803[3] {1}
1	1.2403[8] (4) {0.110}	0.0335[6] (3) {0.043}	0.6303[8] (4) {0.065}	0.792[3] {1}
2	1.3164[8] (5) {0.033}	0.0349[8] (5) {0.042}	0.6704[7] (4) {0.098}	0.784[3] {1}
3	1.3882[10] (7) {0.210}	0.0350[8] (5) {0.043}	0.7062[7] (4) {0.110}	0.783[3] {1}

Table 3: Critical exponents of the  $O(N)$ -symmetric  $\phi^4$ -theory from our new resummation method. The numbers in square brackets indicate the total errors. They arise from the error of the resummation at fixed values of  $\omega$  indicated in parentheses, and the errors coming from the inaccurate knowledge of  $\omega$ . The former are estimated from the scattering of the approximants around the graphically determined large- $L$  limit, the latter follow from the errors in  $\omega$  and the derivatives of the critical exponents with respect to changes of  $\omega$  indicated in the curly brackets.

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[5] For a detailed description of this theory in quantum mechanics see the textbook H. Kleinert, *Path Integrals in Quantum Mechanics Statistics and Polymer Physics*, World Scientific, Singapore 1995 (<http://www.physik.fu-berlin.de/~kleinert/b3>). The extension to quantum field theory is described in Chapter 19 of the textbook H. Kleinert and V. Schulte-Frohlinde, *Critical Properties of  $\phi^4$ -Theories*, World Scientific, Singapore 2000. (<http://www.physik.fu-berlin.de/~kleinert/b8>).

There were two main predecessors to variational perturbation theory coming from two different directions. From the mathematical side, the seminal paper was

R. Seznec and J. Zinn-Justin, J. Math. Phys. **20**, 1398 (1979).

From the physical side, inspiration came from

R.P Feynman and H. Kleinert, Phys. Rev. A **34**, 5080 (1986) (<http://www.physik.fu-berlin.de/~kleinert/159>),

and its systematic extension in

H. Kleinert, Phys. Lett. A **173**, 332 (1993) (quant-ph/9511020). For the contributions of numerous other authors see Notes and References of Chapters 5 and 17 in the first textbook. A small excerpt is given below.

The crucial breakthrough which opened up the previous quantum mechanical variational approaches to quantum field theory came in three steps. First, still in quantum mechanics, by exploiting previously unused even approximants which do not have an extremum, as explained in Chapter 5 of the textbook. For applications to quantum field theory, two more ingredients were important, as pointed out in Refs. [8] and [9]: the determination of the exponent  $\omega$  by the leading power behavior in the strong-coupling limit, and an extrapolation procedure to infinite order on the basis of the theoretically known analytic order dependence. These developments

were essential in obtaining accurate critical exponents rivaling the powerful combination of renormalization group and Borel-type resummation methods. Variational perturbation theory also yields directly  $\epsilon$ -expansions for the critical exponents without the renormalization group formalism, as shown in Ref. [10].

A selected list of important contributions is

T. Barnes and G.I. Ghandour, Phys. Rev. D **22**, 924 (1980);

B.S. Shaverdyan and A.G. Ushveridze, Phys. Lett. B **123**, 316 (1983);

P.M. Stevenson, Phys. Rev. D **30**, 1712 (1985);

D **32**, 1389 (1985);

P.M. Stevenson and R. Tarrach, Phys. Lett. B **176**, 436 (1986);

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W. Namgung, P.M. Stevenson, and J.F. Reed, Z. Phys. C **45**, 47 (1989);

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M.H. Thoma, Z. Phys. C **44**, 343 (1991);

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R. Tarrach, Phys. Lett. B **262**, 294 (1991);

H. Haugerud and F. Raunda, Phys. Rev. D **43**, 2736 (1991);

A.N. Sissakian, I.L. Solivtosv, and O.Y. Sheychenko, Phys. Lett. B **313**, 367 (1993);

A. Duncan and H.F. Jones, Phys. Rev. D **47**, 2560 (1993);

For the anharmonic oscillator, the highest accuracy in the strong-coupling limit was reached with exponentially fast convergence in Ref. [18]. That paper contains references to earlier less accurate calculations of strong-coupling expansion coefficients from weak-coupling perturbation theory, in particular

F.M. Fernández and R. Guardiola, J. Phys. A **26**, 7169 (1993);

F.M. Fernández, Phys. Lett. A **166**, 173 (1992);

R. Guardiola, M.A. Solís, and J. Ros, Nuovo Cimento B **107**, 713 (1992). A.V. Turbiner and A.G. Ushveridze, J. Math. Phys. **29**, 2053 (1988);

B. Bonnier, M. Hontebeyrie, and E.H. Ticemba, J. Math. Phys. **26**, 3048 (1985);

Those works were unable to extract the exponential law of convergence from their data. This was shown to be related to the convergence radius of the strong-coupling expansion by

H. Kleinert and W. Janke, Phys. Lett. A **206**, 283 (1995) (quant-ph/9509005);  
and in the last paper in Ref. [7].

Predecessors of these works which did not yet explain the exponentially fast convergence in the strong-couplings limit are

I.R.C. Buckley, A. Duncan, and H.F. Jones, Phys. Rev. D **47**, 2554 (1993);

C.M. Bender, A. Duncan, and H.F. Jones, Phys. Rev. D **49**, 4219 (1994);

A. Duncan and H.F. Jones, Phys. Rev. D **47**, 2560 (1993);

C. Arvanitis, H.F. Jones, and C.S. Parker, Phys. Rev. D **52**, 3704 (1995) (hep-th/9502386);

R. Guida, K. Konishi, and H. Suzuki, Annals Phys. **241**, 152 (1995) (hep-th/9407027).

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*methods.* These comprise transformation  $B(y) = \sum f_k y^k / \mu_k$ , where the  $\mu_k$  are given by a Stieltjes integral  $\mu_k = \int_0^\infty x^k d\chi(x)$  and  $\chi$  is a bounded and increasing function of  $x$  guaranteeing the convergence of the Stieltjes integral. This definition includes our transformation for the somewhat complicated choice  $d\chi(x) = (\Gamma(\beta_0)/2\pi i)x^{-s-1} \oint_C dt e^{t+x^\omega t^{1-\omega}} t^{s(1-1/\omega)-\beta_0} dx$ .

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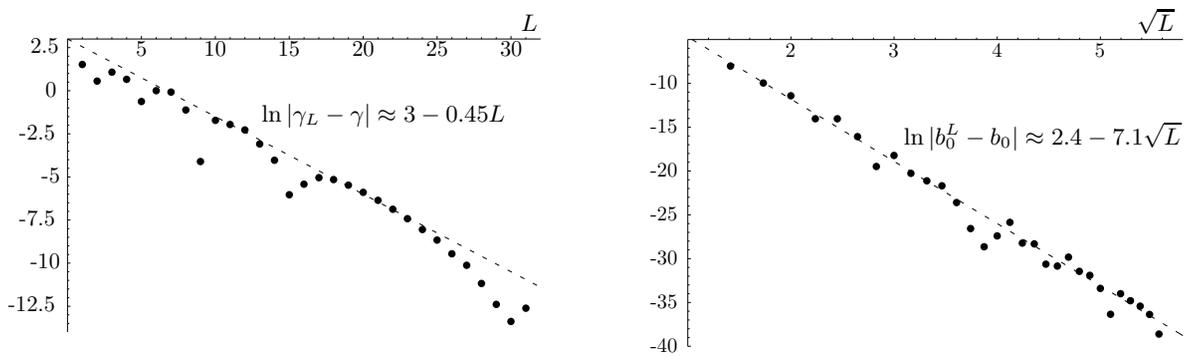


Figure 1: Logarithmic plot of the convergence behavior of the successive approximations to the prefactor  $\gamma^L$  in the large-order behavior (87), and of the leading strong coupling coefficient  $b_0^L$ .

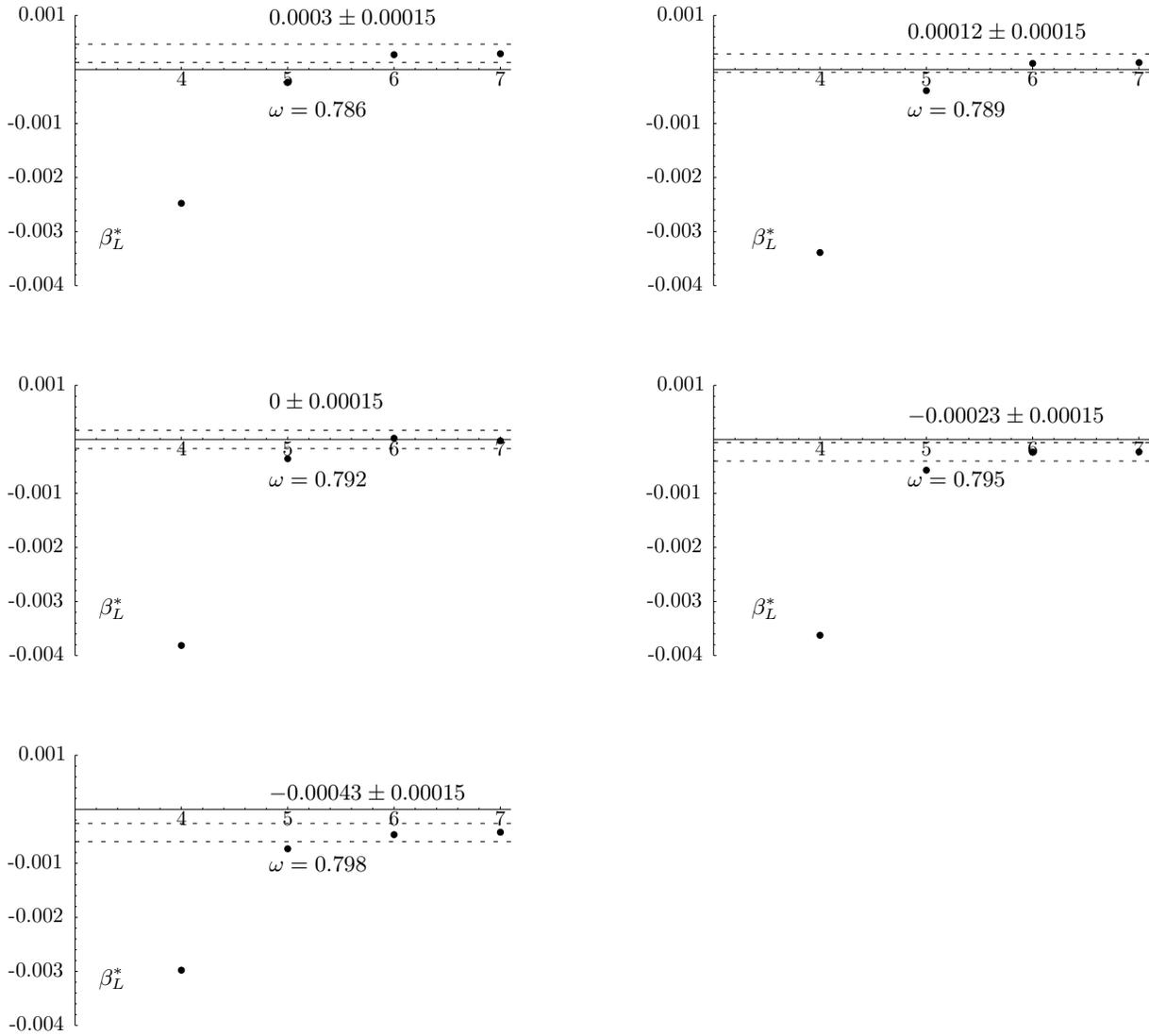


Figure 2: Convergence of strong-coupling limits of the  $\beta$ -function (82) for  $N = 1$  and different values of  $\omega$ . The upper and lower dashed lines denote the range of the  $L \rightarrow \infty$  limit of  $\beta_L^*$  from which the value of  $\omega$  is deduced in Fig. 3.

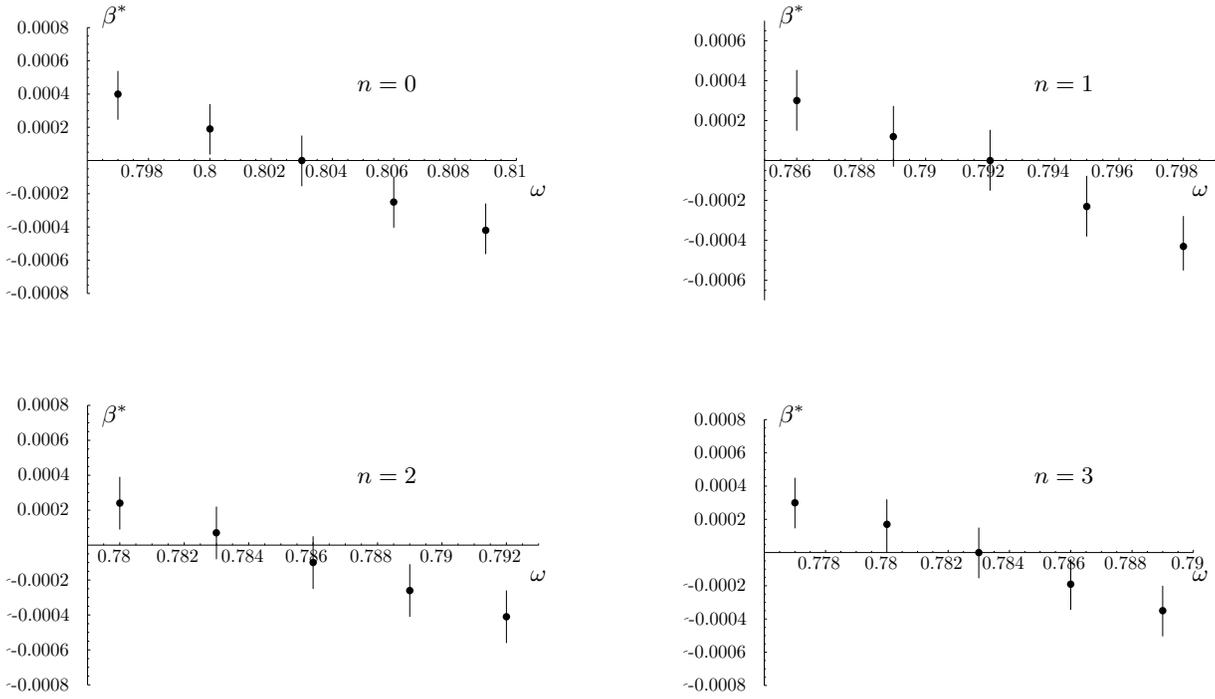


Figure 3: Plot of resummed values of  $\beta^*$  against  $\omega$ . The true value of  $\omega$  is deduced from the condition  $\beta^* = 0$  and the errors are determined from the range of  $\omega$  where the error bars from the resummation of  $\beta^*$  intersect with the  $x$ -axis.

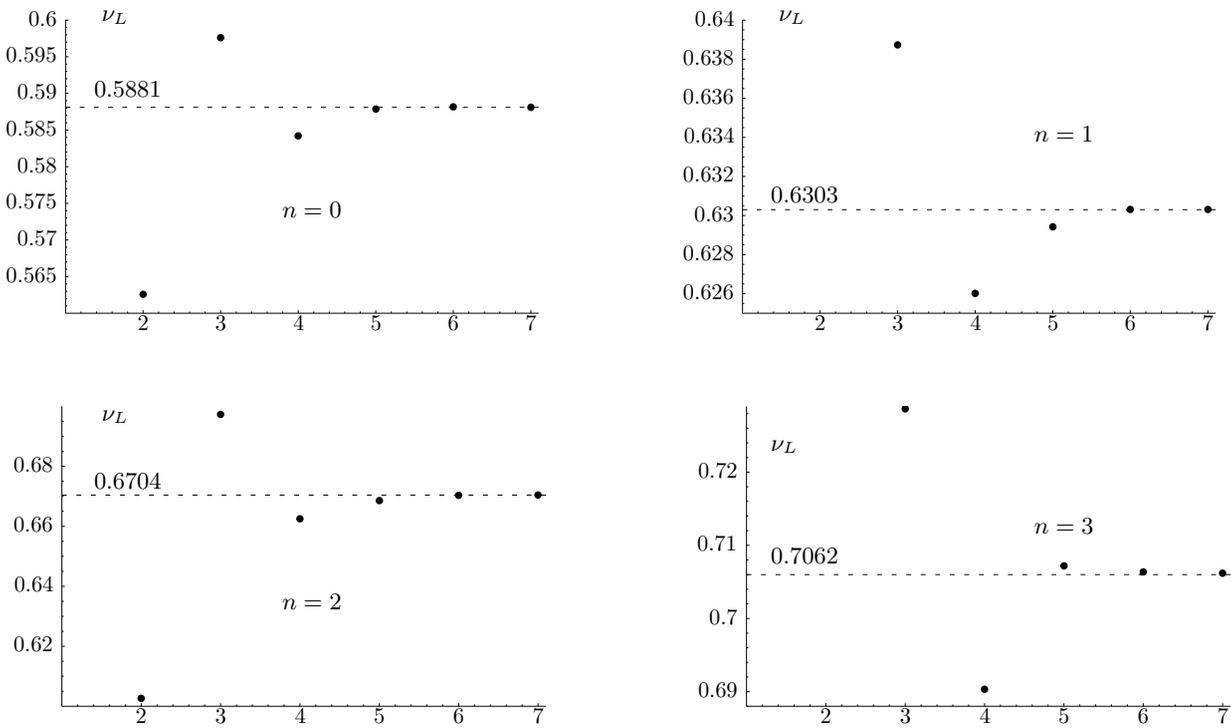


Figure 4: Convergence of the approximations  $\nu_L$  to the critical exponent  $\nu$  for different values of  $N$ .

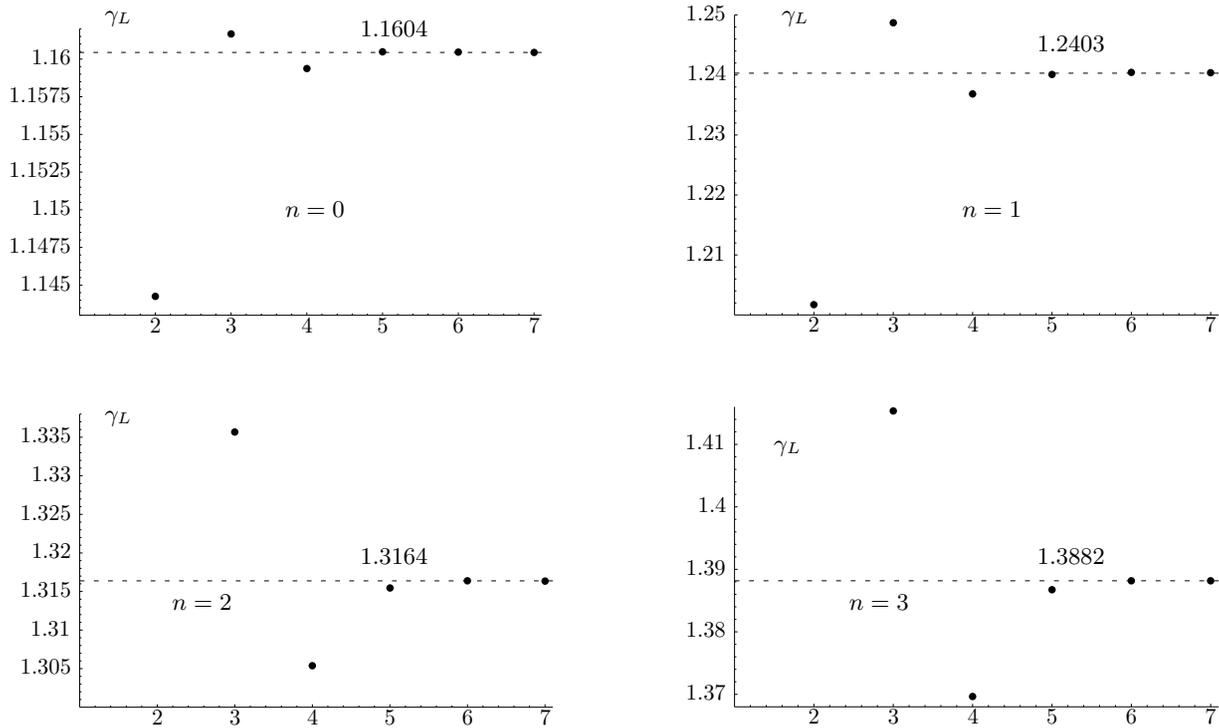


Figure 5: Convergence of the approximations  $\gamma_L$  to the critical exponent  $\gamma$  for different values of  $N$ .

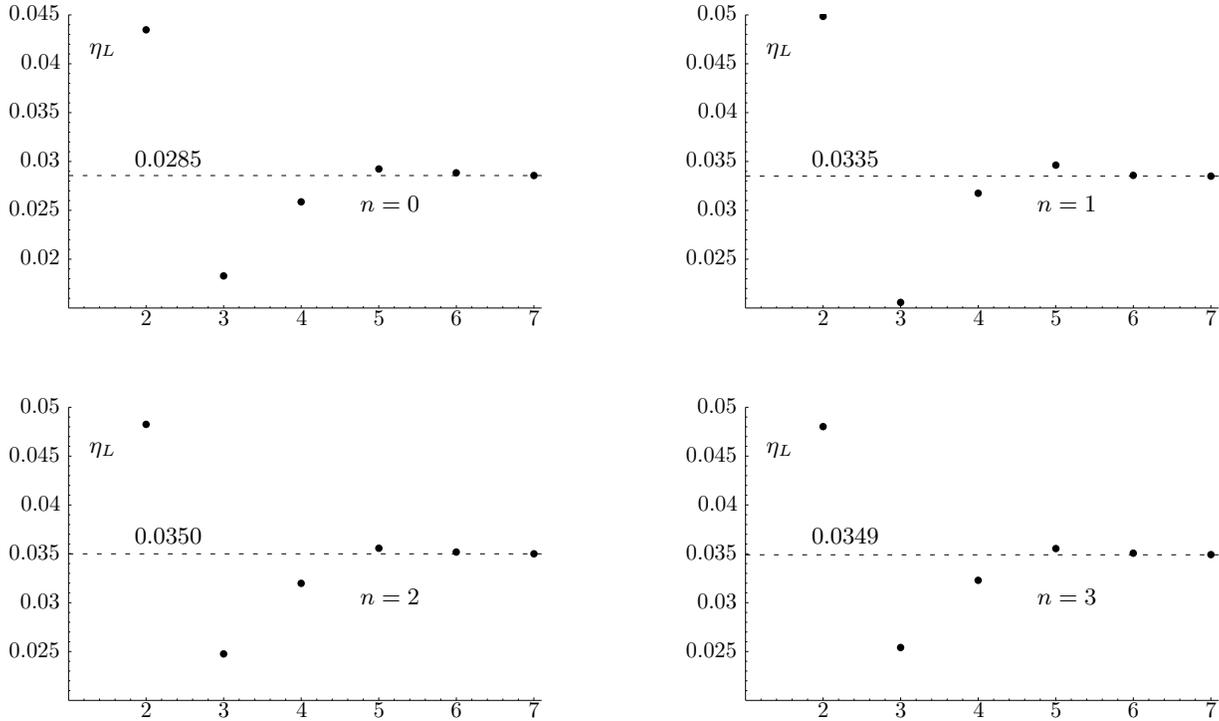


Figure 6: Convergence of the approximations  $\eta_L$  to the critical exponent  $\eta$  for different values of  $N$ .

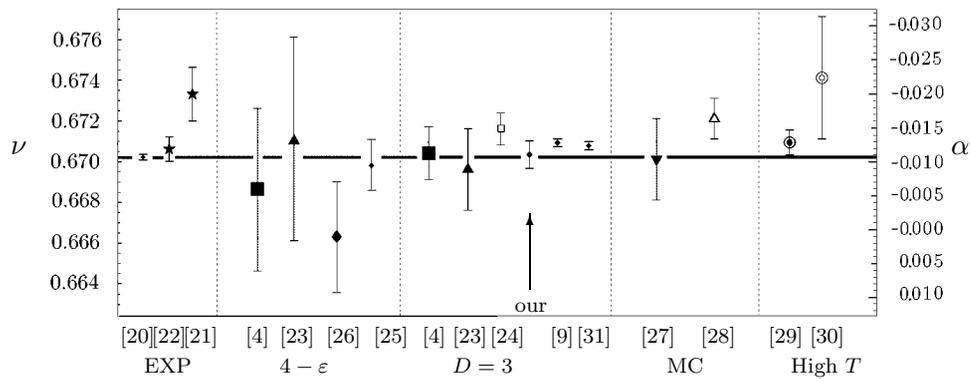


Figure 7: Comparison of our result (96) for critical exponents  $\alpha$  of superfluid helium with experiments and other theories.