BACKWARD DISPERSION RELATIONS FOR $\pi N \to \pi\Delta$ SCATTERING AND $N\Delta\gamma$ COUPLINGS

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Received 15 March 1971

Using unsubtracted dispersion relations for the invariant amplitudes of the process $\pi N \to \pi\Delta$ in forward and backward direction we derive sum rules relating t channel to s and u channel resonances. We saturate these sum rules by ρ, N, and Δ and obtain for the ρ coupling constants

$$C_3 m = \frac{5}{\sqrt{3}}; \qquad C_4 m^2 = C_5 m^2 = \frac{5}{4\sqrt{3}}$$

where m is the average mass of N and Δ. Assuming vector meson dominance of the electromagnetic vertex $N\Delta\gamma$ these become the values of the Gourdin-Salin coupling constants in reasonable agreement with experiment.

The assumption that the t channel cut of backward dispersion relations of pion nucleon scattering can be dominated by a ρ meson for $I_t = 1$ and that the coupling constants of ρ can be related to the electromagnetic couplings of the nucleons via the vector meson dominance hypothesis has led to three successful sum rules for κ^V, the anomalous isovector magnetic moment [1]. These sum rules emerge by equating the value of the amplitudes $A^{(-)}$, $B^{(-)}$ and $A'^{(-)}$ (the t channel helicity flip amplitude) at threshold obtained from a dispersion relation at $\theta_s = \pi$ with the value obtained from a forward dispersion relation and saturating the integrals with sharp resonances. We have found that a similar procedure applied to the corresponding invariant amplitudes of $\pi N - \pi\Delta$ scattering leads to good sum rules for the $N\Delta\gamma$ vertex. The kinematics of these sum rules are, in general, quite involved. The calculation becomes, however, very simple if one works at zero pion mass, is content with a saturation by means of N and Δ only, and assume, moreover, that the masses of these particles are degenerate.

Let the invariant amplitudes* be defined by

$$\langle \pi^b(q')\,\Delta(p')\,|\,T\,|\,\pi^a(q)\,N(p)\rangle \equiv \bar{U}^\mu(p')[(A_1^{ba} + \not{Q}B_1^{ba})\,Q_\mu + (A_2^{ba} + \not{Q}B_2^{ba})K^\mu]u(p) \tag{1}$$

with

$$Q \equiv \tfrac{1}{2}(q+q') \qquad K \equiv \tfrac{1}{2}(q-q') \tag{2}$$

and

$$A_1^{ba} = A_1^{(-)} I_{(-)}^{ba} + A_1^{(+)} I_{(+)}^{ba} \tag{3}$$

etc., where $I_{(+)}$ and $I_{(-)}$ are the isospin matrices of $I_t = 1$ and $I_t = 2$ [2].
Then we find that A and B can be expressed in terms of the s channel helicity amplitudes

$$\hat{T}^{(s)}_{\lambda'\lambda} \equiv \cos^{-|\lambda'+\lambda|}\tfrac{1}{2}\theta_s \,\sin^{-|\lambda'-\lambda|}\tfrac{1}{2}\theta_s \; T^{(s)}_{\lambda',\lambda} \tag{4}$$

in the form

$$\begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} = \mathcal{M}(s,\theta_s) \begin{pmatrix} \hat{T}^{(s)}_{3/2\;1/2} \\ \sqrt{3}\;\hat{T}^{(s)}_{1/2\;1/2} \\ \sqrt{3}\;\hat{T}^{(s)}_{-1/2\;1/2} \\ \hat{T}^{(s)}_{-3/2\;1/2} \end{pmatrix} \tag{5}$$

where $\mathcal{M}$ is given in forward direction by $(W \equiv \sqrt{s})$

* Our normalization is $\langle \pi(\mathbf{q}')\,|\,\pi(\mathbf{q})\rangle = 2q_0(2\pi)^3\delta^3(\mathbf{q}'-\mathbf{q})$ and $\langle N(\mathbf{p}')\,|\,N(\mathbf{p})\rangle = \frac{p_0}{m}(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})$ while T is defined by $S \equiv 1 - i(2\pi)^4\delta^{(4)}(P - P_1)$

$$\mathcal{M}(s,0) = -\frac{2\sqrt{2}mW}{(s-m^2)^2}\begin{Bmatrix} m & 0 & m & 0 \\ -m & 0 & -m & -2W \\ 0 & \frac{m}{W} & 0 & 0 \\ 2 & -\frac{m}{W} & 0 & 0 \end{Bmatrix}$$

while for backward angle $\theta_s = \pi$ we find

$$\mathcal{M}(s,\pi) = -\frac{2\sqrt{2}mW}{(s-m^2)^2}\begin{Bmatrix} 0 & 0 & m & -\frac{s+m^2}{W} \\ 0 & 0 & -m & -\frac{s-m^2}{W} \\ \frac{s+m^2}{s} & \frac{m}{W} & 0 & \frac{m}{W} \\ \frac{s-m^2}{s} & -\frac{m}{W} & 0 & -\frac{m}{W} \end{Bmatrix}$$

Here m has to be taken as the average mass of nucleon and Δ resonance.

Consider the $I_t = 1$ amplitudes. From the definition (1) we see that $A_1^{(-)}$, $B_2^{(-)}$ are even functions in $s-u$ while $A_2^{(-)}$ and $B_1^{(-)}$ are odd. As usual, we divide the factor $2/(s-u)$ out of $A_2^{(-)}$ and $B_1^{(-)}$ in order to obtain functions of $(s-u)^2$ and t only. Applying the Regge pole hypothesis we find that

$$A_1^{(-)},\ \frac{2}{s-u}A_2^{(-)},\ \frac{2}{s-u}B_1^{(-)},\ B_2^{(-)} \tag{8}$$

behave asymptotically as

$$s^{\alpha_\rho - 1},\ s^{\alpha_\rho - 1},\ s^{\alpha_\rho - 3},\ s^{\alpha_\rho - 1}$$

in forward direction where α_ρ is the intercept of the ρ trajectory (≈ 0.5). Hence, we can write an unsubtracted dispersion relation in s for the threshold point $s = m^2$, $t = 0$**.

In the backward direction the same amplitudes have only dynamic singularities in the variable t. To see this, we just note that $(s-u)^2 = t(t-4m^2)$, such that no kinematic singularity comes from the variable $(s-u)^2$. The baryon trajectory with $\alpha_B \approx 0.19$ governs the asymptotic behaviour of $A_1^{(-)} - A_2^{(-)}$, $B_1^{(-)} - B_2^{(-)}$, $A_1^{(-)} + A_2^{(-)}$ and $B_1^{(-)} + B_2^{(-)}$ by

$$t^{\alpha_B - 3/2},\ t^{\alpha_B - 3/2},\ t^{\alpha_B - 1/2},\ t^{\alpha_B - 1/2}$$

Hence, another set of unsubtracted dispersion relations can be written in the variable t for the threshold point $t = 0$.

Let us calculate the imaginary parts of $\hat{T}^{(s)}_{\lambda'\lambda}$ at a baryon pole. We find it most convenient to use PCAC and state the result in terms of the collinear invariant matrix elements $\chi_a(\lambda)$ of the axial current at $q^2 = (p'-p)^2 = 0$ between baryons of helicity λ which satisfy SU(2)×SU(2) algebra [3]

$$[\chi_b(\lambda),\ \chi_a(\lambda)] = i\,\Sigma_{bac}\,T_c \tag{9}$$

One finds for a baryon resonance of mass m_γ and spin J

$$\mathrm{Im}\,\hat{T}^{(s)}_{\lambda'\lambda} = -\frac{\pi}{4mF_\pi^2}\,\delta(s-m_\gamma^2)(m^2-m_\gamma^2)^2\,[\chi_b(\lambda)_{\beta\gamma}\,\chi_a(\lambda)_{\gamma\alpha}]\,\hat{d}^J_{\lambda\lambda'}(\theta_s) \tag{10}$$

where the square bracket denotes antisymmetrization with respect to the pion indices b and a and $\hat{d}_{\lambda\lambda'}$ is defined in terms of the standard $d_{\lambda\lambda'}(\theta)$ functions analogously to (4). As a consequence, the baryon resonance contributes to the unsubtracted dispersion relation of

$$\mathscr{A}^{ba} \equiv \begin{pmatrix} & A_1^{(1)} \\ \frac{2}{s-u} & A_2^{(1)} \\ \frac{2}{s-u} & B_1^{(-)} \\ & B_2^{(-)} \end{pmatrix} I^{ba}_{(-)}$$

in forward direction

$$\mathscr{A}^{ba}_\gamma(s=m^2, t=0)_f = \mathcal{M}(m_\gamma^2, 0)\begin{Bmatrix} (m_\gamma^2 - m^2)\sqrt{(J+\frac{3}{2})(J-\frac{1}{2})}[\chi_b(\frac{3}{2})\chi_a(\frac{1}{2})] \\ \sqrt{3}\,[\chi_b(\frac{1}{2}), \chi_a(\frac{1}{2})] \\ -n\sqrt{3}\,[\chi_b(\frac{1}{2}), \chi_a(\frac{1}{2})] \\ (m_\gamma^2 - m^2)n^{J+\frac{1}{2}}\sqrt{(J+\frac{3}{2})(J-\frac{1}{2})}[\chi_b(\frac{3}{2})\chi_a(\frac{1}{2})] \end{Bmatrix} \tag{11}$$

while for the backward direction it gives

$$\mathscr{A}^{ba}_\gamma(s=m^2, t=0)_b = -\mu\mathcal{M}(m_\gamma^2, \pi)\begin{Bmatrix} \frac{m_\gamma^4 - m^4}{2m_\gamma^3}\,n^{\frac{J+\frac{1}{2}}{2}}\sqrt{(J+\frac{3}{2})(J-\frac{1}{2})}[\chi_b(\frac{3}{2}), \chi_a(\frac{1}{2})] \\ -n\sqrt{3}[\chi_b(\frac{1}{2})\chi_a(\frac{1}{2})] \\ \sqrt{3}[\chi_b(\frac{1}{2})\chi_a(\frac{1}{2})] \\ \frac{m_\gamma^4 - m^4}{2m_\gamma^2}\sqrt{(J+\frac{3}{2})(J-\frac{1}{2})}[\chi_b(\frac{3}{2})\chi_a(\frac{1}{2})] \end{Bmatrix} \tag{12}$$

** In this letter, we shall not discuss the more stringent superconvergence relations obeyed by $2/(s-u)B_1^{(-)}$ and $2/(s-u)A_2^{(-)}$.

Here η is the parity of the intermediate state while n denotes the normality $n \equiv \eta\,(-)^{J-1/2}$ *. The threshold value obtained by subtracting all backward contributions (12) from all forward ones (11) should be equal to the t channel cut of the backward amplitude

$$\mathcal{A}_t^{\mathrm{ba}} = \sum_\gamma \mathcal{A}_{\gamma \mathrm{f}}^{\mathrm{ba}} - \sum_\gamma \mathcal{A}_{\gamma \mathrm{b}}^{\mathrm{ba}} \tag{13}$$

Let us assume that this cut is dominated by a ρ meson. We introduce the ρ couplings to N and Δ according to Gourdin and Salin [4]**.
Then the unsubtracted t channel contribution is given by***

$$\mathcal{A}_t^{(-)}(s=m^2, t=0) = \sqrt{6}\frac{1}{F_\pi^2}\begin{pmatrix} -2mC_3 + \frac{m_\rho^2}{2}(C_4 - C_5) \\ -(C_4 + C_5) \\ 0 \\ -2C_3 \end{pmatrix}$$

Inserting eqs. (11), (12) and (14) into eq. (13), we can write down explicitly our sum rule for the N$\Delta\gamma$ coupling constants C_3, C_4, C_5. We shall not do so here, however, but shall proceed directly to the approximation of saturating this sum rule with a nucleon and a Δ only. Then ‡

$$\mathcal{A}_t^{\mathrm{ba}}(s=m^2, t=0) = \tag{15}$$

$$\frac{1}{F_\pi^2}\frac{\sqrt{6}}{2m}\left\{\begin{pmatrix} -m \\ 0 \\ 0 \\ -1 \end{pmatrix}[\chi_{\mathrm{b}}(\tfrac{1}{2})\chi_{\mathrm{a}}(\tfrac{1}{2})]_{\mathrm{N}} + \begin{pmatrix} 5m \\ \frac{3}{2}m \\ -\frac{3}{2}m \\ 5 \end{pmatrix}[\chi_{\mathrm{b}}(\tfrac{1}{2})\chi_{\mathrm{a}}(\tfrac{1}{2})]_{\Delta}\right\}$$

Introduce the coupling constants $G_{\beta\alpha}$ of a pion to particles β and α ‡‡.

* Also we have used the property of $\chi(\lambda)$ [3].
$\chi_{\mathrm{a}}(\lambda)_{\beta\alpha} = -\eta_\rho n_\alpha \chi_{\mathrm{a}}(-\lambda)_{\beta\alpha}$

** $\langle \Delta(p')\lambda' | j_\rho^\mu | \mathrm{N}(p)\lambda\rangle \equiv 2\gamma_\rho \bar{U}^\nu(p')[C_3(qg_\nu^\mu - q_\nu\gamma^\mu) + C_4(qp'g_\nu^\mu - q_\nu p'^\mu) + C_5(qpg_\nu^\mu - q_\nu p^\mu)]U(p)$
where in our convention $2\gamma_\rho = g_{\rho\pi\pi} \approx 2.5$.

*** We have used the KSFR relation $\gamma_\rho = (m_\rho^2/8F_\pi^2)$ to eliminate γ_ρ.

‡ The $\chi_{\Delta\Delta}(3/2)$ matrix element does not occur due to the helicity condition for elastic matrix elements

$\chi_{\alpha\alpha}(\lambda)/\chi_{\alpha\alpha}(\lambda') = \lambda/\lambda'$

which follows directly from the definition (10).

‡‡ For the normalization of $G_{\beta\alpha}$ see ref. [5].

Thus we find

$$\mathcal{A}_t^{(-)} = \frac{1}{F_\pi^2}\frac{3}{2\sqrt{2}}\begin{cases} -m\,(G_{\Delta\mathrm{N}}G_{\mathrm{NN}} + 25G_{\Delta\Delta}G_{\Delta\mathrm{N}}) \\ -\frac{1}{m}\frac{15}{2}G_{\Delta\Delta}G_{\Delta\mathrm{N}} \\ \frac{1}{m}\frac{15}{2}G_{\Delta\Delta}G_{\Delta\mathrm{N}} \\ -G_{\Delta\mathrm{N}}G_{\mathrm{NN}} + 25G_{\Delta\Delta}G_{\Delta\mathrm{N}}) \end{cases} \tag{16}$$

We have to insert values for the coupling constants. Within the spirit of our approximation we shell not use the experimental values but those are determined from the SU(2)×SU(2) algebra (9) for the NΔ system only. These coupling constants satisfy the algebraic relations ‡‡‡

$$G_{\mathrm{NN}}^{\;2} - G_{\mathrm{N}\Delta}^{\;2} = 1 \tag{17}$$

$$G_{\Delta\Delta}^{\;2} - \tfrac{1}{2}G_{\Delta\mathrm{N}}^{\;2} = 1 \tag{18}$$

$$G_{\Delta\mathrm{N}}G_{\mathrm{NN}} - 5\,G_{\Delta\Delta}\,G_{\Delta\mathrm{N}} = 0 \tag{19}$$

They are solved by

$$(G_{\mathrm{NN}}, G_{\mathrm{N}\Delta}, G_{\Delta\Delta}) = (\tfrac{5}{3}, \tfrac{4}{3}, 1). \tag{20}$$

With these values $\mathcal{A}_\rho^{(-)}$ becomes

$$\mathcal{A}_t^{(-)} = \frac{20}{\sqrt{2}F_\pi^2}\begin{Bmatrix} 1 \\ 1/4m^2 \\ -1/4m^2 \\ 1/m \end{Bmatrix} \tag{21}$$

Comparing with (14) we obtain our final result

$$C_3 m = \frac{5}{\sqrt{3}}, \quad C_4 m^2 = C_5 m^2 = \frac{1}{4}\frac{5}{\sqrt{3}} \tag{22}$$

The sum rule for $B_1^{(-)}$ cannot be fulfilled since the nucleon does not contribute and, at least, one resonance of spin $\geqslant 3/2$ is needed to obtain the required zero. The ΔNρ coupling constants (22) can be compared with experiment if one assumes that the electro-magnetic current of the NΔ transition is dominated by a ρ meson.
In this case, our C's become directly the Gourdin-Salin electromagnetic coupling constants. They have been measured to be [4,6]

$$C_3 m \approx 2.3\ , \quad C_4 m^2 + C_5 m^2 \approx 0.54 \quad [4] \tag{23}$$
$$\approx 2.0 \quad [6]$$

We see that the agreement with the theoretical value in (22) is as good as the approximation involved.

‡‡‡ Obviously, these are the Adler-Weisberg sum rules for πN, $\pi\Delta$ and πN → $\pi\Delta$ scattering.

A lot more algebraic effort is necessary to lift the degeneracy of N and Δ and to include higher resonances. The discussion of such a calculation will be presented elsewhere.

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