No Massless Pions in Nambu-Jona-Lasinio Model due to Chiral Fluctuations

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In contrast to common belief, the chirally symmetric Nambu–Jona-Lasinio model does not contain massless pions, due to strong chiral fluctuations. Although quarks acquire spontaneously a nonzero constituent mass M, pions have a nonzero mass equal to the mass of σ-mesons, both being of the order of M. This result is found in several cutoff schemes. Our derivation is nonperturbative, but involves a simple approximation (London limit) which should, however, receive only quantitative, no qualitative corrections.

I. INTRODUCTION

The chirally symmetric Nambu–Jona-Lasinio model [1] was invented to explain the small mass of the pions as a result of a spontaneous breakdown of chiral symmetry in hadron physics. The first realistic formulation of the model which included flavored quarks, possessed chiral symmetry $SU(3) \times SU(3)$, and a spectrum of σ, π, ρ, A_1 mesons and their $SU(3)$ partners, was formulated and investigated in 1976 by one of the authors [2], and has been followed by many papers in nuclear physics in the past twenty-three years [3].

In two important respects, however, the model was unsatisfactory. First, it was not renormalizable in four dimensions, but required a momentum space cutoff Λ to produce finite results. Moreover, to obtain physical quantities of the correct size, the cutoff had to be rather small, below one GeV, thus limiting the applicability of the model to very low energies. Second, the model could not account for quark confinement.

The first problem, the nonrenormalizability, was removed in [2] by replacing the four-fermion interaction by the exchange of a massive vector meson V_μ. The different attractive meson channels were obtained by a Fierz transformation of the effective four-fermion vector-vector interaction. The mass of V_μ took over the role of the cutoff. Although this led to a bona-fide field theory, the range of applicability was still limited by the second problem, the lack quark confinement.

The purpose of this note is to point out a much more severe problem with the model: If chiral fluctuations are properly taken into account, the spontaneous symmetry breakdown disappears, and the pions acquire a nonzero mass equal to that of the σ-mesons. The argument is nonperturbative, and this is the reason why it has been overlooked until now. The chiral properties of the Nambu–Jona-Lasinio model which have been derived and studied in the existing literature turn out to exist only if the quarks would exist with $N_c > 3$ identical replica, instead of the three colored quarks existing in nature.

The nonperturbative arguments used in this paper are analogous to those applied before [4] in a discussion of the Gross-Neveu model [5] in $2 + \varepsilon$ dimensions, where it was shown that this model has two phase transitions, one where quarks become massive, and another one where chiral symmetry is spontaneously broken.

II. NAMBU–JONA-LASINIO MODEL

The model contains N_f massless quark fields $\psi(x)$, each with N_c colors. Since the fluctuation effects to be discussed will be caused by the massless modes, we may restrict ourselves to up and down quarks ($N_f = 2$). The Lagrangian of the model is given by [3]

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi + \frac{g_0}{2N_c} \left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}\lambda_a i\gamma_5 \psi\right)^2 \right]. \tag{1}$$

The three 2×2-dimensional matrices $\lambda_a/2$, generate the fundamental representation of flavor $SU(2)$, and are normalized by $\mathrm{tr}(\lambda_a \lambda_b) = 2\delta_{ab}$.

A Hubbard-Stratonovich transformation makes the model equivalent to a theory of collective scalar and pseudoscalar fields σ and π_a:

$$\mathcal{L} = \bar{\psi} \left(i\not{\partial} - \sigma - i\gamma_5 \lambda_a \pi_a \right) \psi - \frac{N_c}{2g_0} \left(\sigma^2 + \pi_a^2 \right). \tag{2}$$

After integrating out the quark fields, the limit $N_c \to \infty$ leads to an effective action for the ground state

$$\Gamma(\rho) = -\Omega[\Delta v(\rho) + v_0] \tag{3}$$

where Ω is the spacetime volume, v_0 is the divergent energy density of the symmetric state, and $\Delta v(\rho)$ is the condensation energy

$$\Delta v(\rho) = \frac{N_c}{2} \left\{ \frac{1}{g_0}\rho^2 - \frac{2}{(2\pi)^2} \left[\frac{\rho^2\Lambda^2}{2} + \frac{\Lambda^4}{2} \ln\left(1 + \frac{\rho^2}{\Lambda^2}\right) - \frac{\rho^4}{2} \ln\left(1 + \frac{\Lambda^2}{\rho^2}\right) \right] \right\} \tag{4}$$

at a constant $\sigma^2 + \pi_a^2 \equiv \rho^2$. As in the original treatment [1], the momentum integral is regularized by a cutoff Λ

*On leave from absence of Physique Nucléaire Théorique, B5, Université de Liège Sart-Tilman, 4000 Liège, Belgium in euclidean momentum space. The condensation energy is extremal at $\rho = \rho_0$, which solves the *gap equation*

$$\frac{1}{g_0} = \frac{2}{(2\pi)^2}\left[\Lambda^2 - {\rho_0}^2 \ln\left(1 + \frac{\Lambda^2}{{\rho_0}^2}\right)\right]. \tag{5}$$

The quantity ρ_0 plays the role of an *order parameter* for the condensed state. It also determines the constituent mass M_0 of the quark fields in the limit $N_c \to \infty$.

III. CHIRAL FLUCTUATIONS

Since the physical number of quarks N_c is finite, the fields perform fluctuations around the extremal field value, which we may assumed to point in the σ-direction: $(\sigma, \pi_a) = (\rho_0, 0)$. As long as N_c can be considered as a large number, the deviations $(\sigma', \pi'_a) \equiv (\sigma - \rho_0, \pi_a)$ are small, and the action can be expanded in powers of (σ', π'_a). The quadratic terms in this expansion define the propagators of the collective fields (σ', π'_a), whereas the higher expansion terms define the interactions. In momentum space, the quadratic terms are

$$\mathcal{A}_0[\sigma', \pi'] = \frac{1}{2}\int d^4q \left[\begin{pmatrix}\pi'_a(q)\\ \sigma'(q)\end{pmatrix}^T \begin{pmatrix}G_\pi^{-1}(q) & 0\\ 0 & G_\sigma^{-1}(q)\end{pmatrix}\begin{pmatrix}\pi'_a(-q)\\ \sigma'(-q)\end{pmatrix}\right],$$

with the inverse bosonic propagators $G_{\sigma,\pi}^{-1}(q)$

$$N_c\left\{2 \times 2^{D/2}\int\frac{d^4p_E}{(2\pi)^4}\frac{(p_E^2 + p_E q_E \mp {\rho_0}^2)}{(p_E^2 + {\rho_0}^2)[(p_E + q_E)^2 + {\rho_0}^2]} - \frac{1}{g_0}\right\}.$$

With the help of the gap equation (5), we eliminate the term $1/g_0$ and obtain in $D = 4$ dimensions the inverse propagators

$$G_{\pi,\sigma}^{-1}(q_E^2) = N_c\left\{8\int_0^1 dy \int_0^{\Lambda^2}\frac{dp_E^2\, p_E^2}{16\pi^2} \times \frac{q_E^2(1-y) + (0, 2\rho_0^2)}{[p_E^2 + q_E^2 y(1-y) + {\rho_0}^2]^2}\right\}. \tag{6}$$

For small momenta, these behave like

$$G_\pi^{-1} \approx Z(\rho_0) q_E^2, \quad G_\sigma^{-1} \approx Z(\rho_0)(q_E^2 + 4{\rho_0}^2), \tag{7}$$

where $Z(\rho_0)$ is the wave function renormalization constant

$$Z(\rho_0) = \frac{N_c}{(2\pi)^2}\left[\ln\left(1 + \frac{\Lambda^2}{{\rho_0}^2}\right) - \frac{\Lambda^2}{\Lambda^2 + {\rho_0}^2}\right]. \tag{8}$$

As a consequence of the spontaneous symmetry breakdown, the fluctuations of the pseudoscalar fields are massless *Goldstone bosons*. These fields appear in the x-space version of the above quadratic action in a pure gradient form, thus performing violent fluctuations. The fluctuations possess a large entropy, leading to a destruction of the ordered state and a restoration of the chiral symmetry, unless N_c is unphysically large.

For simplicity, we shall ignore the massive size fluctuations of the field ρ, as in the *London limit* of superconductivity (also called *hydrodynamic limit*). We introduce a unit vector field $n_i \equiv (n_0, n_a) \equiv (\sigma, \pi_a)/\rho$ in the mesonic field space, whose long-wavelength fluctuations have, for symmetry reasons, the gradient action

$$\mathcal{A}_0[n_i] = \frac{\beta(\rho)\rho^2}{2}\int d^4x [\partial n_i(x)]^2. \tag{9}$$

The prefactor $\beta(\rho)$ in (9) is called the *stiffness* of the directional fluctuations [4,6–8]. The first of Eqs. (7) shows that the stiffness is

$$\beta(\rho) = Z(\rho), \tag{10}$$

to be evaluated at the extremum $\rho = \rho_0 = M$, in the London limit.

For shorter wavelengths, the action (9) becomes

$$\mathcal{A}_1[n_i] = \frac{\rho^2}{2}\int d^4x\, n_i(x) G_\pi^{-1}(-\partial^2) n_i(x), \tag{11}$$

with $G_\pi^{-1}(-\partial^2)$ from Eq. (6).

We now demonstrate that the stiffness (10) in the Nambu–Jona-Lasinio model for three colored quarks is far too small to let the spontaneous symmetry breakdown survive the strong fluctuations of the directional field $n_i(x)$. We do this first in the long-wave approximation (9), where the discussion is simplest. Rewriting (9) with the help of a Lagrange multiplier field as

$$\frac{\beta(\rho)\rho^2}{2}\int d^4x \left\{[\partial n_i(x)]^2 + \lambda(x)\left[n_i^2(x) - 1\right]\right\}, \tag{12}$$

we integrate out the $n_i(x)$-fields, and find

$$\tilde{\mathcal{A}}_0[\lambda] = -\beta(\rho)\rho^2\int d^4x \frac{\lambda(x)}{2} + \frac{N_n}{2}\mathrm{Tr}\ln\left[-\partial^2 + \lambda(x)\right], \tag{13}$$

where N_n is the number of components of $n_i(x)$, and Tr denotes the functional trace. In going from $\mathcal{A}_0[n_i]$ to $\tilde{\mathcal{A}}_0[\lambda]$, we have summed up infinitely many diagrams of the ordinary perturbation theory of σ- and π_a-fields, whose expansion parameter is $1/N_c$.

For large N_n, the fluctuations are suppressed. In the ground state, the field $\lambda(x)$ becomes a constant, satisfying a *second gap equation*

$$\beta(\rho) = \frac{N_n}{\rho^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2 + \lambda}. \tag{14}$$

There exists a phase transition at a critical stiffness

$$\beta^{\rm cr} = \frac{N_n}{\rho^2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2}. \tag{15}$$

For $\beta(\rho) < \beta^{\rm cr}$, chiral fluctuations are so violent that the system goes into a disordered state with $\lambda \neq 0$, where the N_n fields $n_i(x)$ have a nonzero square mass λ, which plays the role of an order parameter in the directional phase transition. Since the fields $n_i(x)$ are the normalized σ- and π_a-fields, λ is a nonzero common square mass of the associated particles. Thus chiral fluctuations have restored the chiral symmetry which was broken in the initial large-N_c approximation.

Note that an important consequence of the initial large-N_c approximation persists: the quarks are still massive. This does not contradict the Goldstone theorem. A nonzero quark mass is perfectly compatible with chiral symmetry since an arbitrary chiral rotation of the mass term in the Dirac equation, performed into any pion direction, produces a term $m(\cos\chi + i\gamma_5 \sin\chi)$, and thus describes quarks of the same mass $M = \sqrt{\sigma^2 + \pi_a^2}$. Physically, the mass term is a consequence of the *formation* of the pairs which for small N_c are strongly bound. The phase transition taking place at $\beta(\rho) = \beta_c$, on the other hand, describes the Bose condensation of these pairs, which is a completely different process for small N_c. The separate occurrence of the two transition follows from a simple fluctuation criterion [9].

In the model, the number N_n is equal to four, such that the large-N_n approximation in the calculation of $\beta^{\rm cr}$ may be questioned. However, Monte-Carlo studies have shown that $N_n = 4$ is large enough to ensure the existence of the transition, and that the critical stiffness obtained from (15) is correct to within 2% [10,11].

For a first crude estimate of the critical number of colors $N_c^{\rm cr}$ where symmetry restoration takes place, we cut the divergent integral (15) off at the same Λ as the fermion loop integrals. This will be referred to as approximation 1. Then we obtain, for $N_n = 4$,

$$\beta^{\rm cr} = 4\frac{\Lambda^2}{16\pi^2}, \tag{16}$$

from which we find, by comparison with (10),

$$N_c^{\rm cr} = \left(\frac{\Lambda}{\rho_0}\right)^2 \left\{ \ln\left[1 + \left(\frac{\Lambda}{\rho_0}\right)^2\right] - \frac{(\Lambda/\rho_0)^2}{1 + (\Lambda/\rho_0)^2} \right\}^{-1}. \tag{17}$$

The model only possesses a phase in which pions are Goldstone bosons if the number of colors exceeds $N_c^{\rm cr}$. The critical number (17) is plotted in Fig. 1 as a short-dashed curve. We see that N_c would have to exceed the unphysical number of colors 5 to have massless pions.

Let us refine this crude estimate by going to approximation 2. Here we first replace $1/k^2$ in Eq. (15) by the full pion propagator $G_\pi(k^2)/\rho_0^2$ associated with the action (11). Second, we choose a more physical cutoff Λ_π to make the integral over pion momenta finite. The pion fields are composite, and will certainly not be defined over length scales much shorter than the inverse binding energy of the pair wave function which is equal to $2M = 2\rho_0$. Thus we perform the integral in the modified Eq. (15) up to the cutoff $4M^2$. This yields the solid curve in Fig. 1. The intercept of this curve with the $N_c = 3$ -line shows that there exists a phase with broken symmetry for three colors only if the cutoff of the quark loop integration is $\Lambda^2 > 11M^2$. Such a large cutoff, however, is incompatible with the experimental value of the pion decay constant $f_\pi \approx 0.093$. Within the present model, this constant is in the large-N_c limit $f_\pi/M = Z^{1/2}(M)$. For the typical estimates of constituent quark masses $M \in (300, 400)$ MeV [2], we find that Λ^2/M^2 should lie in the range $(3.3, 7.3)$, the highest value corresponding to the lowest possible mass 300 MeV.

If we cut the momentum integral over $G_\pi(k^2)/\rho_0^2$ off at a larger value $8M^2$, which we call approximation 3, we obtain the long-dashed curve in Fig. 1. It cuts the $N_c = 3$ -line at an even higher unphysical value $\Lambda^2/M^2 \approx 60$.

Another natural way of cutting off the integral is by Taylor-expanding the denominator of $G_\pi(k^2)/\rho_0^2$ up to k^6 and doing the, now convergent, momentum integral up to infinity. This yields approximation 4, pictured as a shorter-dashed curve in Fig. 1, again with no symmetry-broken phase for a physically acceptable range of cutoffs Λ.

Thus we conclude that the Nambu–Jona-Lasinio model cannot properly be used to describe the chiral symmetry breakdown of quark physics in quantum chromodynamics.

It is interesting to see that the same conclusion cannot be reached in the dimensional regularization scheme [12]. In that scheme, the integral in (15) determining the critical stiffness vanishes. This is one of the typical unphysical features of dimensional regularization [13], which can meaningfully be used only in renormalizable theories, where all quantities which diverge with a power of the momentum space cutoff Λ can be absorbed into unobservable bare quantities of the theory, thus being physically irrelevant. In the present nonrenormalizable theory, they are not!

Let us briefly sketch the calculation of the common nonzero mass λ of σ and π_a-fields in the phase of restored chiral symmetry. For this we consider the change of the effective potential caused by chiral fluctuations. They add to $\Delta v(\rho)$ in (3) the action (13) at a constant $\lambda(x) = \lambda$, but with $-\partial^2$ replaced by $G_\pi^{-1}(-\partial^2)/Z(\rho)$.

$$\Delta' v(\rho, \lambda) = -\frac{1}{2}\lambda Z(\rho)\rho^2 + \frac{N_n}{2}\int_0^\infty \frac{dq_E^2\, q_E^2}{16\pi^2} \log[G^{-1}(q_E^2)/Z(\rho) + \lambda] \tag{18}$$

Extremizing $\Delta v(\rho) + \Delta' v(\rho, \lambda)$ yields two coupled gap equations replacing (5) and (14). They can be written down explicitly for approximation 1. They read

$$x_0 \ln\left(1+x_0^{-1}\right) + \frac{y}{2}\frac{d}{dx}\left(x\bar{Z}\right) = x\ln\left(1+x^{-1}\right), \quad (19)$$

$$N_c x\bar{Z} = 1 - y\ln\left(1+y^{-1}\right), \quad (20)$$

with the reduced quantities $\bar{Z} = \ln\left(1+x^{-1}\right) - (1+x)^{-1}$ and $x \equiv \rho^2/\Lambda^2, y \equiv \lambda/\Lambda^2$. The coupling constant g_0 has been eliminated, with the help of (5), in favor of the constituent quark mass $\rho_0 = M_0$ which characterizes the model uniquely above $N_c^{\rm cr}$. For $\lambda = 0$, Eq. (19) reduces to (5). Equation (20), on the other hand, determines the common square mass λ of σ and π_a as a function of N_c, which begins developing for $N_c < N_c^{\rm cr}$.

Note that the inclusion of other flavors does not prevent the restoration of chiral symmetry, since the associated pseudoscalar mesons are too massive to make their fluctuations relevant to the described phenomenon.

The nonperturbative phenomenon can, of course, not be reproduced in the typical $1/N_c$-expansions of the model to any finite order [14].

IV. CONCLUSION

In a nonperturbative treatment we have shown that for three colored quarks, the Nambu–Jona-Lasinio model does not really display the spontaneous symmetry breakdown for whose illustration it was invented. The violent chiral fluctuations in the degenerate potential valley on the surface of a sphere in the four-dimensional field space of σ- and π_a-mesons restore chiral symmetry, making σ and π equally massive, with a mass of the order of the constituent quark mass. It will be interesting to see how our limits on the critical number of colors will change with various corrections to our approximation.

More details will be published elsewhere.

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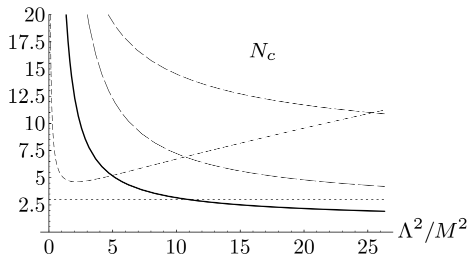

FIG. 1. Lowest critical number of colors N_c for which there exists a symmetry-broken phase with zero-mass pions in the Nambu-Jona-Lasinio model, as a function of Λ^2/M^2, where Λ is the cutoff for the quark fields with their constituent mass M. The solid line is from the exact propagator with a cutoff for the pion field $\Lambda_\pi^2 = 4M^2$ (approximation 2), the long-dashed line for $\Lambda_\pi^2 = 8M^2$ (approximation 3). The shorter-dashed curve comes from an expansion of the denominator of the exact propagator up to q^4 and an infinite cutoff Λ_π (approximation 4). The short-dashed curve would result from a pure $1/Z(\rho_0)q^2$ approximation of the pion propagator with a cutoff $\Lambda_\pi = \Lambda$ (approximation 1). Except for an excessively large cutoff $\Lambda^2 \approx 11M^2$ in the exact propagator, which is incompatible with the experimental value of $f_\pi \approx 0.093$, the symmetry will always be restered for $N_c = 3$ (dotted line).