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Relativistic Current of the H Atom in $O(4,2)$ Dynamics*

HAGEN KLEINERT†
International Centre for Theoretical Physics, Trieste, Italy
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A relativistic current operator is given in the $O(4,2)$ description of the H atom whose matrix elements reduce to the correct Galilean form in the limit $c\rightarrow\infty$. This current is used to obtain a relativistic wave equation which contains the H-atom states as solutions and in which the electromagnetic field is coupled minimally.

I. INTRODUCTION

RECENTLY, the study of the electromagnetic interactions of the hydrogen atom from the group-theoretical standpoint has revealed high symmetries in the structure of form factors, or generalized oscillator strengths, as the atomic physicists call them.[1,2] A nonrelativistic wave equation has been found which describes in complete equivalence to Schrödinger theory a hydrogen atom that has been accelerated from rest to momentum $\mathbf{q}$ by the impact of an external photon.[3,4] This equation defines a current operator if one introduces the electromagnetic field through minimal coupling in the standard way. The transition amplitudes to higher states which are excited by the photon impact are then given in a natural manner as the matrix elements of the current operator between the initial and final state. The final state moving with momentum $\mathbf{q}$ is described by means of a certain Galilean transformation corresponding to the acceleration process.[5]

The structure of this description of the electromagnetic current has been postulated to apply also to the form factors of elementary particles. In this case, however, a relativistic current operator is needed to couple to the electromagnetic field. The results of such an approach have been very encouraging. Electromagnetic form factors up to high momentum transfers[6,7] and pion baryon form factors (as tested by pionic decay rates of baryon resonances)[8] are found in excellent agreement with experiment.

The relativistic currents in such calculations were first, for simplicity, assumed to be completely algebraic,[6] and when the agreement with experiment turned out to be not yet satisfactory, convective types of currents were added.[7]

In order to have some guide as to what types of currents may occur in theories of this kind, it is quite instructive to turn back to the exactly soluble case of the hydrogen atom, and to investigate the same problem there.

It is the purpose of this paper to give a relativistic current of the hydrogen atom that has the property of reducing to the correct nonrelativistic current in the limit of infinite light velocity. Using this current a relativistic wave equation of the hydrogen atom is set up to which the electromagnetic field is coupled minimally.[9] The current turns out to be the sum of an algebraic and a convective vector operator.

II. NONRELATIVISTIC CURRENT

Let us recall briefly how the nonrelativistic current for hydrogen-atom transitions is calculated. If $\mathbf{x}_e$, $\mathbf{x}_p$, $\mathbf{X}$, $\mathbf{x}$ denote electron, proton, center-of-mass, and relative coordinates, respectively, then a hydrogen atom moving with momentum $\mathbf{q}$ is described by the state

$$\phi_{\mathbf{q}}(\mathbf{x}_e,\mathbf{x}_p)=e^{i\mathbf{q}\cdot\mathbf{X}}\psi_n(\mathbf{x})\,, \tag{1}$$

with obvious notation. The electronic contribution to the current for the transition from momentum $\mathbf{q}$ to rest is then given by the Fourier transforms of charge and current densities:

$$\rho(\mathbf{q})=\frac{1}{V}\int\phi_0'^*(\mathbf{x}_e,\mathbf{x}_p)\phi_{-\mathbf{q}}(\mathbf{x}_e,\mathbf{x}_p)e^{i\mathbf{q}\cdot\mathbf{x}_e}d\mathbf{x}_e d\mathbf{x}_p\,, \tag{2}$$

$$j^i(\mathbf{q})=\frac{1}{V}\frac{1}{2m_e i}\int\left[\phi_0'^*(\mathbf{x}_e,\mathbf{x}_p)\times\frac{\overleftrightarrow{\partial}}{\partial x_e{}^i}\phi_{-\mathbf{q}}(\mathbf{x}_e,\mathbf{x}_p)\right]e^{i\mathbf{q}\cdot\mathbf{x}_e}d\mathbf{x}_e d\mathbf{x}_p\,. \tag{3}$$

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† Permanent address: Department of Physics, University of Colorado, Boulder, Colo. Present address: Department of Physics, Montana State University, Bozeman, Mont.

[1] Hagen Kleinert, Fortschr. Physik **16**, 1 (1968).

[2] A. O. Barut and Hagen Kleinert, Phys. Rev. **160**, 1149 (1967).

[3] C. Fronsdal, Phys. Rev. **156**, 1665 (1967).

[4] A. O. Barut and Hagen Kleinert, Phys. Rev. **156**, 1541 (1967); **157**, 1180 (1967).

[5] This is the *active* Galilean transformation in the sense of Ref. 1. It can be characterized in the following manner: The interaction causing the acceleration can be written as a product of *local fields* transforming covariantly in this representation of the Galilean group.

[6] Hagen Kleinert, Phys. Rev. **163**, 1807 (1967).

[7] A. O. Barut, D. Corrigan, and Hagen Kleinert, Phys. Rev. **167**, 1527 (1968); Phys. Rev. Letters **20**, 167 (1968).

[8] A. O. Barut and Hagen Kleinert, Phys. Rev. Letters **18**, 743 (1967); Hagen Kleinert, *ibid.* **18**, 1027 (1967).

[9] This proves that the current is conserved.

Changing variables to **X** and **x**, we find from this,

$$\rho(\mathbf{q})=\int\psi_{n'}{}^*(\mathbf{x})e^{i\mathbf{q}[m_p/(m_p+m_e)]\cdot\mathbf{x}}\psi_n(\mathbf{x})d\mathbf{x}\,, \qquad (4)$$

$$j^i(\mathbf{q})=I^i(\mathbf{q})-(\mathbf{q}/2m_e)\rho(\mathbf{q})\,, \qquad (5)$$

with

$$I^i(\mathbf{q})=\frac{1}{m_e i}\int\psi_{n'}{}^*(\mathbf{x})\frac{\partial}{\partial x^i}e^{i\mathbf{q}[m_p/(m_p+m_e)]\cdot\mathbf{x}}\psi_n(\mathbf{x})d\mathbf{x}\,. \qquad (6)$$

These equations can be cast in group language on an $O(4,2)$ representation space generated by $L_{\alpha\beta}$ ($\alpha,\beta=1,\cdots,6$) with the commutation rules

$$[L_{\alpha\beta},L_{\alpha\gamma}]=ig_{\alpha\alpha}L_{\beta\gamma};\quad g=\begin{pmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&1&&\\&&&&-1&\\&&&&&-1\end{pmatrix}. \qquad (7)$$

One identifies L_{ij} as the angular momentum, L_{i4} as the Lenz vector, and L_{56} as the principal-quantum-number operator, and the Casimir operators of the representation have to be chosen as follows:

$$\begin{aligned}C_2&=L_{\alpha\beta}L^{\alpha\beta}=6\,,\\C_3&=\epsilon_{\alpha\beta\gamma\delta\epsilon\tau}L^{\alpha\beta}L^{\gamma\delta}L^{\epsilon\tau}=0\,,\\C_4&=L_{\alpha\beta}L^{\beta\gamma}L_{\gamma\delta}L^{\delta\alpha}=-12\,.\end{aligned} \qquad (8)$$

In this case, the states of the hydrogen atom with the parabolic quantum numbers n_1, n_2, m can be represented by means of creation and annihilation operators $a_r{}^\dagger$, a_r, $b_r{}^\dagger$, b_r satisfying

$$[a_r,a_s{}^\dagger]=\delta_{r,s},\quad [b_r,b_s{}^\dagger]=\delta_{r,s}\quad (r,s=1,2)$$

in the form

$$|n_1n_2m\rangle=[n_1!(n_1+m)!n_2!(n_2+m)!]^{-1/2}\times a_1{}^{\dagger n_2+m}a_2{}^{\dagger n_1}b_1{}^{\dagger n_1+m}b_2{}^{\dagger n_2}|0\rangle \qquad (9)$$

for $m\geq 0$. For $m<0$, the same formula holds with n_1 and n_2 being replaced by n_1-m and n_2-m, respectively. On these states the generators $L_{\alpha\beta}$ become

$$\begin{aligned}L_{ij}&=\tfrac{1}{2}(a^\dagger\sigma_k a+b^\dagger\sigma_k b)\equiv L_k\,,\\L_{i4}&=-\tfrac{1}{2}(a^\dagger\sigma_i a-b^\dagger\sigma_i b)\equiv R_i\,,\\L_{i5}&=-\tfrac{1}{2}(a^\dagger\sigma_i Cb^\dagger-aC\sigma_i b)\,,\\L_{i6}&=(1/2i)(a^\dagger\sigma_i Cb^\dagger+aC\sigma_i b)\,,\\L_{45}&=(1/2i)(a^\dagger Cb^\dagger-aCb)\,,\\L_{46}&=\tfrac{1}{2}(a^\dagger Cb^\dagger+aCb)\,,\\L_{56}&=\tfrac{1}{2}(a^\dagger a+b^\dagger b+2)\,,\end{aligned} \qquad (10)$$

where σ_i are the Pauli matrices and

$$C=\begin{pmatrix}&1\\-1&\end{pmatrix}=i\sigma_2\,. \qquad (11)$$

From these equations we find that the third components of the angular momentum and the Lenz vector are diagonal with the eigenvalues m and (n_1-n_2), respectively, while L_{56} gives the value of the principal quantum number n:

$$n=n_1+n_2+|m|+1\,. \qquad (12)$$

The transition from parabolic states to the usual ones with definite angular momentum L^2 are found in the following way. One observes from Eq. (10) that L is the direct sum of "a spin"

$$\mathbf{J}^a=\tfrac{1}{2}a^\dagger\sigma a \qquad (13)$$

and "b spin"

$$\mathbf{J}^b=\tfrac{1}{2}b^\dagger\sigma b\,, \qquad (14)$$

which are diagonal on the states (1) with the eigenvalues

$$\begin{aligned}(\mathbf{J}^a)^2&=\tfrac{1}{4}(n^2-1),\quad J_3{}^a=\tfrac{1}{2}(m-n_1+n_2)\,,\\(\mathbf{J}^b)^2&=\tfrac{1}{4}(n^2-1),\quad J_3{}^b=\tfrac{1}{2}(m+n_1-n_2)\,.\end{aligned} \qquad (15)$$

But then the states $|nlm\rangle$ of angular momentum l are simply the combinations

$$|nlm\rangle=(-)^m(2l+1)^{1/2}\times\begin{pmatrix}\tfrac{1}{2}(n-1)&\tfrac{1}{2}(n-1)&l\\\tfrac{1}{2}(m-n_1+n_2)&\tfrac{1}{2}(m+n_1-n_2)&-m\end{pmatrix}|n_1n_2m\rangle\,. \qquad (16)$$

Using this representation, it has been shown that the Schrödinger theory of the hydrogen atom can be completely translated into group-theoretical language if one represents the physical states in the form

$$|\tilde{n}lm\rangle\equiv(1/n)\,e^{-iL_{45}\ln(an)}|nlm\rangle \qquad (17)$$

and the quantum-mechanical operators x_i and p_i by

$$\begin{aligned}x_i&=(1/a)(L_{i5}-L_{i4})\,,\\p_i&=a(L_{56}-L_{46})^{-1}L_{i6}\,,\end{aligned} \qquad (18)$$

where a is a completely arbitrary parameter.[10] In these equations atomic units have been used with $\mu\equiv m_em_p/(m_e+m_p)=e=\hbar=1$.

The physical scalar product has to be evaluated using the metric $(L_{56}-L_{46})/a$ in the form

$$(\psi',\mathcal{O}\psi)=\langle\tilde{n}'l'm'|(1/a)(L_{56}-L_{46})\mathcal{O}|\tilde{n}lm\rangle\,. \qquad (19)$$

The operation

$$T_n\equiv e^{-iL_{45}\ln(an)} \qquad (20)$$

is called "tilting" and turns out to be very important in applications to particle physics. It is interesting to note that the tilting angles

$$\Theta_n\equiv-\ln(an) \qquad (21)$$

have just the right values to make the physical states

[10] Hagen Kleinert, *Lectures in Theoretical Physics*, edited by W. E. Brittin and A. O. Barut (Gordon and Breach Science Publishers, Inc., New York, 1967).

$|\bar{n}lm\rangle$ orthogonal with respect to the scalar product (19).[11]

Using the translation rules (17)–(19), the currents can be rewritten in the form

$$\begin{aligned}\rho(\mathbf{q})&=\langle\bar{n}'|(1/a)(L_{56}-L_{46})e^{i(q^k/m_e a)(L_{k5}-L_{k4})}|\bar{n}\rangle,\\ I^i(\mathbf{q})&=(1/m_e)\langle\bar{n}'|L_{i6}\,e^{i(q^k/m_e a)(L_{k5}-L_{k4})}|\bar{n}\rangle,\end{aligned}\tag{22}$$

where the indices l,m have been dropped for simplicity. These expressions can be simplified by observing that the tilting operator T_n dilates $(L_{56}-L_{46})$ and $(L_{k5}-L_{k4})$ by the factor an while it leaves L_{i6} invariant, i.e.,

$$\begin{aligned}T_n^{-1}(L_{56}-L_{46})T_n&=an(L_{56}-L_{46}),\\ T_n^{-1}(L_{k5}-L_{k4})T_n&=an(L_{k5}-L_{k4}).\end{aligned}\tag{23}$$

Thus Eqs. (22) take the form

$$\begin{aligned}\rho(\mathbf{q})&=(1/n)\langle n'|(L_{56}-L_{46})\\ &\quad\times e^{-i[\ln(n/n')]L_{45}}\,e^{i(q^k/m_e)n(L_{k5}-L_{k4})}|n\rangle,\\ I^i(\mathbf{q})&=(1/n'nm_e)\langle n'|L_{i6}\\ &\quad\times e^{-i[\ln(n/n')]L_{45}}\,e^{i(q^k/m_e)n(L_{k5}-L_{k4})}|n\rangle,\end{aligned}\tag{24}$$

in which the currents can easily be evaluated using $O(2,1)$ subgroups.[1,10] For completeness we have given the results in the Appendix.

III. RELATIVISTIC CURRENT

Consider now the current operator defined on the tilted states $|\bar{n}lm\rangle$ by

$$J^\mu=(\alpha/m_e)[\Gamma^\mu-(P^\mu/2m_p)L_{46}],\tag{25}$$

where Γ^μ is the algebraic vector

$$\Gamma^\mu=(-L_{56},L_{i6}),\tag{26}$$

P^μ is the sum of initial and final four-momentum of the hydrogen atom at the interaction vertex, and α is the fine-structure constant. Here we have used natural units which are convenient for relativistic calculations ($\mu=\hbar=c=1$). This operator J^μ is clearly a four-vector if one defines Lorentz transformations by

$$\Lambda=e^{-i\zeta^k L_{k5}},\tag{27}$$

where ξ^k is the rapidity parameter $[=\tanh^{-1}(v^k/c)]$ of the motion. The matrix elements of J^μ for the transition of the atom from rapidity $(-\zeta)$ to rest are given by

$$j^\mu(\zeta)\equiv\langle\bar{n}'|J^\mu\,e^{i\zeta^k L_{k5}}|\bar{n}\rangle,\tag{28}$$

where $|\bar{n}\rangle$ are tilted states just like in Eq. (17), but a has now the specific value

$$a=2m_p/\alpha.\tag{29}$$

We then prove that these matrix elements reduce in the nonrelativistic limit $c\to\infty$ exactly to the result (24). To see this, let us perform the operations corresponding to (23). We find

$$\begin{aligned}T_n^{-1}L_{56}T_n&=e^{-i\theta_n L_{45}}L_{56}\,e^{i\theta_n L_{45}}\\ &=L_{56}\cosh\theta_n+L_{46}\sinh\theta_n,\end{aligned}\tag{30}$$

which becomes with (21) and (29)

$$\begin{aligned}T_n^{-1}L_{56}T_n&=(m_p/\alpha)n(L_{56}-L_{46})\\ &\quad\times(\alpha/4m_pn)(L_{56}+L_{46}).\end{aligned}\tag{31}$$

Similarly, we obtain

$$\begin{aligned}T_n^{-1}L_{46}T_n&=-(m_p/\alpha)n(L_{56}-L_{46})\\ &\quad+(\alpha/4m_pn)(L_{56}+L_{46}),\\ T_n^{-1}L_{i6}T_n&=L_{i6},\\ T_n^{-1}L_{k5}T_n&=(m_p/\alpha)n(L_{k5}-L_{k4})\\ &\quad+(\alpha/4m_pn)(L_{k5}+L_{k4}).\end{aligned}\tag{32}$$

Thus, up to the lowest order in α^2, the current can be rewritten as

$$\begin{aligned}j^0(\xi)&=\left(\frac{P^0}{2m_e}-\frac{m_p}{m_e}\right)\frac{1}{n}\langle n'|(L_{56}-L_{46})\\ &\quad+e^{-i[\ln(n/n')]L_{45}}\,e^{i(\xi^k/\alpha)m_pn(L_{k5}-L_{k4})}|n\rangle\\ &\quad+O^0(\alpha^2),\\ j^i(\xi)&=(1/nn')(\alpha/m_e)\langle n'|L_{i6}\\ &\quad\times e^{-i[\ln(n/n')]L_{45}}\,e^{i(\xi^k/\alpha)m_pn(L_{k5}-L_{k4})}|n\rangle\\ &\quad+(q^i/2m_e)\cdot\alpha\cdot j^0(\xi)+O^i(\alpha^3).\end{aligned}\tag{33}$$

Let us in this equation go to the nonrelativistic limit $c\to\infty$. This means that also

$$\alpha\to0\tag{34}$$

and we can forget about the functions $O^0(\alpha^2)$ and $O^i(\alpha^3)$. The sum of initial and final mass of the atom at the vertex, P^0/c^2, is explicitly

$$\frac{1}{c^2}P^0=2(m_p+m_e)-\frac{\mu\alpha^2}{2n'^2}-\frac{\mu\alpha^2}{2n^2}+\left(\frac{q}{c}\right)^2\Big/2\left(m_p+m_e-\frac{\mu\alpha^2}{2n^2}\right)$$

and therefore converges to $2(m_p+m_e)$. Further, the parameter of the Lorentz transformation becomes

$$(\zeta^k/\alpha)m_p\to(v^k/c\alpha)m_p=(1/c\alpha)(q^k/m_e)\mu.\tag{35}$$

If we now introduce the charge density

$$\rho(\mathbf{q})=(1/c)j^0(\mathbf{q})\tag{36}$$

and use atomic units (which is simply done by dividing the currents by α and inserting $c=1/\alpha$ everywhere) we indeed recover Eq. (24).

[11] This follows from conservation of the current and is shown explicitly in Ref. 1. Note, however, that the physical states $|\bar{n}lm\rangle$ *do not span* the representation space given by the states $|nlm\rangle$, even though they are in one-to-one correspondence with $|nlm\rangle$ and orthogonal in the physical scalar product. For completeness the states of the continuum of the H atom have to be added which are tilted eigenstates of L_{46} rather than L_{56}. For a detailed discussion of this point see Ref. 0

Thus the operator J^μ given in Eq. (25) represents a relativistic current possessing the correct nonrelativistic matrix elements in the limit of infinite light velocity. It is the sum of an algebraic vector which lies completely in the Lie algebra and a convective current $L_{46}P^\mu$.

For comparison we write down the relativistic current that has been used to fit the electromagnetic properties of the nucleons.[7] This current is

$$J^\mu = a_1\Gamma^\mu + (a_2 + a_3L_{46})P^\mu + ia_4\Lambda^{\mu\nu}q_\nu\,, \qquad (37)$$

where $\Lambda^{\mu\nu}$ are the generators of the Lorentz group, i.e.,

$$\Lambda^{ij} = L_k, \quad \Lambda^{0i} = -L_{i5}\,, \qquad (38)$$

and

$$q^\nu = (p' - p)^\nu\,. \qquad (39)$$

We notice that like in the hydrogenic current all terms are at most linear in the Lie algebra and the external momenta. But while the first three terms of the baryon current have the same structure, we have found for the hydrogen atom, the third term is new in character and similar to the anomalous part in the Dirac current of the nucleons $\frac{1}{2}\kappa\sigma^{\mu\nu}q_\nu$. We shall discuss this current in the next section.

IV. RELATIVISTIC WAVE EQUATION

The current J^μ can be used to construct a relativistic wave equation of the type[12]

$$(J^\mu P_\mu - \text{scalar})\psi(x) = 0\,. \qquad (40)$$

The most general scalar in the $O(4,2)$ representation space is

$$\text{scalar} = \beta L_{46} + \gamma\,. \qquad (41)$$

We now show that there exists a choice of β and γ such that the physical states used in (28) with the tilting angles (29) become eigenstates of this equation.

Consider the equation at rest, when $P = (m,0,0,0)$. The general solution is then

$$\psi(X) = |\tilde{n}\rangle \bar{e}^{ip\cdot X}\,. \qquad (42)$$

On the states $|\tilde{n}\rangle$ the equation can therefore be written explicitly as

$$\left[\frac{\alpha}{m_e}\left(\Gamma^0 m - L_{46}\frac{m^2}{2m_p}\right) - \beta L_{46} - \gamma\right]|\tilde{n}\rangle = 0\,. \qquad (43)$$

Let us now take β and γ to be

$$\beta = \alpha(m_p^2 - m_e^2)/2m_pm_e \qquad (44)$$

and

$$\gamma = -\alpha^2\,. \qquad (45)$$

Then we see that the tilting operation

$$|\tilde{n}\rangle = e^{-i[\ln 2(m_p/\alpha)n]L_{45}}|n\rangle \qquad (46)$$

reduces Eq. (43) to

$$\left\{\frac{\alpha}{m_e}\left[m^2 - \left(\frac{m^2}{2m_p} + \frac{m_p^2 - m_e^2}{2m_p}\right)^2\right]^{1/2} L_{56} - \alpha^2\right\}|n\rangle = 0\,, \qquad (47)$$

from which we find the mass spectrum

$$m_n^2 = m_p^2 + m_e^2 \pm 2m_pm_e(1 - \alpha^2/n^2)^{1/2}\,. \qquad (48)$$

The upper sign of this spectrum yields

$$m_n = +[m_p^2 + m_e^2 + 2m_pm_e(1-\alpha^2/n^2)^{1/2}]^{1/2}$$
$$= +[(m_p + m_e) - (\mu\alpha^2/2n^2) + O(\alpha^4)]\,, \qquad (49)$$

which are the masses of the hydrogen atom. Using the lower sign we obtain

$$m_n = +[m_p^2 + m_e^2 - 2m_pm_e(1-\alpha^2/n^2)^{1/2}]$$
$$= +\left[(m_p - m_e) + \frac{m_pm_e}{m_p - m_e}\frac{\alpha^2}{2n^2} + O(\alpha^4)\right], \qquad (50)$$

which is an unphysical spectrum converging to $m_p - m_e$ from above. The eigenstates of (47) corresponding to these masses are orthogonal to the hydrogen states in the physical scalar product (19), since the Hamiltonian corresponding to the wave equation is Hermitian.

Thus we observe that our equation describes more than the observed states.[13] This seems to be characteristic for many relativistic wave equations.[3,12,14] In order to interpret the Eq. (40) physically, we therefore have to understand it in terms of a projection onto the physical subspace. A purely algebraic definition of currents and matrix elements as we have given it in (25) and (28) without the use of wave equations avoids this difficulty by working from begin with only in the physical Hilbert space.[15]

If we now introduce the electromagnetic field in the wave equation, postulating minimal electromagnetic coupling in the form

$$P^\mu \rightarrow P^\mu - eA^\mu\,, \qquad (51)$$

we obtain the correct first-order interaction $eJ^\mu A_\mu$ we started out with. The electromagnetic field therefore couples minimally to our relativistic wave equation.

In contradistinction to this, consider now the current of the baryons (37). Also there one can find a wave equation that contains the physical states as solutions:

$$\{[a_1\Gamma^\mu + (a_2 + a_3L_{46})P^\mu]P_\mu - \beta L_{46} - \gamma\}\psi(x) = 0\,. \qquad (52)$$

[12] Equations of this structure were first discussed by Y. Nambu, Progr. Theoret. Phys. (Kyoto), 1966, commemorative issue in honor of S. Tomonaga.

[13] Besides this, the Eq. (40) contains a continuous set of solutions part of which gives the free wave functions of the H atom while the other part arises from exchanging m_e by $-m_e$. Further there are also unphysical solutions with spacelike momenta. We shall not discuss these points here (see Ref. 14).

[14] If one requires currents to satisfy factorized $SU(3)+SU(3)$ commutation rules such a projection is not possible. The currents of infinite component wave equations satisfy the current algebra if the complete set of solutions, physical as well as unphysical, is used as intermediate states.

[15] Shan-Tin Chang and L. O'Raiteartaigh, Princeton Report (unpublished); I. T. Grodskey and R. F. Streater, Trieste Report No. IC/68/8 (unpublished).

However, as one can easily see, this equation produces a current containing only the first three terms of the expression (37), if the electromagnetic field is coupled minimally. In fact, one can prove that there is no wave equation at all that can give a current term $ia_4\Lambda^{\mu\nu}q_\nu$ by minimal coupling.[7]

The reason for this can be understood rather simply. Relativistic wave equations constructed with a current J^μ are nothing else but another way of expressing the conservation of this current. Thus if

$$(J^\mu P_\mu - \text{scalar})|\bar{n},P\rangle = 0\,, \tag{53}$$

then obviously

$$q^\mu\langle\bar{n}',P'|J^\mu|\bar{n},P\rangle = \langle\bar{n}',P'|J^\mu P_\mu' - J^\mu P_\mu|\bar{n},P\rangle = 0\,. \tag{54}$$

Conversely, if the current J^μ is conserved, i.e., Eq. (54) holds for all n', n, P', P, then $J^\mu P_\mu$ must be equal to a scalar operator that does not depend on P, and with this operator the states $|\bar{n},P\rangle$ satisfy a wave equation (53). Now observe that the term $ia_4\Lambda^{\mu\nu}q_\nu$ always contributes a conserved current no matter what the physical states are. Therefore it cannot change the wave equation for the physical states. Hence it also can never be obtained from such a wave equation by minimal electromagnetic coupling.

If the current (37) pertains to give a good description of the electromagnetic form factors also for the higher mass baryons, then either the idea of a wave equation or its minimal coupling will have to be abandoned.

V. CONCLUSION

We have seen that the relativistic current of the hydrogen atom has the following properties: (a) It is the sum of an algebraic and a convective part. (b) It is minimal in the sense that one can construct with it an infinite-component wave equation which contains the electromagnetic field in minimal coupling. Only the first property is satisfied by the electromagnetic current of the baryons that has been used to fit the form factors of the nucleons up to high momentum transfers. In addition, the baryon current contains an essentially nonminimal term that cannot be generated by minimal coupling.

The theory of the electromagnetic properties of the hydrogen atom has until now been the exclusive model in guiding the construction of a similar theory for baryons. With respect to the new nonminimal current term the model has been left behind. If one likes the idea of minimal coupling and infinite-component wave equations this is a rather unesthetic feature of the theory. A more intensive test of the baryon transition form factors will be necessary to find out which way nature has chosen.

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APPENDIX

We introduce the auxiliary functions

$$v_{mn}{}^p(\beta) = \Theta_{mn}{}^p \cosh^{-m-n}(\tfrac{1}{2}\beta)\, \sinh^{m-n}(\tfrac{1}{2}\beta) F(p-n,\, 1-n-p,\, 1+m-n,\, -\sinh^2(\tfrac{1}{2}\beta))\,, \tag{A1}$$

where $F(a,b,c,z)$ is the hypergeometric function and

$$\Theta^p_{mn} = \frac{1}{(m-n)!}\left[\frac{(m-p)!(m+p-1)!}{(n-p)!(n+p-1)!}\right]^{1/2}, \tag{A2}$$

for $m \geq n$ [for $m<n$ use ${}_{nm}(-\beta)$ instead]. Further, let

$$\begin{Bmatrix} h_{k',k}{}^{+,(i)}(\alpha,\gamma) \\ h_{k',k}{}^{0,(i)}(\alpha,\gamma) \\ h_{k',k}{}^{-,(i)}(\alpha,\gamma) \end{Bmatrix} = \begin{bmatrix} \cos \\ -i\sin \end{bmatrix} \left\{\left[2k' + \begin{Bmatrix} +1 \\ 0 \\ -1 \end{Bmatrix}\right]\alpha + 2k\gamma\right\} \quad \text{for} \quad (-)^i = \begin{bmatrix} +1 \\ -1 \end{bmatrix}. \tag{A3}$$

Then one finds for the charge density

$$\rho(\mathbf{q}) = \frac{1}{n}[(2l'+1)(2l+1)]^{1/2} \sum_{k',k} \begin{pmatrix} \frac{1}{2}(n'-1) & \frac{1}{2}(n'-1) & l' \\ \frac{1}{2}m-k' & \frac{1}{2}m+k' & -m \end{pmatrix} \begin{pmatrix} \frac{1}{2}(n-1) & \frac{1}{2}(n-1) & l \\ \frac{1}{2}m-k & \frac{1}{2}m+k & -m \end{pmatrix}$$
$$\times\{n' h_{k',k}{}^{0,(l'-l)}(\alpha,\gamma) v_{\frac{1}{2}n'+k',\frac{1}{2}n+k}{}^{\frac{1}{2}(m+1)}(\beta) + \tfrac{1}{2}[(n'+1+2k')^2-m^2]^{1/2} h_{k',k}{}^{+,(l'-l)}(\alpha,\gamma)$$
$$\times v_{\frac{1}{2}n'+k'+1,\frac{1}{2}n+k}{}^{\frac{1}{2}(m+1)}(\beta) + \tfrac{1}{2}[(n'-1+2k')^2-m^2]^{1/2} h_{k',k}{}^{-,(l'-l)}(\alpha,\gamma)$$
$$\times v_{\frac{1}{2}n'+k'-1,\frac{1}{2}n+k}{}^{\frac{1}{2}(m+1)}(\beta) v_{\frac{1}{2}n'-k',\frac{1}{2}n-k}{}^{\frac{1}{2}(m+1)}(-\beta)\}\,, \tag{A4}$$

while the spatial parts of the current are

$$I^{\pm}(\mathbf{q})=\mp\frac{i}{n'n}\frac{1}{m_e}[(2l'+1)(2l+1)]^{1/2}\sum_{k'k}\begin{pmatrix}\frac{1}{2}(n'-1) & \frac{1}{2}(n'-1) & l' \\ \frac{1}{2}(m\pm1)-k' & \frac{1}{2}(m\pm1)+k' & -m\mp1\end{pmatrix}\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & l \\ \frac{1}{2}m-k & \frac{1}{2}m+k & -m\end{pmatrix}$$

$$\times\{[(\tfrac{1}{2}(n'+m))^2-k'^2]^{1/2}h_{k',k}{}^{0,(l'-l+1)}(\alpha,\gamma)v_{\frac{1}{2}(n'\mp1)+k',\frac{1}{2}(n)+k}{}^{\frac{1}{2}(m+1)}(\beta)v_{\frac{1}{2}(n'\mp1)-k',\frac{1}{2}(n)-k}{}^{\frac{1}{2}(m+1)}(-\beta)$$

$$+[(\tfrac{1}{2}(n'-m))^2-k'^2]^{1/2}h_{k',k}{}^{0,(l'-l+1)}(\alpha,\gamma)v_{\frac{1}{2}(n'\pm1)+k',\frac{1}{2}(n)+k}{}^{\frac{1}{2}(m+1)}(\beta)v_{\frac{1}{2}(n'\pm1)-k',\frac{1}{2}(n)-k}{}^{\frac{1}{2}(m+1)}(-\beta)\} \quad \text{(A5)}$$

and

$$I^{3}(\mathbf{q})=\frac{i}{n'n}\frac{1}{m_e}[(2l'+1)(2l+1)]^{1/2}\sum_{k'k}\begin{pmatrix}\frac{1}{2}(n'-1) & \frac{1}{2}(n'-1) & l' \\ \frac{1}{2}m-k' & \frac{1}{2}m+k' & -m\end{pmatrix}\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & l \\ \frac{1}{2}m-k & \frac{1}{2}m+k & -m\end{pmatrix}$$

$$\times\{\tfrac{1}{2}[(n'+1+2k')^2-m^2]^{1/2}h_{k',k}{}^{+,(l'-l+1)}(\alpha,\gamma)v_{\frac{1}{2}n'+k'+1,\frac{1}{2}n+k}(\beta)$$

$$-\tfrac{1}{2}[(n'-1+2k')^2-m^2]^{1/2}h_{k',k}{}^{-,(l'-l+1)}(\alpha,\gamma)v_{\frac{1}{2}n'+k'-1,\frac{1}{2}n+k}(\beta)\}v_{\frac{1}{2}n'-k',\frac{1}{2}n-k}(-\beta). \quad \text{(A6)}$$

The angles α,β,γ are defined in terms of the momentum transfer q^2 and the principal numbers n and n' of initial and final states by

$$\sinh(\tfrac{1}{2}\beta)=\frac{1}{2(n'n)^{1/2}}[(n'-n)^2+q^2n'^2n^2]^{1/2}$$

and

$$\alpha=\text{arc}\sin(nq/\sinh\beta)\,,$$
$$\gamma=-\text{arc}\sin(n'q/\sinh\beta)\,,$$

where the principal value of arc sin has to be taken for $n'\leq n$, while for $n'>n$, α starts out at $q=0$ with the value π, and γ with $-\pi$. A plot of some of these form factors is given in Ref. 10.