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Functional differential equations for the free energy and the effective energy in the broken-symmetry phase of ϕ^4-theory and their recursive graphical solution

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Abstract

Extending recent work on QED and the symmetric phase of the euclidean multicomponent scalar ϕ^4-theory, we construct the vacuum diagrams of the free energy and the effective energy in the *ordered* phase of ϕ^4-theory. By regarding them as functionals of the free correlation function and the interaction vertices, we graphically solve nonlinear functional differential equations, obtaining loop by loop all connected and one-particle irreducible vacuum diagrams with their proper weights.

1. Introduction

Some time ago, one of us proposed a program for a systematical construction of all Feynman diagrams of a field theory together with their proper weights by graphically solving a set of functional differential equations [1]. It relies on considering a Feynman diagram as a functional of its graphical elements, i.e., its lines and vertices. Functional derivatives with respect to these graphical elements are represented by removing lines or vertices of a Feynman diagram in all possible ways. With these graphical operations, the program proceeds in four steps. First, a nonlinear functional differential equation for the free energy is derived as a consequence of the field equations. Subsequently, this functional differential equation is converted into a recursion relation for the loop

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expansion coefficients of the free energy. From its graphical solution, the connected vacuum diagrams are constructed. Finally, all diagrams of n-point functions are obtained from removing lines or vertices from the connected vacuum diagrams.

This program was recently used to systematically generate all connected Feynman diagrams of QED [2], of Ginzburg–Landau theory [3] and of the euclidean multicomponent scalar ϕ^4 theory [4,5]. In the disordered, symmetric phase of the latter theory, where the field expectation value vanishes, the energy functional consists only of even powers of the field. To generate all connected diagrams of the n-point functions, it was sufficient to work with the functional derivative with respect to the free correlation function [4]. In the ordered phase, however, where the symmetry is spontaneously broken by a nonzero field expectation value, the situation is more complicated as the energy functional also contains odd powers of the field. To handle these, it is necessary to extend the symmetric treatment by a second type of functional derivative. This was first done in Ref. [5] using functional derivatives with respect to both the free correlation function and the external current by keeping the number of derivatives at a minimum. The procedure led to *two* coupled nonlinear graphical recursion relations for each of the connected and the one-particle irreducible vacuum diagrams, respectively. In this paper, we show that all these vacuum diagrams can be obtained from a *single* nonlinear graphical recursion relation which is derived via functional derivatives with respect to both the free correlation function and the 3-vertex. Thus, we obtain a result which is relevant for the calculation of universal amplitude ratios [6,7] by an additive renormalization of the vacuum energy above [8,9] and below the critical point. As is explained, for instance, in Ref. [10], this calculation can be performed by evaluating the one-particle irreducible vacuum diagrams of the effective energy with the help of the background method.

2. Negative free energy

Consider a self-interacting scalar field ϕ with N components in d euclidean dimensions whose thermal fluctuations are controlled by the energy functional

$$E[\phi] = E[0] - \int_1 J_1\phi_1 + \frac{1}{2}\int_{12} G_{12}^{-1}\phi_1\phi_2 + \frac{1}{6}\int_{123} K_{123}\phi_1\phi_2\phi_3 + \frac{1}{24}\int_{1234} L_{1234}\phi_1\phi_2\phi_3\phi_4 \,. \tag{1}$$

In this short-hand notation, the spatial and tensorial arguments of the field ϕ, the current J, the bilocal kernel G^{-1}, as well as the cubic and the quartic interactions K and L are indicated by simple number indices, i.e.,

$$1 \equiv \{x_1,\alpha_1\}, \quad \int_1 \equiv \sum_{\alpha_1}\int d^d x_1, \quad \phi_1 \equiv \phi_{\alpha_1}(x_1)\,,$$
$$J_1 \equiv J_{\alpha_1}(x_1), \quad G_{12}^{-1} \equiv G_{\alpha_1,\alpha_2}^{-1}(x_1,x_2), \quad K_{123} \equiv K_{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3)\,,$$
$$L_{1234} \equiv L_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x_1,x_2,x_3,x_4)\,. \tag{2}$$

The kernel is a functional matrix G^{-1}, while K and L are functional tensors, all being symmetric in their respective indices. The energy functional (1) describes generically d-dimensional euclidean ϕ^4-theories. These are models for a family of universality classes of continuous phase transitions, such as the $O(N)$-symmetric ϕ^4-theory, which serves to derive the critical phenomena in dilute polymer solutions ($N=0$), Ising- and Heisenberg-like magnets ($N=1,3$), and superfluids ($N=2$). In the disordered phase above the critical point, where the system displays the full $O(N)$ symmetry and the field expectation value vanishes, the energy functional (1) consists of even powers of the field and is specified by

$$
\begin{aligned}
&E[0]=0\,,\\
&J_{\alpha_1}(x_1)=0\,,\\
&G^{-1}_{\alpha_1,\alpha_2}(x_1,x_2)=\delta_{\alpha_1,\alpha_2}(-\partial^2_{x_1}+m^2)\delta(x_1-x_2)\,,\\
&K_{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3)=0\,,\\
&L_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x_1,x_2,x_3,x_4)=\frac{g}{3}\{\delta_{\alpha_1,\alpha_2}\delta_{\alpha_3,\alpha_4}+\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4}+\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}\}\\
&\qquad\times\delta(x_1-x_2)\delta(x_1-x_3)\delta(x_1-x_4)\,.
\end{aligned}
\tag{3}
$$

The bare mass m^2 is proportional to the temperature distance from the critical point, and g is the coupling strength. In the ordered phase below the critical point, where the symmetry is spontaneously broken by a nonzero field expectation value, one has to allow also for odd powers of the field. This situation is modelled by the energy functional (1) and (3) if an additional shift of the field ϕ around some background field χ is taken into account according to the replacement $\phi\to\chi+\phi$ (cf. [10, Section 5.3]). Thus the energy functional (1) is specified by

$$
\begin{aligned}
&E[0]=\frac{1}{2}\sum_{\alpha_1}\int d^dx_1\chi_{\alpha_1}(x_1)(-\partial^2_{x_1}+m^2)\chi_{\alpha_1}(x_1)+\frac{g}{24}\sum_{\alpha_1,\alpha_2}\int d^dx_1\chi^2_{\alpha_1}(x_1)\chi^2_{\alpha_2}(x_1)\,,\\
&J_{\alpha_1}(x_1)=-(-\partial^2_{x_1}+m^2)\chi_{\alpha_1}(x_1)-\frac{g}{6}\chi_{\alpha_1}(x_1)\sum_{\alpha_2}\chi^2_{\alpha_2}(x_1)\,,\\
&G^{-1}_{\alpha_1,\alpha_2}(x_1,x_2)=\left\{\delta_{\alpha_1,\alpha_2}\left(-\partial^2_{x_1}+m^2+\frac{g}{6}\sum_{\alpha_3}\chi^2_{\alpha_3}(x_1)\right)\right.\\
&\qquad\left.+\frac{g}{3}\chi_{\alpha_1}(x_1)\chi_{\alpha_2}(x_1)\right\}\delta(x_1-x_2)\,,\\
&K_{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3)=\frac{g}{3}\{\delta_{\alpha_1,\alpha_2}\chi_{\alpha_3}(x_1)+\delta_{\alpha_1,\alpha_3}\chi_{\alpha_2}(x_1)+\delta_{\alpha_2,\alpha_3}\chi_{\alpha_1}(x_1)\}\delta(x_1-x_2)\\
&\qquad\times\delta(x_1-x_3)\,,\\
&L_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x_1,x_2,x_3,x_4)=\frac{g}{3}\{\delta_{\alpha_1,\alpha_2}\delta_{\alpha_3,\alpha_4}+\delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4}+\delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}\}\\
&\qquad\times\delta(x_1-x_2)\delta(x_1-x_3)\delta(x_1-x_4)\,.
\end{aligned}
\tag{4}
$$

In the following, we shall leave J, G^{-1}, K, and L completely general, except for the symmetry with respect to their indices, and insert the physical values (3) or (4) only at the end when we are looking at the disordered or the ordered phase, respectively. By doing so we regard the energy (1) as a functional of its arguments J, G^{-1}, K, L, i.e.,

$$E[\phi] = E[\phi, J, G^{-1}, K, L] , \tag{5}$$

so that the same functional dependences are inherited by all field-theoretic quantities derived from it. In particular, we are interested in studying the dependence of the partition function, which is determined as a functional integral over a Boltzmann weight in natural units

$$Z[J, G^{-1}, K, L] = \int \mathscr{D}\phi \, \mathrm{e}^{-E[\phi, J, G^{-1}, K, L]} \tag{6}$$

and its logarithm, the negative free energy

$$W[J, G^{-1}, K, L] = \ln \, Z[J, G^{-1}, K, L] . \tag{7}$$

By performing a loop expansion of the partition function (6), the contributions to the negative free energy (7) consist of all connected vacuum diagrams constructed according to Feynman rules. A single dot represents the energy shift

$$\bullet \equiv -E[0] \tag{8}$$

a cross an integral over the current

$$\times\!\!\!- \equiv \int_1 J_1 \tag{9}$$

and a line represents the free correlation function

$$1 \;\text{——}\; 2 \;\equiv\; G_{12} , \tag{10}$$

which is the functional inverse of the kernel G^{-1} in the energy functional (1), defined by

$$\int_2 G_{12} G_{23}^{-1} = \delta_{13} . \tag{11}$$

A 3-vertex represents an integral over the cubic interaction

$$\equiv - \int_{123} K_{123} \tag{12}$$

and a 4-vertex stands for an integral over the quartic interaction

$$\equiv - \int_{1234} L_{1234} . \tag{13}$$

If the cubic and the quartic interactions K and L in (1) vanish, the functional integral in (6) is Gaussian and can be immediately calculated to obtain for the negative free energy

$$W^{(0)}[J, G^{-1}, 0, 0] = -E[0] - \frac{1}{2} \operatorname{Tr} \ln G^{-1} + \frac{1}{2} \int_{12} G_{12} J_1 J_2 , \tag{14}$$

where the trace of the logarithm of the kernel is defined by the series [10, p. 16]

$$\mathrm{Tr}\ln G^{-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_{1\ldots n} \{G^{-1}_{12} - \delta_{12}\} \cdots \{G^{-1}_{n1} - \delta_{n1}\} \, . \tag{15}$$

The zeroth-order contribution (14) to the negative free energy will be graphically represented by

$$W^{(0)} = \bullet + \frac{1}{2} \bigcirc + \frac{1}{2} \times\!\!\!-\!\!\!-\!\!\!\times \, . \tag{16}$$

In order to find the connected vacuum diagrams of the negative free energy together with their weights for nonvanishing cubic and quartic interactions K and L, we proceed as follows. We first introduce, in Section 2.1, functional derivatives with respect to the graphical elements J, G^{-1}, K, L of Feynman diagrams. With these we derive, in Section 2.2, a single nonlinear functional differential equation for the negative free energy. This is converted into a graphical relation in Section 2.3 which is solved graphically in Section 2.4.

2.1. Functional derivatives

Each Feynman diagram may be considered as a functional of the quantities in (1) characterizing the field theory, i.e., of the current J, the kernel G^{-1}, and the interactions K and L. In this section, we introduce functional derivatives with respect to these, identify their associated graphical representations, and study field-theoretic relations between them.

2.1.1. Graphical representation

We start with studying the functional derivative with respect to the current J, whose basic rule is

$$\frac{\delta J_2}{\delta J_1} = \delta_{12} \, . \tag{17}$$

We represent this graphically by extending the elements of Feynman diagrams by an open dot with two labelled line ends representing the delta function:

$$1 \,-\!\circ\!-\, 2 = \delta_{12} \, . \tag{18}$$

Thus, we can write differentiation (17) graphically as

$$\frac{\delta}{\delta \times\!\!-\!\!-\, 1} \times\!\!-\!\!-\!\!-\, 2 \;=\; 1 \,-\!\circ\!-\, 2 \, . \tag{19}$$

Differentiating a cross with respect to the current replaces the cross by the spatial index of the current.

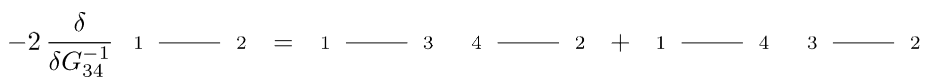
Since ϕ is a real scalar field, the kernel G^{-1} is a symmetric functional matrix. This property has to be taken into account when performing functional derivatives with respect to the kernel G^{-1}, whose basic rule is [4,5]

$$\frac{\delta G^{-1}_{12}}{\delta G^{-1}_{34}} = \frac{1}{2} \{\delta_{13}\delta_{42} + \delta_{14}\delta_{32}\} \, . \tag{20}$$

From identity (11) and the functional chain rule, we find the effect of this derivative on the free propagator

$$-2\frac{\delta G_{12}}{\delta G_{34}^{-1}} = G_{13}G_{42} + G_{14}G_{32} \ . \tag{21}$$

This has the graphical representation



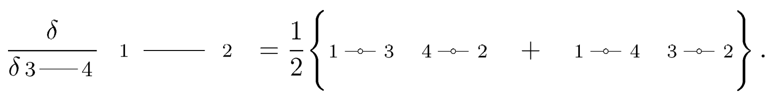
(22)

Thus, differentiating a free correlation function with respect to the kernel G^{-1} amounts to cutting the associated line into two pieces. The differentiation rule (20) ensures that the spatial indices of the kernel are symmetrically attached to the newly created line ends in the two possible ways. Differentiating a general Feynman diagram with respect to G^{-1}, the product rule of functional differentiation leads to diagrams in each of which one of the lines of the original Feynman diagram is cut.

We now study the graphical effect of functional derivatives with respect to the free propagator G, where the basic differentiation rule reads

$$\frac{\delta G_{12}}{\delta G_{34}} = \frac{1}{2}\left\{\delta_{13}\delta_{42} + \delta_{14}\delta_{32}\right\} \ . \tag{23}$$

This can be written graphically as follows:



(24)

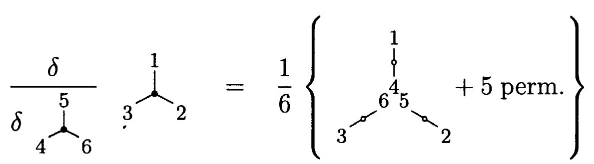
Thus differentiating a line with respect to the free correlation function removes the line, leaving in a symmetrized way the spatial indices of the free correlation function on the vertices to which the line was connected.

As the interactions K and L are functional tensors which are symmetric in their respective indices, their functional derivatives are

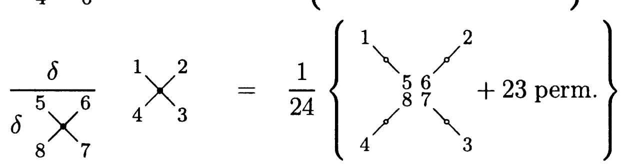
$$\frac{\delta K_{123}}{\delta K_{456}} = \frac{1}{6}\left\{\delta_{14}\delta_{25}\delta_{36} + 5 \text{ perm.}\right\} \ , \tag{25}$$

$$\frac{\delta L_{1234}}{\delta L_{5678}} = \frac{1}{24}\left\{\delta_{15}\delta_{26}\delta_{37}\delta_{48} + 23 \text{ perm.}\right\} \ . \tag{26}$$

They have the graphical representations



(27)



(28)

Thus, differentiating a 3- or a 4-vertex with respect to the cubic or the quartic interaction removes this vertex, leaving in a symmetrized way the spatial indices of the interaction on the line ends to which the vertex was connected.

2.1.2. Compatibility relations

The functional derivative of the energy functional (1) with respect to the current J_1 gives the field ϕ_1:

$$\phi_1 = -\frac{\delta E[\phi]}{\delta J_1} \, . \tag{29}$$

Products of fields, on the other hand, can be obtained in various ways from functional derivatives of the energy functional (1)

$$\phi_1\phi_2 = \frac{\delta^2 E[\phi]}{\delta J_1 \delta J_2} = 2\frac{\delta E[\phi]}{\delta G_{12}^{-1}} \, , \tag{30}$$

$$\phi_1\phi_2\phi_3 = -\frac{\delta^3 E[\phi]}{\delta J_1 \delta J_2 \delta J_3} = -2\frac{\delta^2 E[\phi]}{\delta G_{12}^{-1}\delta J_3} = 6\frac{\delta E[\phi]}{\delta K_{123}} \, , \tag{31}$$

$$\phi_1\phi_2\phi_3\phi_4 = \frac{\delta^4 E[\phi]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} = 2\frac{\delta^3 E[\phi]}{\delta G_{12}^{-1}\delta J_3 \delta J_4} = -6\frac{\delta^2 E[\phi]}{\delta K_{123}\delta J_4}$$

$$= 4\frac{\delta^2 E[\phi]}{\delta G_{12}^{-1}\delta G_{34}^{-1}} = 24\frac{\delta E[\phi]}{\delta L_{1234}} \, , \tag{32}$$

as follows from (17), (20), (25) and (26). Applying these rules to the functional integral of the partition function (6), we obtain for the functional derivatives of the negative free energy (7) various compatibility relations, for instance

$$\frac{\delta W}{\delta G_{12}^{-1}} = -\frac{1}{2}\left\{\frac{\delta^2 W}{\delta J_1 \delta J_2} + \frac{\delta W}{\delta J_1}\frac{\delta W}{\delta J_2}\right\} \, , \tag{33}$$

$$\frac{\delta W}{\delta K_{123}} = \frac{1}{3}\left\{\frac{\delta^2 W}{\delta G_{12}^{-1}\delta J_3} + \frac{\delta W}{\delta G_{12}^{-1}}\frac{\delta W}{\delta J_3}\right\} \, , \tag{34}$$

$$\frac{\delta W}{\delta L_{1234}} = -\frac{1}{6}\left\{\frac{\delta^2 W}{\delta G_{12}^{-1}\delta G_{34}^{-1}} + \frac{\delta W}{\delta G_{12}^{-1}}\frac{\delta W}{\delta G_{34}^{-1}}\right\} \, . \tag{35}$$

In the following it will turn out that the compatibility relation (34) is crucial for deriving a single functional differential equation for W.

2.2. Functional differential equation for $W = \ln Z$

We start from the identity

$$\int \mathscr{D}\phi \frac{\delta}{\delta \phi_1}\{\phi_2 \mathrm{e}^{-E[\phi]}\} = 0 \, , \tag{36}$$

which follows by direct functional integration from the vanishing of the exponential at infinite fields. Taking into account the explicit form of the energy functional (1), we perform the functional derivative with respect to the field and obtain

$$\int \mathscr{D}\phi \left\{ \delta_{12} + J_1\phi_2 - \int_3 G_{13}^{-1}\phi_2\phi_3 \right. \\ \left. - \frac{1}{2}\int_{34} K_{134}\phi_2\phi_3\phi_4 - \frac{1}{6}\int_{345} L_{1345}\phi_2\phi_3\phi_4\phi_5 \right\} \mathrm{e}^{-E[\phi]} = 0\ . \tag{37}$$

Substituting field products by functional derivatives according to (29)–(32), we keep the number of these derivatives at a minimum and express the resulting equation in terms of the partition function (6):

$$\delta_{12}Z + J_1\frac{\delta Z}{\delta J_2} + 2\int_3 G_{13}^{-1}\frac{\delta Z}{\delta G_{23}^{-1}} + 3\int_{34} K_{134}\frac{\delta Z}{\delta K_{123}} + 4\int_{345} L_{1345}\frac{\delta Z}{\delta L_{2345}} = 0\ . \tag{38}$$

Going over from Z to $W = \ln Z$, we obtain the linear functional differential equation

$$\delta_{12} + J_1\frac{\delta W}{\delta J_2} + 2\int_3 G_{13}^{-1}\frac{\delta W}{\delta G_{23}^{-1}} + 3\int_{34} K_{134}\frac{\delta W}{\delta K_{123}} + 4\int_{345} L_{1345}\frac{\delta W}{\delta L_{2345}} = 0\ . \tag{39}$$

Applying the compatibility relations (34) and (35), this linear functional differential equation turns into the nonlinear one

$$\delta_{12} + J_1\frac{\delta W}{\delta J_2} + 2\int_3 G_{13}^{-1}\frac{\delta W}{\delta G_{23}^{-1}} \\ = -\int_{34} K_{134}\left\{ \frac{\delta^2 W}{\delta G_{23}^{-1}\delta J_4} + \frac{\delta W}{\delta G_{23}^{-1}}\frac{\delta W}{\delta J_4} \right\} \\ + \frac{2}{3}\int_{345} L_{1345}\left\{ \frac{\delta^2 W}{\delta G_{23}^{-1}\delta G_{45}^{-1}} + \frac{\delta W}{\delta G_{23}^{-1}}\frac{\delta W}{\delta G_{45}^{-1}} \right\}\ , \tag{40}$$

which is identical with Eq. (55) in Ref. [5]. In order to eliminate functional derivatives with respect to the current J, we consider the second identity

$$\int \mathscr{D}\phi \frac{\delta}{\delta\phi_1}\mathrm{e}^{-E[\phi]} = 0\ , \tag{41}$$

which leads to

$$\int \mathscr{D}\phi \left\{ -J_1 + \int_2 G_{12}^{-1}\phi_2 + \frac{1}{2}\int_{23} K_{123}\phi_2\phi_3 + \frac{1}{6}\int_{234} L_{1234}\phi_2\phi_3\phi_4 \right\} \mathrm{e}^{-E[\phi]} \\ = 0\ . \tag{42}$$

Applying again the substitution rules (29)–(31) while keeping the number of functional derivatives at a minimum, and taking into account the partition function (6), we obtain for the negative free energy $W = \ln Z$:

$$\frac{\delta W}{\delta J_1} = \int_2 G_{12} J_2 + \int_{234} K_{234} G_{12} \frac{\delta W}{\delta G_{34}^{-1}} + \int_{2345} L_{2345} G_{12} \frac{\delta W}{\delta K_{345}} \, . \tag{43}$$

Differentiatin

Note that due to (34) the functional derivative with respect to the cubic interaction K in (43) is compatible with functional derivatives with respect to the current J and the kernel G^{-1}. Inserting (34) in (43) would lead to Eq. (54) in Ref. [5],

$$\frac{\delta W}{\delta J_1} = \int_2 G_{12} J_2 + \int_{234} K_{234} G_{12} \frac{\delta W}{\delta G_{34}^{-1}} + \frac{1}{3} \int_{2345} L_{2345} G_{12} \left\{ \frac{\delta^2 W}{\delta G_{12}^{-1} \delta J_3} + \frac{\delta W}{\delta G_{12}^{-1}} \frac{\delta W}{\delta J_3} \right\} , \tag{46}$$

such that functional derivatives with respect to the current J in Eq. (40) can no longer be eliminated. This line of approach has been pursued in Ref. [5] where the two coupled nonlinear differential equations (40) and (46) for the negative free energy are used for deriving all connected vacuum diagrams.

2.3. Graphical relation

With the help of the functional chain rule, the first and second derivatives with respect to the kernel G^{-1} are rewritten as

$$\frac{\delta}{\delta G_{12}^{-1}} = - \int_{34} G_{13} G_{24} \frac{\delta}{\delta G_{34}} \tag{47}$$

and

$$\frac{\delta^2}{\delta G_{12}^{-1} \delta G_{34}^{-1}} = \int_{5678} G_{15} G_{26} G_{37} G_{48} \frac{\delta^2}{\delta G_{56} \delta G_{78}} + \frac{1}{2} \int_{56} \{ G_{13} G_{25} G_{46} + G_{14} G_{25} G_{36} + G_{23} G_{15} G_{46} + G_{24} G_{15} G_{36} \} \frac{\delta}{\delta G_{56}} , \tag{48}$$

respectively. The functional differential equation (45) for W takes then the form

$$\delta_{11} \int_1 + \int_{12} G_{12} J_1 J_2 - 2 \int_{12} G_{12} \frac{\delta W}{\delta G_{12}} = \int_{1234} K_{123} G_{12} G_{34} J_4 + 2 \int_{123456} K_{123} G_{14} G_{25} G_{36} J_4 \frac{\delta W}{\delta G_{56}} - \int_{12345} L_{1234} G_{15} J_5 \frac{\delta W}{\delta K_{234}}$$

$$- 2 \int_{12345678} K_{123} K_{456} G_{14} G_{25} G_{37} G_{68} \frac{\delta W}{\delta G_{78}}$$

$$- \int_{12345678} K_{123} K_{456} G_{12} G_{34} G_{57} G_{68} \frac{\delta W}{\delta G_{78}}$$

$$- \int_{123456789\bar{1}} K_{123} K_{456} G_{14} G_{27} G_{38} G_{59} G_{6\bar{1}} \left\{ \frac{\delta^2 W}{\delta G_{78} \delta G_{9\bar{1}}} + \frac{\delta W}{\delta G_{78}} \frac{\delta W}{\delta G_{9\bar{1}}} \right\}$$

$$+ \int_{1234567} K_{123} L_{4567} G_{12} G_{34} \frac{\delta W}{\delta K_{567}}$$

$$+ \int_{123456789} K_{123} K_{4567} G_{14} G_{27} G_{38} \left\{ \frac{\delta^2 W}{\delta K_{567} \delta G_{89}} + \frac{\delta W}{\delta K_{567}} \frac{\delta W}{\delta G_{89}} \right\}$$

$$+ \frac{4}{3} \int_{123456} L_{1234} G_{12} G_{35} G_{46} \frac{\delta W}{\delta G_{56}}$$

$$+ \frac{2}{3} \int_{12345678} L_{1234} G_{15} G_{26} G_{37} G_{48} \left\{ \frac{\delta^2 W}{\delta G_{56} \delta G_{78}} + \frac{\delta W}{\delta G_{56}} \frac{\delta W}{\delta G_{78}} \right\} . \tag{49}$$

If the cubic and the quartic interactions K and L vanish, Eq. (49) is solved by the zeroth-order contribution to the negative free energy (14) which has the functional derivatives

$$\frac{\delta W^{(0)}}{\delta J_1} = \int_2 G_{12} J_2 , \qquad \frac{\delta W^{(0)}}{\delta G_{12}} = \frac{1}{2} G_{12}^{-1} + \frac{1}{2} J_1 J_2 ,$$

$$\frac{\delta^2 W^{(0)}}{\delta G_{12} \delta G_{34}} = - \frac{1}{4} \left\{ G_{13}^{-1} G_{24}^{-1} + G_{14}^{-1} G_{23}^{-1} \right\} . \tag{50}$$

For nonvanishing cubic and quartic interactions K and L, the right-hand side in Eq. (49) produces corrections to (14) which we shall denote by $W^{(\mathrm{int})}$. Thus, the negative free energy W decomposes according to

$$W = W^{(0)} + W^{(\mathrm{int})} . \tag{51}$$

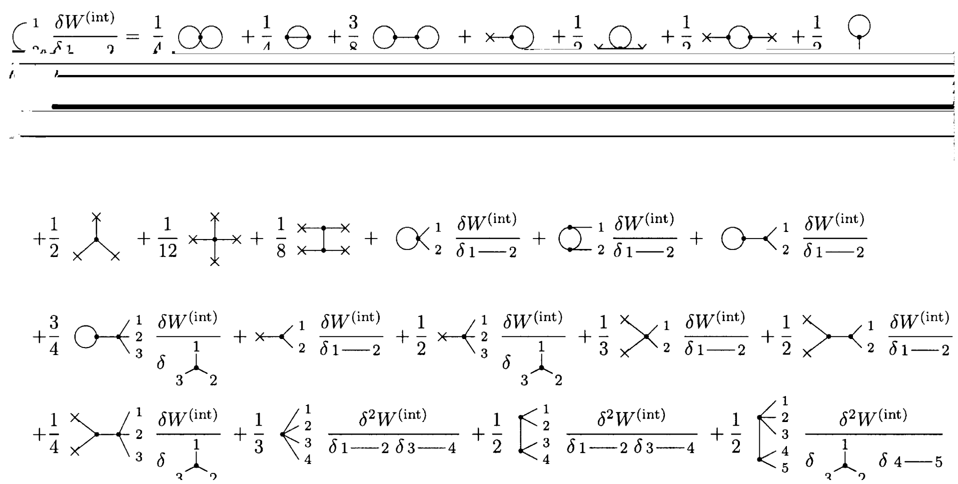
Inserting this into (49) and using (50), we obtain the following functional differential equation for the interaction negative free energy $W^{(\mathrm{int})}$:

$$\int_{12} G_{12} \frac{\delta W^{(\mathrm{int})}}{\delta G_{12}}$$

$$= - \frac{1}{4} \int_{1234} L_{1234} G_{12} G_{34} + \frac{1}{4} \int_{123456} K_{123} K_{456} G_{14} G_{25} G_{36}$$

$$
\begin{aligned}
&+\frac{3}{8}\int_{123456} K_{123}K_{456}G_{14}G_{23}G_{56} - \int_{1234} K_{123}G_{12}G_{34}J_4 \\
&-\frac{1}{2}\int_{123456} L_{1234}G_{12}G_{35}G_{46}J_5J_6 + \frac{1}{2}\int_{12345678} K_{123}K_{456}G_{14}G_{25}G_{37}G_{68}J_7J_8 \\
&+\frac{1}{2}\int_{12345678} K_{123}K_{456}G_{12}G_{34}G_{57}G_{68}J_7J_8 - \frac{1}{2}\int_{123456} K_{123}G_{14}G_{25}G_{36}J_4J_5J_6 \\
&-\frac{1}{2}\int_{12345678} L_{1234}G_{15}G_{26}G_{37}G_{48}J_5J_6J_7J_8 \\
&+\frac{1}{8}\int_{123456789\bar{1}} K_{123}K_{456}G_{14}G_{27}G_{38}G_{59}G_{6\bar{1}}J_7J_8J_9J_{\bar{1}} \\
&-\int_{123456} L_{1234}G_{12}G_{35}G_{46}\frac{\delta W^{(\mathrm{int})}}{\delta G_{56}} + \int_{12345678} K_{123}K_{456}G_{14}G_{25}G_{37}G_{48}\frac{\delta W^{(\mathrm{int})}}{\delta G_{78}} \\
&+\int_{12345678} K_{123}K_{456}G_{12}G_{34}G_{57}G_{68}\frac{\delta W^{(\mathrm{int})}}{\delta G_{78}} \\
&-\frac{3}{4}\int_{1234567} K_{123}L_{4567}G_{12}G_{34}\frac{\delta W^{(\mathrm{int})}}{\delta K_{567}} \\
&-\int_{123456} K_{123}G_{14}G_{25}G_{36}J_4\frac{\delta W^{(\mathrm{int})}}{\delta G_{56}} + \frac{1}{2}\int_{12345} L_{1234}G_{15}J_5\frac{\delta W^{(\mathrm{int})}}{\delta K_{234}} \\
&+\frac{1}{3}\int_{12345678} L_{1234}G_{15}G_{26}G_{37}G_{48}J_5J_6\frac{\delta W^{(\mathrm{int})}}{\delta G_{78}} \\
&+\frac{1}{2}\int_{123456789\bar{1}} K_{123}K_{456}G_{14}G_{27}G_{38}G_{59}G_{6\bar{1}}J_7J_8\frac{\delta W^{(\mathrm{int})}}{\delta G_{9\bar{1}}} \\
&-\frac{1}{4}\int_{123456789} K_{123}L_{4567}G_{14}G_{27}G_{38}J_7J_8\frac{\delta W^{(\mathrm{int})}}{\delta K_{567}} \\
&-\frac{1}{3}\int_{12345678} L_{1234}G_{15}G_{26}G_{37}G_{48}\left\{\frac{\delta^2 W^{(\mathrm{int})}}{\delta G_{56}\delta G_{78}} + \frac{\delta W^{(\mathrm{int})}}{\delta G_{56}}\frac{\delta W^{(\mathrm{int})}}{\delta G_{78}}\right\} \\
&+\frac{1}{2}\int_{123456789\bar{1}} K_{123}K_{456}G_{14}G_{27}G_{38}G_{59}G_{6\bar{1}}\left\{\frac{\delta^2 W^{(\mathrm{int})}}{\delta G_{78}\delta G_{9\bar{1}}} + \frac{\delta W^{(\mathrm{int})}}{\delta G_{78}}\frac{\delta W^{(\mathrm{int})}}{\delta G_{9\bar{1}}}\right\} \\
&-\frac{1}{2}\int_{123456789} K_{123}L_{4567}G_{14}G_{27}G_{38}\left\{\frac{\delta^2 W^{(\mathrm{int})}}{\delta K_{567}\delta G_{89}} + \frac{\delta W^{(\mathrm{int})}}{\delta K_{567}}\frac{\delta W^{(\mathrm{int})}}{\delta G_{89}}\right\} .
\end{aligned}
\tag{52}
$$

With the help of the graphical rules (9), (10), (12), (13), this can be written diagrammatically as follows:



(53)

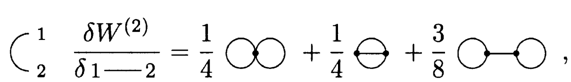
The effect of the term on the left-hand side is to count the number of lines of each connected vacuum diagram. Indeed, the functional derivative $\delta/\delta G_{12}$ removes successively the lines which are, subsequently, reinserted by the operation $\int_{12} G_{12}$. The right-hand side contains altogether 25 terms, 10 without $W^{(\text{int})}$, 12 linear in $W^{(\text{int})}$ and 3 bilinear in $W^{(\text{int})}$.

2.4. Loopwise recursive graphical solution

Now we show how Eq. (53) is solved graphically for the current-free connected vacuum diagrams. To this end we expand the interaction negative free energy $W^{(\text{int})}$ with respect to the loop order l

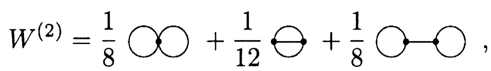
$$W^{(\text{int})} = \sum_{l=2}^{\infty} W^{(l)} \,. \tag{54}$$

With the help of (54) we convert Eq. (53) into a recursive graphical solution for the current-free expansion coefficients $W^{(l)}$. For $l = 2$, Eq. (53) reduces to



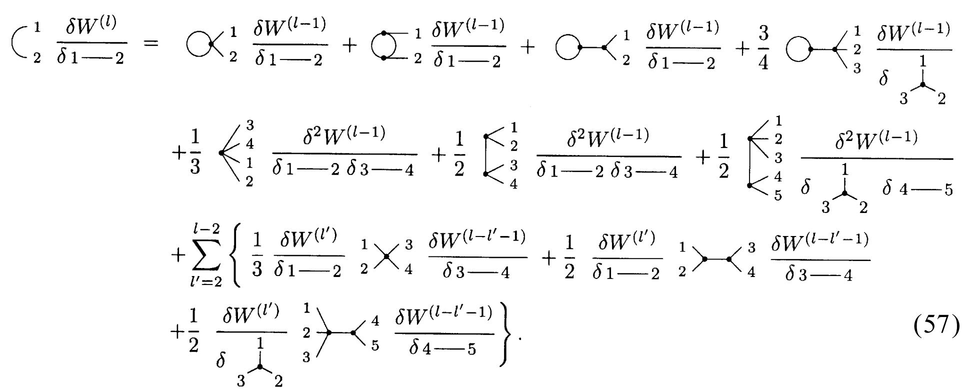
(55)

which is immediately solved by



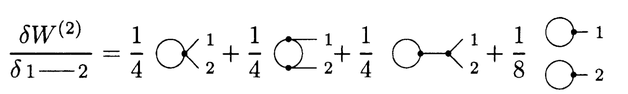
(56)

as the first vacuum diagram contains 2 and the last two vacuum diagrams 3 lines. For $l \geq 3$ we obtain the graphical recursion relation



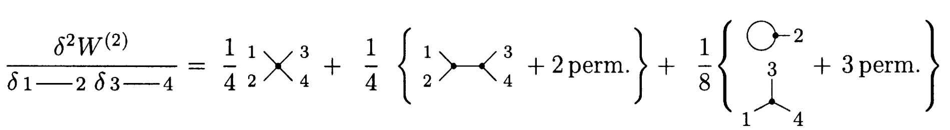
(57)

We observe that for either a vanishing cubic or quartic interaction the graphical recursion relation (57) only involves the graphical operation of removing lines. Proceeding to the loop order $l=3$, we have to evaluate from the vacuum diagrams (56) a one-line amputation



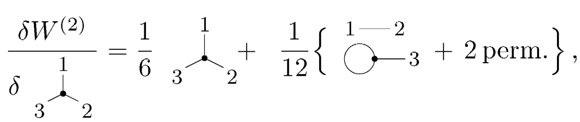
(58)

a two-line amputation



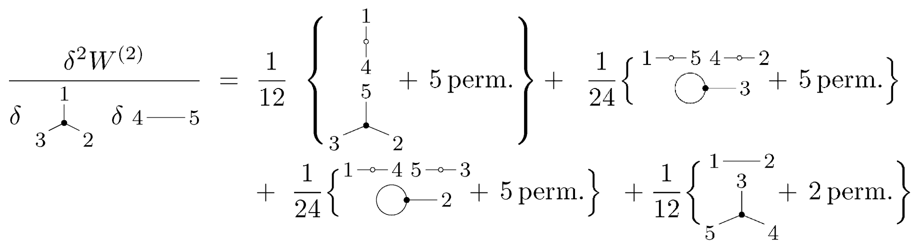
(59)

a 3-vertex amputation



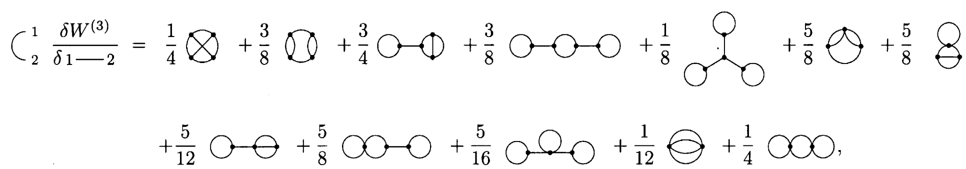
(60)

and the amputation of one line and one 3-vertex



(61)

Then we obtain from Eq. (57) for $l = 3$:



(62)

which leads to the connected vacuum diagrams listed in Table 1 together with the subsequent loop order $l = 4$. In a similar way, the graphical relation (53) can be recursively iterated to construct the connected vacuum diagrams which involve currents.

The topology of each connected diagram in Table 1 can be characterized by the 5 component vector $(S, D, T, F; N)$. Here S, D, T, F denote the number of self-, double, triple and fourfold connections, whereas N stands for the number of identical vertex permutations where the 3- and 4-vertices as well as the currents remain attached to the lines emerging from them in the same way as before. The proper weights of the connected vacuum diagrams in the ϕ^3–ϕ^4-theory are then given by the formula [4,11,12]

$$W = \frac{1}{2!^{S+D}3!^T 4!^F N} \, . \tag{63}$$

For higher orders, it becomes more and more difficult to identify by inspection the number N of identical vertex permutations. A mnemonic rule states that the number N of identical vertex permutations is given by twice the number of symmetry axes, if the diagram is imagined in a suitable maximally symmetric way in some higher dimensional space. A more systematic determination of N is possible by introducing a matrix notation for the diagrams as explained in detail in Refs. [4,12].

3. Effective energy

In field theory one is often interested in the functional Legendre transform of the negative free energy $W[J, G^{-1}, K, L]$ with respect to the current J [12–15]. To this end

Table 1
Connected vacuum diagrams and their weights of the ϕ^3–ϕ^4-theory without currents up to four loops

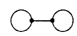
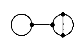
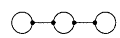
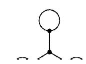
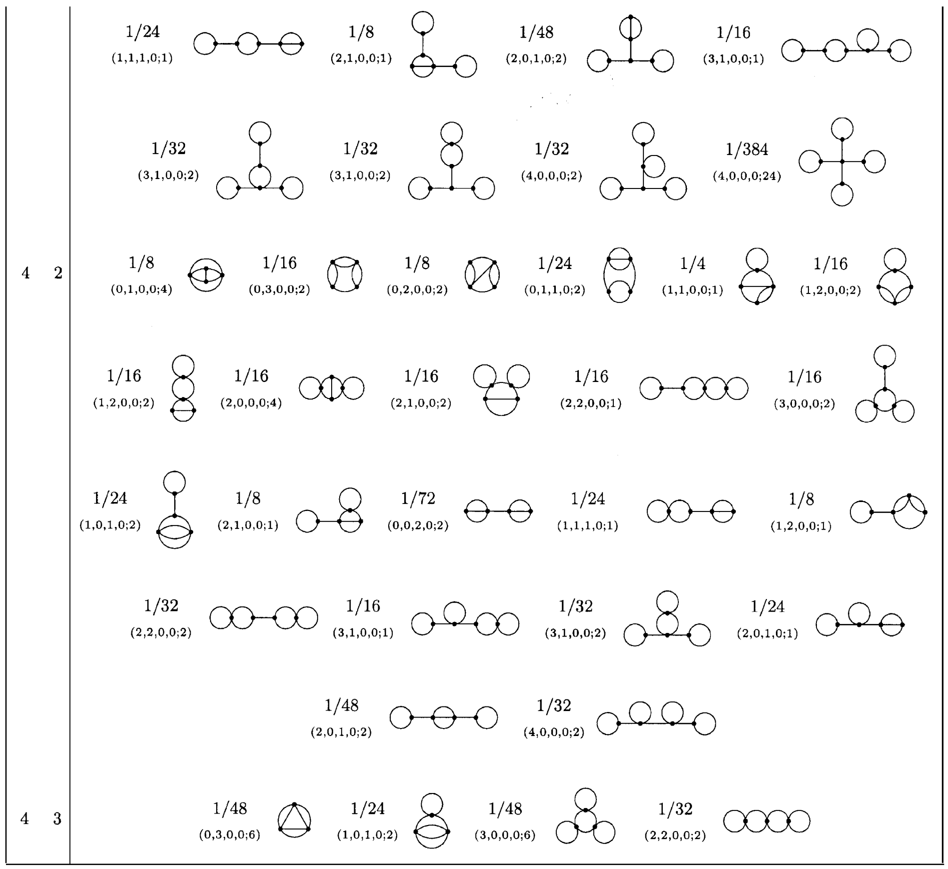
l	p	$W^{(l,p)}$
2	0	1/12 (0,0,1,0;2)  1/8 (2,0,0,0;2) 
2	1	1/8 (2,1,0,0;1) 
3	0	1/24 (0,0,0,0;24)  1/16 (0,2,0,0;4)  1/8 (1,1,0,0;2)  1/16 (2,1,0,0;2)  1/48 (3,0,0,0;6) 

Table 1. Continued



Within each loop order l the diagrams are distinguished with respect to the number p of 4-vertices. Each diagram is characterized by the vector $(S, D, T, F; N)$ whose components specify the number of self-, double, triple, fourfold connections, and of the identical vertex permutations, respectively.

one introduces the new field

$$\Phi_1[J, G^{-1}, K, L] = \left.\frac{\delta W[J, G^{-1}, K, L]}{\delta J_1}\right|_{G^{-1}, K, L}, \tag{64}$$

which implicitly defines J as a functional of Φ:

$$J_1 = J_1[\Phi, G^{-1}, K, L]\,. \tag{65}$$

From the Eqs. (6) and (7) we read off that Φ coincides with the 1-point function of the theory, i.e., the field expectation value of the fluctuations around the background field χ in Eq. (4) in the presence of the current J. The functional Legendre transform of the negative free energy $W[J, G^{-1}, K, L]$ with respect to the current J results in the effective energy

$$\Gamma[\Phi, G^{-1}, K, L] = \int_1 J_1[\Phi, G^{-1}, K, L] \left. \frac{\delta W[J[\Phi, G^{-1}, K, L], G^{-1}, K, L]}{\delta J_1[\Phi, G^{-1}, K, L]} \right|_{G^{-1}, K, L} - W[J[\Phi, G^{-1}, K, L], G^{-1}, K, L] \, , \tag{66}$$

which simplifies due to (64):

$$\Gamma[\Phi, G^{-1}, K, L] = \int_1 J_1[\Phi, G^{-1}, K, L]\Phi_1 - W[J[\Phi, G^{-1}, K, L], G^{-1}, K, L] \, . \tag{67}$$

Taking into account the functional chain rule, it leads to the equation of state

$$\left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L]}{\delta \Phi_1} \right|_{G^{-1}, K, L} = J_1[\Phi, G^{-1}, K, L] \, . \tag{68}$$

Performing a loop expansion, the respective contributions to the effective energy (67) may be displayed as one-particle irreducible vacuum diagrams which are constructed according to the Feynman rules (8), (10), (12), (13). In addition a dot with a wiggled line represents an integral over the field expectation value

 $\equiv \int_1 \Phi_1 \, .$ (69)

For instance, if the cubic and the quartic interactions K and L vanish, the zeroth order contribution to the negative free energy (14) leads with (64) to the field expectation value

$$\Phi_1^{(0)}[J, G^{-1}, 0, 0] = \int_2 G_{12} J_2 \, , \tag{70}$$

which is inverted as

$$J_1^0[\Phi, G^{-1}, 0, 0] = \int_2 G_{12}^{-1} \Phi_2 \tag{71}$$

to result in the zeroth-order contribution to the effective energy

$$\Gamma^{(0)}[\Phi, G^{-1}, 0, 0] = E[0] + \frac{1}{2} \operatorname{Tr} \ln G^{-1} + \frac{1}{2} \int_{12} G_{12}^{-1} \Phi_1 \Phi_2 \, . \tag{72}$$

Its graphical representation reads by definition

$-\Gamma^{(0)} =$ $+ \frac{1}{2}$  $+ \frac{1}{2}$. (73)

In order to find the one-particle irreducible vacuum diagrams of the effective energy together with their weights for nonvanishing cubic and quartic interactions K and L, we proceed as follows. We start in Section 3.1 with investigating the consequences of the functional Legendre transform with respect to the current for functional derivatives and their compatibility relations. With these results we derive in Section 3.2 a single nonlinear functional differential equation for the effective energy which is converted into a graphical relation in Section 3.3 and recursively solved in Section 3.4.

3.1. Functional Legendre transform

In order to investigate in detail the field-theoretic consequences of the functional Legendre transform, we start with the effective energy $\Gamma[\Phi, G^{-1}, K, L]$ and introduce the current J via the equation of state (68). As this implicitly defines the field expectation value as a functional of the current, i.e.,

$$\Phi_1 = \Phi_1[J, G^{-1}, K, L] \,, \tag{74}$$

the negative free energy is recovered according to

$$W[J, G^{-1}, K, L] = \int_1 J_1 \Phi_1[J, G^{-1}, K, L] - \Gamma[\Phi[J, G^{-1}, K, L], G^{-1}, K, L] \,. \tag{75}$$

With this we derive useful relations between the functional derivatives of the negative free energy W and the effective energy Γ, respectively.

3.1.1. Functional derivatives

Taking into account the functional chain rule, the first functional derivatives of the negative free energy W read (64) and

$$\left.\frac{\delta W[J, G^{-1}, K, L]}{\delta G^{-1}_{12}}\right|_{J,K,L} = -\left.\frac{\delta \Gamma[\Phi[J, G^{-1}, K, L], G^{-1}, K, L]}{\delta G^{-1}_{12}}\right|_{\Phi,K,L} \,, \tag{76}$$

$$\left.\frac{\delta W[J, G^{-1}, K, L]}{\delta K_{123}}\right|_{J,G^{-1},L} = -\left.\frac{\delta \Gamma[\Phi[J, G^{-1}, K, L], G^{-1}, K, L]}{\delta K_{123}}\right|_{\Phi,G^{-1},L} \,, \tag{77}$$

$$\left.\frac{\delta W[J, G^{-1}, K, L]}{\delta L_{1234}}\right|_{J,G^{-1},K} = -\left.\frac{\delta \Gamma[\Phi[J, G^{-1}, K, L], G^{-1}, K, L]}{\delta L_{1234}}\right|_{\Phi,G^{-1},K} \,. \tag{78}$$

To evaluate second functional derivatives of the negative free energy W is more involved. At first we observe

$$\begin{aligned}
\left.\frac{\delta^2 W[J, G^{-1}, K, L]}{\delta J_2 \delta J_1}\right|_{G^{-1},\,K,\,L} &= \left.\frac{\delta \Phi_1[J, G^{-1}, K, L]}{\delta J_2}\right|_{G^{-1},\,K,\,L} \,, \\
&= \left(\left.\frac{\delta J_2[\Phi[J, G^{-1}, K, L], G^{-1}, K, L]}{\delta \Phi_1[J, G^{-1}, K, L]}\right|_{G^{-1},\,K,\,L} \right)^{-1} \\
&= \left(\left.\frac{\delta^2 \Gamma[\Phi[J, G^{-1}, K, L], G^{-1}, K, L]}{\delta \Phi_1[J, G^{-1}, K, L] \delta \Phi_2[J, G^{-1}, K, L]}\right|_{G^{-1},\,K,\,L} \right)^{-1} \,,
\end{aligned} \tag{79}$$

where we used (64), (68) and the fact that the derivative of a functional equals the inverse of the derivative of the inverse functional. To precise the meaning of relation (79), we rederive it from another point of view. To this end we consider the functional identity

$$\left.\frac{\delta J_1[\Phi[J,G^{-1},K,L],G^{-1},K,L]}{\delta J_2}\right|_{G^{-1},\,K,\,L} = \delta_{12} \tag{80}$$

and apply the functional chain rule together with (64) and (68), so that we result in

$$\int_3 \left.\frac{\delta^2 \Gamma[\Phi[J,G^{-1},K,L],G^{-1},K,L]}{\delta\Phi_1[J,G^{-1},K,L]\delta\Phi_3[J,G^{-1},K,L]}\right|_{G^{-1},\,K,\,L} \times \left.\frac{\delta^2 W[J,G^{-1},K,L]}{\delta J_3 \delta J_2}\right|_{G^{-1},\,K,\,L} = \delta_{12}\,. \tag{81}$$

Furthermore we obtain from (76) by applying again the functional chain rule and relation (79)

$$\begin{aligned} &\frac{\delta}{\delta J_3}\left(\left.\frac{\delta W[J,G^{-1},K,L]}{\delta G^{-1}_{12}}\right|_{J,K,L}\right)_{G^{-1},K,L} \\ &\quad = -\int_4 \left(\left.\frac{\delta^2 \Gamma[\Phi[J,G^{-1},K,L],G^{-1},K,L]}{\delta\Phi_3[J,G^{-1},K,L]\delta\Phi_4[J,G^{-1},K,L]}\right|_{G^{-1},\,K,\,L}\right)^{-1} \\ &\qquad \times \frac{\delta}{\delta\Phi_4[J,G^{-1},K,L]}\left(\left.\frac{\delta\Gamma[\Phi[J,G^{-1},K,L],G^{-1},K,L]}{\delta G^{-1}_{12}}\right|_{\Phi,K,L}\right)_{G^{-1},K,L} \end{aligned} \tag{82}$$

and, correspondingly,

$$\begin{aligned} &\left.\frac{\delta^2 W[J,G^{-1},K,L]}{\delta G^{-1}_{34}\delta G^{-1}_{12}}\right|_{J,K,L} \\ &\quad = -\left.\frac{\delta^2 \Gamma[\Phi[J,G^{-1},K,L],G^{-1},K,L]}{\delta G^{-1}_{34}\delta G^{-1}_{12}}\right|_{\Phi,K,L} \\ &\qquad + \int_{56} \left(\left.\frac{\delta^2 \Gamma[\Phi[J,G^{-1},K,L],G^{-1},K,L]}{\delta\Phi_5[J,G^{-1},K,L]\delta\Phi_6[J,G^{-1},K,L]}\right|_{G^{-1},\,K,\,L}\right)^{-1} \\ &\qquad \times \frac{\delta}{\delta\Phi_5[J,G^{-1},K,L]}\left(\left.\frac{\delta\Gamma[\Phi[J,G^{-1},K,L],G^{-1},K,L]}{\delta G^{-1}_{12}}\right|_{\Phi,K,L}\right)_{G^{-1},K,L} \\ &\qquad \times \frac{\delta}{\delta\Phi_6[J,G^{-1},K,L]}\left(\left.\frac{\delta\Gamma[\Phi[J,G^{-1},K,L],G^{-1},K,L]}{\delta G^{-1}_{34}}\right|_{\Phi,K,L}\right)_{G^{-1},K,L}\,. \end{aligned} \tag{83}$$

3.1.2. Compatibility relations

Performing the functional Legendre transform with respect to the current, the compatibility relation (33) between functional derivatives with respect to the current J and the kernel G^{-1}, we result in

$$\left(\left. \frac{\delta^2 \Gamma[\Phi, G^{-1}, K, L]}{\delta \Phi_2 \delta \Phi_1} \right|_{G^{-1}, K, L} \right)^{-1} = 2 \left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L]}{\delta G^{-1}_{12}} \right|_{\Phi, K, L} - \Phi_1 \Phi_2 \tag{84}$$

due to (64), (76) and (79). The compatibility relation (34) between functional derivatives with respect to the current J, the kernel G^{-1} and the 3-vertex K is converted by using (64), (76), (77) and (82) to

$$\int_4 \frac{\delta}{\delta \Phi_4} \left(\left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L]}{\delta G^{-1}_{12}} \right|_{\Phi, K, L} \right)_{G^{-1}, K, L} \left(\left. \frac{\delta^2 \Gamma[\Phi, G^{-1}, K, L]}{\delta \Phi_3 \delta \Phi_4} \right|_{G^{-1}, K, L} \right)^{-1}$$
$$= 3 \left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L]}{\delta K_{123}} \right|_{\Phi, G^{-1}, L} - \left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L]}{\delta G^{-1}_{12}} \right|_{\Phi, K, L} \Phi_3 \,. \tag{85}$$

Furthermore we perform the functional Legendre transform of the compatibility relation (35) between functional derivatives with respect to the kernel G^{-1} and the 4-vertex L which leads with (76), (78) and (83) to

$$\int_{56} \frac{\delta}{\delta \Phi_5} \left(\left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L]}{\delta G^{-1}_{12}} \right|_{\Phi, K, L} \right)_{G^{-1}, K, L} \frac{\delta}{\delta \Phi_6} \left(\left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L}{\delta G^{-1}_{34}} \right|_{\Phi, K, L} \right)_{G^{-1}, K, L}$$
$$\times \left(\left. \frac{\delta^2 \Gamma[\Phi, G^{-1}, K, L]}{\delta \Phi_5 \delta \Phi_6} \right|_{G^{-1}, K, L} \right)^{-1} = 6 \left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L]}{\delta L_{1234}} \right|_{\Phi, G^{-1}, K}$$
$$+ \left. \frac{\delta^2 \Gamma[\Phi, G^{-1}, K, L]}{\delta G^{-1}_{12} \delta G^{-1}_{34}} \right|_{\Phi, K, L} - \left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L]}{\delta G^{-1}_{12}} \right|_{\Phi, K, L} \left. \frac{\delta \Gamma[\Phi, G^{-1}, K, L]}{\delta G^{-1}_{34}} \right|_{\Phi, K, L} \,. \tag{86}$$

3.2. Functional differential equation for Γ

Now we aim at deriving a functional differential equation for the effective energy Γ. To this end we start with the first functional differential equation (40) for W, which originates from the identity (36), and perform the functional Legendre transform with respect to the current J. Inserting (64),(68), (76), (82) and (83) by taking into account the compatibility relation (84) between functional derivatives with respect to the field expectation value Φ and the kernel G^{-1}, we thus obtain

$$\delta_{12} + \Phi_1 \frac{\delta\Gamma}{\delta\Phi_2} - 2\int_3 G^{-1}_{13} \frac{\delta\Gamma}{\delta G^{-1}_{23}}$$

$$= \int_{34} K_{134} \frac{\delta\Gamma}{\delta G^{-1}_{23}} \Phi_4 + \int_{345} K_{134} \frac{\delta^2\Gamma}{\delta G^{-1}_{23}\delta\Phi_5} \left\{ 2\frac{\delta\Gamma}{\delta G^{-1}_{45}} - \Phi_4\Phi_5 \right\}$$

$$+ \frac{2}{3}\int_{345} L_{1345} \left\{ -\frac{\delta^2\Gamma}{\delta G^{-1}_{23}\delta G^{-1}_{45}} + \frac{\delta\Gamma}{\delta G^{-1}_{23}} \frac{\delta\Gamma}{\delta G^{-1}_{45}} \right\}$$

$$+ \frac{2}{3}\int_{34567} L_{1345} \frac{\delta^2\Gamma}{\delta G^{-1}_{23}\delta\Phi_6} \frac{\delta^2\Gamma}{\delta G^{-1}_{45}\delta\Phi_7} \left\{ 2\frac{\delta\Gamma}{\delta G^{-1}_{67}} - \Phi_6\Phi_7 \right\} , \qquad (87)$$

which corresponds to Eq. (109) in Ref. [5]. Then we reduce the number of functional derivatives by inserting a combination of the two compatibility relations (84) and (85), i.e.,

$$\int_4 \frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta\Phi_4} \left\{ 2\frac{\delta\Gamma}{\delta G^{-1}_{34}} - \Phi_3\Phi_4 \right\} = 3\frac{\delta\Gamma}{\delta K_{123}} - \frac{\delta\Gamma}{\delta G^{-1}_{12}}\Phi_3 \qquad (88)$$

in the last term of Eq. (87), so that we result in

$$\delta_{11}\int_1 + \int_1 \Phi_1 \frac{\delta\Gamma}{\delta\Phi_1} - 2\int_{12} G^{-1}_{12} \frac{\delta\Gamma}{\delta G^{-1}_{12}}$$

$$= \int_{123} K_{123}\Phi_3 \frac{\delta\Gamma}{\delta G^{-1}_{12}} + \int_{1234} K_{123} \frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta\Phi_4} \left\{ 2\frac{\delta\Gamma}{\delta G^{-1}_{34}} - \Phi_3\Phi_4 \right\}$$

$$+ \frac{2}{3}\int_{1234} L_{1234} \left\{ -\frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta G^{-1}_{34}} + \frac{\delta\Gamma}{\delta G^{-1}_{12}} \frac{\delta\Gamma}{\delta G^{-1}_{34}} \right\}$$

$$+ \frac{2}{3}\int_{12345} L_{1234} \frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta\Phi_5} \left\{ 3\frac{\delta\Gamma}{\delta K_{345}} - \frac{\delta\Gamma}{\delta G^{-1}_{34}}\Phi_5 \right\} . \qquad (89)$$

In order to eliminate functional derivatives with respect to the field expectation value Φ, we consider the second functional differential equation (43) for W, which stems from identity (41). Applying (64), (68), (76) and (77), we obtain

$$\frac{\delta\Gamma}{\delta\Phi_1} = \int_2 G^{-1}_{12}\Phi_2 + \int_{23} K_{123} \frac{\delta\Gamma}{\delta G^{-1}_{23}} + \int_{234} L_{1234} \frac{\delta\Gamma}{\delta K_{234}} , \qquad (90)$$

which leads to

$$\frac{\delta^2\Gamma}{\delta G^{-1}_{12}\Phi_3} = \frac{1}{2}\left\{ \delta_{13}\Phi_2 + \delta_{23}\Phi_1 \right\} + \int_{45} K_{345} \frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta G^{-1}_{45}}$$

$$+ \int_{456} L_{3456} \frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta K_{456}} . \qquad (91)$$

Thus we can, indeed, eliminate functional derivatives with respect to the field expectation value Φ on the right-hand side of Eq. (89). In this way, we end up with a single nonlinear functional differential equation for the effective energy Γ which involves on the right-hand side functional derivatives with respect to the kernel G^{-1} and the cubic interaction K:

$$\begin{aligned}
&\delta_{11}\int_1 + \int_1 \Phi_1 \frac{\delta\Gamma}{\delta\Phi_1} - 2\int_{12} G^{-1}_{12}\frac{\delta\Gamma}{\delta G^{-1}_{12}} \\
&\quad = \int_{123456} K_{123}K_{456}\frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta G^{-1}_{45}}\left\{2\frac{\delta\Gamma}{\delta G^{-1}_{36}} - \Phi_3\Phi_6\right\} \\
&\qquad + 3\int_{123} K_{123}\Phi_3\frac{\delta\Gamma}{\delta G^{-1}_{12}} - \int_{123} K_{123}\Phi_1\Phi_2\Phi_3 \\
&\qquad + \int_{1234567} K_{123}L_{4567}\frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta K_{456}}\left\{2\frac{\delta\Gamma}{\delta G^{-1}_{37}} - \Phi_3\Phi_7\right\} \\
&\qquad + \frac{2}{3}\int_{1234} L_{1234}\left\{-\frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta G^{-1}_{34}} + \frac{\delta\Gamma}{\delta G^{-1}_{12}}\frac{\delta\Gamma}{\delta G^{-1}_{34}}\right\} \\
&\qquad + 2\int_{1234} L_{1234}\Phi_4\frac{\delta\Gamma}{\delta K_{123}} - \frac{2}{3}\int_{1234} L_{1234}\Phi_3\Phi_4\frac{\delta\Gamma}{\delta G^{-1}_{12}} \\
&\qquad + 2\int_{1234567} K_{123}L_{4567}\frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta G^{-1}_{45}}\frac{\delta\Gamma}{\delta K_{367}} \\
&\qquad - \frac{2}{3}\int_{1234567} K_{123}L_{4567}\Phi_3\frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta G^{-1}_{45}}\frac{\delta\Gamma}{\delta G^{-1}_{67}} \\
&\qquad + 2\int_{12345678} L_{1234}L_{5678}\frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta K_{567}}\frac{\delta\Gamma}{\delta K_{348}} \\
&\qquad - \frac{2}{3}\int_{12345678} L_{1234}L_{5678}\Phi_8\frac{\delta^2\Gamma}{\delta G^{-1}_{12}\delta K_{567}}\frac{\delta\Gamma}{\delta G^{-1}_{34}}\ . \qquad (92)
\end{aligned}$$

Note that applying the compatibility relation (85) between functional derivatives with respect to the field expectation value Φ, the kernel G^{-1} and the 3-vertex K to (90) would lead to Eq. (108) in Ref. [5],

$$\begin{aligned}
\frac{\delta\Gamma}{\delta\Phi_1} &= \int_2 G^{-1}_{12}\Phi_2 + \int_{23} K_{123}\frac{\delta\Gamma}{\delta G^{-1}_{23}} + \frac{1}{3}\int_{234} L_{1234}\Phi_4\frac{\delta\Gamma}{\delta G^{-1}_{23}} \\
&\quad + \frac{1}{3}\int_{2345} L_{1234}\frac{\delta^2\Gamma}{\delta G^{-1}_{23}\delta\Phi_5}\left\{2\frac{\delta\Gamma}{\delta G^{-1}_{34}} - \Phi_3\Phi_4\right\}\ , \qquad (93)
\end{aligned}$$

so that functional derivatives with respect to the field expectation value Φ in (87) could no longer be eliminated. This procedure has been pursued in Ref. [5], where the two coupled nonlinear functional differential equations (87) and (93) for the effective energy are investigated. Due to the last term in (87) the highest nonlinearity within the approach of Ref. [5] is cubic, whereas our functional differential equation (92) contains at most only quadratic nonlinearities.

3.3. Graphical relation

If the cubic and the quartic interactions K and L vanish, Eq. (92) is solved by the zeroth-order contribution to the effective energy (72) which has the functional derivatives

$$\frac{\delta\Gamma^{(0)}}{\delta\Phi_1} = \int_2 G_{12}^{-1}\Phi_2, \qquad \frac{\delta\Gamma^{(0)}}{\delta G_{12}^{-1}} = \frac{1}{2}\left\{G_{12} + \Phi_1\Phi_2\right\},$$
$$\frac{\delta^2\Gamma^{(0)}}{\delta G_{12}^{-1}\delta G_{34}^{-1}} = -\frac{1}{4}\left\{G_{13}G_{24} + G_{14}G_{23}\right\}. \tag{94}$$

For non-vanishing cubic and quartic interactions K and L, the right-hand side in Eq. (92) produces corrections to (72) which we shall denote with $\Gamma^{(\mathrm{int})}$. Thus the effective energy Γ decomposes according to

$$\Gamma = \Gamma^{(0)} + \Gamma^{(\mathrm{int})}. \tag{95}$$

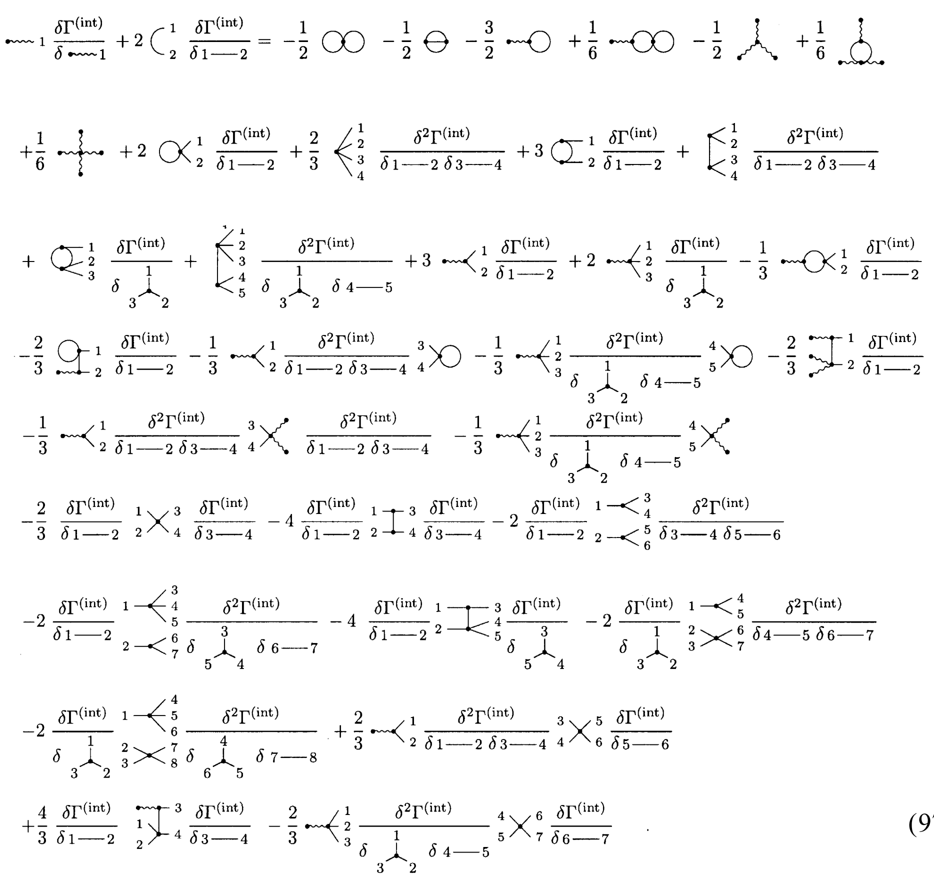
Inserting this into (92) and using (94), we obtain together with (47) and (48) the following function differential equation for the interaction part of the effective energy $\Gamma^{(\mathrm{int})}$:

$$\begin{aligned}
&\int_1 \Phi_1 \frac{\delta\Gamma^{(\mathrm{int})}}{\delta\Phi_1} + 2\int_{12} G_{12}\frac{\delta\Gamma^{(\mathrm{int})}}{\delta G_{12}} \\
&\quad = \frac{1}{2}\int_{1234} L_{1234}G_{12}G_{34} - \frac{1}{2}\int_{123456} K_{123}K_{456}G_{14}G_{25}G_{36} \\
&\qquad + \frac{3}{2}\int_{123} K_{123}G_{12}\Phi_3 + \frac{1}{6}\int_{1234567} K_{123}L_{4567}G_{14}G_{25}G_{67}\Phi_3 \\
&\qquad + \frac{1}{6}\int_{1234567} K_{123}L_{4567}G_{14}G_{25}\Phi_3\Phi_6\Phi_7 + \frac{1}{2}\int_{123} K_{123}\Phi_1\Phi_2\Phi_3 \\
&\qquad - \frac{1}{6}\int_{1234} L_{1234}\Phi_1\Phi_2\Phi_3\Phi_4 - 2\int_{123456} L_{1234}G_{12}G_{35}G_{46}\frac{\delta\Gamma^{(\mathrm{int})}}{\delta G_{56}} \\
&\qquad - \frac{2}{3}\int_{12345678} L_{1234}G_{15}G_{26}G_{37}G_{48}\frac{\delta^2\Gamma^{(\mathrm{int})}}{\delta G_{56}\delta G_{78}} \\
&\qquad + 3\int_{12345678} K_{123}K_{456}G_{14}G_{25}G_{37}G_{68}\frac{\delta\Gamma^{(\mathrm{int})}}{\delta G_{78}}
\end{aligned}$$

$$
\begin{aligned}
&+\int_{123456789\bar{1}} K_{123}K_{456}G_{14}G_{27}G_{38}G_{59}G_{6\bar{1}}\,\frac{\delta^2\Gamma^{(\mathrm{int})}}{\delta G_{78}\delta G_{9\bar{1}}}\\
&-\int_{1234567} K_{123}L_{4567}G_{16}G_{27}\,\frac{\delta\Gamma^{(\mathrm{int})}}{\delta K_{345}}\\
&-\int_{123456789} K_{123}L_{4567}G_{34}G_{18}G_{29}\,\frac{\delta^2\Gamma^{(\mathrm{int})}}{\delta K_{567}\delta G_{89}}\\
&-3\int_{12345} K_{123}\Phi_3G_{14}G_{25}\,\frac{\delta\Gamma}{\delta G_{45}}+2\int_{1234} L_{1234}\Phi_4\,\frac{\delta\Gamma^{(\mathrm{int})}}{\delta K_{123}}\\
&-\frac{1}{3}\int_{123456789} K_{123}L_{4567}\Phi_3G_{16}G_{27}G_{48}G_{59}\,\frac{\delta\Gamma^{(\mathrm{int})}}{\delta G_{89}}\\
&-\frac{2}{3}\int_{123456789} K_{123}L_{4567}\Phi_3G_{45}G_{16}G_{28}G_{79}\,\frac{\delta\Gamma^{(\mathrm{int})}}{\delta G_{89}}\\
&-\frac{1}{3}\int_{123456789\bar{1}\bar{2}} K_{123}L_{4567}\Phi_3G_{45}G_{18}G_{29}G_{6\bar{1}}G_{7\bar{2}}\,\frac{\delta^2\Gamma^{(\mathrm{int})}}{\delta G_{89}\delta G_{\bar{1}\bar{2}}}\\
&+\frac{1}{3}\int_{123456789\bar{1}} L_{1234}L_{5678}\Phi_5G_{34}G_{19}G_{2\bar{1}}\,\frac{\delta^2\Gamma^{(\mathrm{int})}}{\delta K_{678}\delta G_{9\bar{1}}}\\
&-\frac{2}{3}\int_{123456789} K_{123}L_{4567}\Phi_3\Phi_4\Phi_5G_{16}G_{28}G_{79}\,\frac{\delta\Gamma^{(\mathrm{int})}}{\delta G_{89}}\\
&-\frac{1}{3}\int_{123456789\bar{1}\bar{2}} K_{123}L_{4567}\Phi_3\Phi_4\Phi_5G_{18}G_{29}G_{6\bar{1}}G_{7\bar{2}}\,\frac{\delta^2\Gamma^{(\mathrm{int})}}{}
\end{aligned}
$$

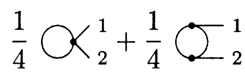
$$-2\int_{123456789\bar{1}\bar{2}\bar{3}} K_{123}K_{456}G_{17}G_{28}G_{59}G_{6\bar{1}}G_{3\bar{2}}G_{4\bar{3}}\,\frac{\delta^2\Gamma^{(\text{int})}}{\delta G_{78}\delta G_{9\bar{1}}}\,\frac{\delta\Gamma^{(\text{int})}}{\delta G_{\bar{2}\bar{3}}}$$
$$+\frac{4}{3}\int_{123456789\bar{1}\bar{2}} K_{123}L_{4567}\Phi_3G_{16}G_{28}G_{79}G_{4\bar{1}}G_{5\bar{2}}\,\frac{\delta\Gamma^{(\text{int})}}{\delta G_{89}}\,\frac{\delta\Gamma^{(\text{int})}}{\delta G_{\bar{1}\bar{2}}}$$
$$+\frac{2}{3}\int_{123456789\bar{1}\bar{2}\bar{3}\bar{4}} K_{123}L_{4567}\Phi_3G_{18}G_{29}G_{6\bar{1}}G_{7\bar{2}}G_{4\bar{3}}G_{5\bar{4}}\,\frac{\delta^2\Gamma^{(\text{int})}}{\delta G_{89}\delta G_{\bar{1}\bar{2}}}\,\frac{\delta\Gamma^{(\text{int})}}{\delta G_{\bar{3}\bar{4}}}$$
$$-\frac{2}{3}\int_{123456789\bar{1}\bar{2}\bar{3}} L_{1234}L_{5678}\Phi_5G_{19}G_{2\bar{1}}G_{3\bar{2}}G_{4\bar{3}}\,\frac{\delta^2\Gamma^{(\text{int})}}{\delta K_{678}\delta G_{9\bar{1}}}\,\frac{\delta\Gamma^{(\text{int})}}{\delta G_{\bar{2}\bar{3}}}\,. \tag{96}$$

With the help of the graphical rules (10), (12), (13), (69), this functional differential equation can be written diagrammatically as follows:



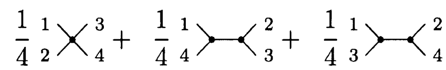
(97)

The effect of the left-hand side is to count the number of field expectation values plus two times the number of lines of each one-particle irreducible vacuum diagram. The right-hanIB[ffBffTd2iC@[W@d2,/HT]]WCBC]IffBffTdid2,/HSSQIVI@CSfiffBffTe2,/HTCSfiffBffT the loop order $l=3$, we have to evaluate from the vacuum diagrams (100) a one-line amputation

$$\frac{\delta\Gamma^{(2)}}{\delta\, 1 \!-\!\!-\! 2} =$$ 

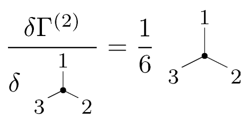
(102)

a two-line amputation

$$\frac{\delta^2\Gamma^{(2)}}{\delta\, 1 \!-\!\!-\! 2\; \delta\, 3 \!-\!\!-\! 4} =$$ 

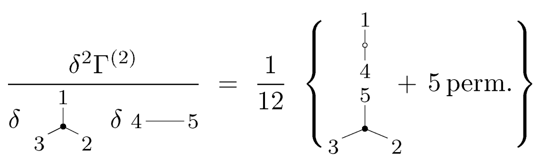
(103)

a 3-vertex amputation



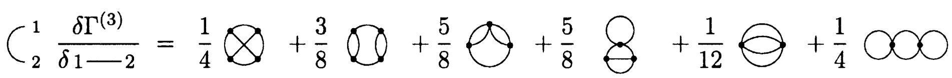
(104)

and the amputation of one line and one 3-vertex



(105)

With this we obtain from Eq. (101) for $l=3$



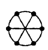
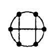
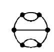
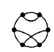
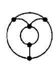
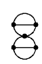
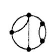
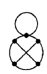
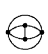
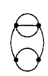
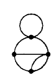
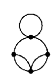
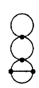
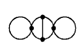
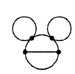
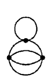
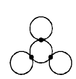
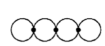
(106)

which leads to the one-particle irreducible vacuum diagrams listed in Table 2 together with the subsequent loop order $l=4$. In a similar way, the graphical relation (97) can be iterated to construct the one-particle irreducible vacuum diagrams which involve field expectation values. Note that Eq. (101) is suitable for an automatized symbolic computation which can be implemented as in Ref. [4], such that one may proceed to higher orders without much effort except for computer time.

4. Summary

In this work we have presented a method for determining the connected and the one-particle irreducible vacuum diagrams together with their proper weights in the ordered phase of the euclidean multicomponent scalar ϕ^4-theory. Whereas in the disordered, symmetric phase it is sufficient to deal with even field powers in the energy functional by using functional derivatives with respect to the free correlation function [4], the situation is more complicated in the ordered, broken-symmetry phase. Due

Table 2
One-particle irreducible vacuum diagrams and their weights of the ϕ^3–ϕ^4-theory without field expectation values up to four loops

l	p	$\Gamma^{(l,p)}$
2	0	1/12 (0,0,1,0;2) 
2	1	1/8 (2,1,0,0;1) 
3	0	1/24 (0,0,0,0;24)  1/16 (0,2,0,0;4) 
3	1	1/8 (0,2,0,0;2)  1/8 (1,1,0,0;2) 
3	2	1/48 (0,0,0,1;2)  1/16 (2,1,0,0;2) 
4	0	1/72 (0,0,0,0;72)  1/12 (0,0,0,0;12)  1/48 (0,3,0,0;6)  1/16 (0,2,0,0;4)  1/8 (0,1,0,0;4) 
4	1	1/8 (0,0,0,0;8)  1/4 (0,1,0,0;2)  1/16 (0,3,0,0;2)  1/32 (0,2,0,0;8)  1/4 (0,2,0,0;1)  1/16 (1,2,0,0;2)  1/8 (1,1,0,0;2)  1/8 (1,0,0,0;4) 
4	2	1/8 (0,1,0,0;4)  1/16 (0,3,0,0;2)  1/8 (0,2,0,0;2)  1/24 (0,1,1,0;2)  1/4 (1,1,0,0;1)  1/16 (1,2,0,0;2)  1/16 (1,2,0,0;2)  1/16 (2,0,0,0;4)  1/16 (2,1,0,0;2) 
4	3	1/48 (0,3,0,0;6)  1/24 (1,0,1,0;2)  1/48 (3,0,0,0;6)  1/32 (2,2,0,0;2) 

Within each loop order l the diagrams are distinguished with respect to the number p of 4-vertices. Each diagram is characterized by the vector $(S, D, T, F; N)$ whose components specify the number of self-, double, triple, fourfold connections, and of the identical vertex permutations, respectively.

to the non-zero field expectation value both odd and even field powers appear in the energy functional, so it is necessary to extend the symmetric treatment by introducing a second type of functional derivative.

We have based the construction on functional derivatives with respect to both the free correlation function and the 3-vertex in contrast to a previous solution of the same problem in Ref. [5], where functional derivatives with respect to both the free correlation function and the external current was used. Our approach has turned out to be conceptually easier, more transparent, and possibly more efficient than the one used in Ref. [5] for the following reasons. Whereas we obtain *one* nonlinear graphical recursion relation for the connected and the one-particle irreducible vacuum diagrams, Ref. [5] had to solve *two* coupled nonlinear graphical recursion relations. In particular, the determination of the loop contributions to the interacting part of the free energy $W^{(\mathrm{int})}$ and the effective energy $\Gamma^{(\mathrm{int})}$ necessitates the construction of the diagrams of one-point functions in an intermediate step, whereas our recursive approach only involves the desired vacuum diagrams. Another advantage of our method is that it only involves *quadratic* nonlinearities for the recursive graphical construction of one-particle irreducible vacuum diagrams, whereas there appears a *cubic* nonlinearity in the corresponding procedure of Ref. [5].

5. Outlook

Recently, it has been shown that the Schwinger-Dyson equations of QED [16] and of ϕ^4-theory in the disordered phase [17] can be exactly closed in a certain functional analytic sense. Using functional derivatives with respect to the free propagators and the interaction allows to derive a closed set of equations for the connected as well as the one-particle irreducible n-point functions. Their conversion to graphical recursion relations leads to a systematic graphical generation of all connected and one-particle irreducible Feynman diagrams. In the future it remains to analyse the closure of the Schwinger-Dyson equations also for the ϕ^4-theory in the broken-symmetry phase.

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