# Fokker-Planck and Langevin Equations from Forward–Backward Path Integral


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Starting from a forward–backward path integral of a point particle in a bath of harmonic oscillators, we derive the Fokker-Planck and Langevin equations with and without inertia. Special emphasis is placed upon the correct operator order in the time evolution operator. The crucial step is the evaluation of a Jacobian with a retarded time derivative by analytic regularization.


## I. INTRODUCTION

In 1963, Feynman and Vernon [1] set up a path integral for the study of a point particle in a thermal bath of harmonic oscillators. Stimulated by work of Dekker [3] and Caldeira and Leggett [4], this has led to a large body of literature on quantum systems with dissipation which are now textbook material. A large number of applications is reviewed in [5] with many references; the foundations are presented pedagogically in the textbook [6], and we shall adhere to the same notation in what follows.

If the particle has a mass $M$, moves in a potential $V(x)$, and is coupled linearly to a large number of oscillators $X_i(t)$ of mass $M_i$ and frequency $\Omega_i$, the probability to run from the spacetime point $x_a t_a$ to $x_b t_b$ is given by the forward–backward path

$$|(x_bt_b|x_at_a)|^2 = \int \mathcal{D}x_+\mathcal{D}x_- \, \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[\frac{M}{2}\left(\dot{x}_+^2-\dot{x}_-^2\right)-V(x_+)+V(x_-)-x_+\sum_i c_iX_+^i+x_-\sum_i c_iX_-^i\right]\right\}, \quad (1.1)$$

where $x_+(t)$ and $x_-(t)$ are two fluctuating paths connecting the initial and final points $x_a$ and $x_b$, and $c_i$ are coupling strengths to be suitably chosen later. The bath oscillators are supposed to be in thermal equilibrium at a temperature $T$. This is taken into account by forming the thermal average of the bath oscillators. For a single oscillator, this is done by considering $c_ix_\pm$ as external currents $j_\pm$ coupled to $X_\pm$, and calculating the Gaussian integral

$$Z_0[j_+,j_-] = \int dX_b\, dX_a\; (X_b\,\hbar\beta|X_a0)_\Omega\; (X_bt_b|X_at_a)_\Omega^{j+}\; (X_bt_b|X_at_a)_\Omega^{j-*}. \quad (1.2)$$

where $(X_b\,\hbar\beta|X_a0)_\Omega$ is the imaginary-time amplitude

$$(X_b\hbar\beta|X_a0) = \frac{1}{\sqrt{2\pi\hbar/M}}\sqrt{\frac{\Omega}{\sinh\hbar\beta}}\exp\left\{-\frac{1}{2\hbar}\frac{M\Omega}{\sinh\hbar\beta\Omega}[(X_b^2+X_a^2)\cosh\hbar\beta\Omega-2X_bX_a]\right\}, \quad (1.3)$$

and $(X_bt_b|X_at_a)_\Omega^j$ the path integral over the bath oscillator

$$\begin{aligned}(X_bt_b|X_at_a)_\Omega^j &= \int\mathcal{D}X(t)\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}dt\left[\frac{M}{2}(\dot{X}^2-\Omega^2X^2)+jX\right]\right\}\\ &= e^{(i/\hbar)\mathcal{A}_{\mathrm{cl},j}}F_{\Omega,j}(t_b,t_a).\end{aligned} \quad (1.4)$$

with a total classical action

$$\begin{aligned}\mathcal{A}_{\mathrm{cl},j} =& \frac{1}{2}\frac{M\Omega}{\sin\Omega(t_b-t_a)}\left[(X_b^2+X_a^2)\cos\Omega(t_b-t_a)-2X_bX_a\right]\\ &+\frac{1}{\sin\Omega(t_b-t_a)}\int_{t_a}^{t_b}dt[X_a\sin\Omega(t_b-t)+X_b\sin\Omega(t-t_a)]j(t),\end{aligned} \quad (1.5)$$

and the fluctuation factor

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$$F_{\Omega,j}(t_b,t_a)=\frac{1}{\sqrt{2\pi i\hbar/M}}\sqrt{\frac{\Omega}{\sin\Omega(t_b-t_a)}}$$
$$\times\exp\left\{-\frac{i}{\hbar M\Omega\sin\Omega(t_b-t_a)}\int_{t_a}^{t_b}dt\int_{t_a}^{t}dt'\sin\Omega(t_b-t)\sin\Omega(t'-t_a)j(t)j(t')\right\}. \tag{1.6}$$

The result of the thermal average in Eq. (1.2) is

$$Z_0[j_+,j_-]=\exp\left\{-\frac{1}{2\hbar^2}\int dt\int dt'\Theta(t-t')\right. \tag{1.7}$$
$$\left.\times\Big[(j_+-j_-)(t)C(t,t')(j_++j_-)(t')+(j_+-j_-)(t)A(t,t')(j_+-j_-)(t')\Big],\right\}$$

where $\Theta(t-t')$ is the completely retarded Heaviside function which is equal to unity for $t>t'$ and vanishes for $t\le t'$, and $C(t,t')$ $A(t,t')$ and $C(t,t')$ are the thermal expectation values of commutator and anticommutator $\langle[\hat{X}(t),\hat{X}(t')]\rangle_T$ and $\langle\{\hat{X}(t),\hat{X}(t')\}\rangle_T$, respectively, in operator language. They are twice the real and imaginary parts of the time-ordered Green function $G(t,t')=\langle\hat{T}\hat{X}(t),\hat{X}(t')\rangle_T$ for $t>t'$:

$$G(t,t')=\frac{1}{2}\left[A(t,t')+C(t,t')\right]=\frac{\hbar}{2M\Omega}\frac{\cosh\frac{\Omega}{2}\left[\hbar\beta-i(t-t')\right]}{\sinh\frac{\hbar\Omega\beta}{2}},\quad t>t'.. \tag{1.8}$$

which is the analytic continuation of the periodic imaginary-time Green function to $\tau=it$.

The thermal average of the probability (1.1) is then given by the forward–backward path integral

$$|(x_bt_b|x_at_a)|^2=\int\mathcal{D}x_+(t)\int\mathcal{D}x_-(t)$$
$$\times\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}dt\left[\frac{M}{2}(\dot{x}_+^2-\dot{x}_-^2)-(V(x_+)-V(x_-))\right]+\frac{i}{\hbar}\mathcal{A}^{\rm FV}[x_+,x_-]\right\}. \tag{1.9}$$

where $\exp\{i\mathcal{A}^{\rm FV}[x_+,x_-]/\hbar\}$ is the Feynman-Vernon *influence functional* defined by

$$Z_0^{\rm b}[x_+,x_-]\equiv\exp\left\{i\mathcal{A}^{\rm FV}[x_+,x_-]/\hbar\right\}\equiv\exp\left\{i\mathcal{A}_D^{\rm FV}[x_+,x_-]/\hbar+i\mathcal{A}_F^{\rm FV}[x_+,x_-]/\hbar\right\} \tag{1.10}$$
$$=\exp\left\{-\frac{1}{2\hbar}\int dt\int dt'\,\Theta(t-t')\Big[(x_+-x_-)(t)C_{\rm b}(t,t')(x_++x_-)(t')+(x_+-x_-)(t)A_{\rm b}(t,t')(x_+-x_-)(t')\Big]\right\}.$$

with $C_{\rm b}(t,t')$ and $A_{\rm b}(t,t')$ being commutator and anticommutator functions of the bath at temperature $T$. The first and second parts of the exponents have been distinguished as dissipative and fluctuating parts $\mathcal{A}_D^{\rm FV}[x_+,x_-]$ and $\mathcal{A}_F^{\rm FV}[x_+,x_-]$ of of the effective influence action $\mathcal{A}^{\rm FV}[x_+,x_-]$.

The bath functions $C_{\rm b}(t,t')$ and $A_{\rm b}(t,t')$ are sums of correlation functions of the individual oscillators of mass $M_i$ frequency $\Omega_i$, each contributing with a weight $c_i^2$. Thus we may write

$$C_{\rm b}(t,t')=\sum_i c_i^2\left\langle[\hat{X}_i(t),\hat{X}_i(t')]\right\rangle_T=-\hbar\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\sigma_{\rm b}(\omega')i\sin\omega'(t-t'),$$
$$A_{\rm b}(t,t')=\sum_i c_i^2\left\langle\left\{\hat{X}_i(t),\hat{X}_i(t')\right\}\right\rangle_T=\hbar\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\sigma_{\rm b}(\omega')\coth\frac{\hbar\omega'}{2k_BT}\;\cos\omega'(t-t'), \tag{1.11}$$

where $\sigma_{\rm b}(\omega')$ is the spectral density of the bath

$$\sigma_{\rm b}(\omega')\equiv 2\pi\sum_i\frac{c_i^2}{2M_i\Omega_i}[\delta(\omega'-\Omega_i)-\delta(\omega'+\Omega_i)] \tag{1.12}$$

For a discussion of the properties of the influence functional, we introduce an auxiliary retarded function

$$\gamma(t-t')\equiv\Theta(t-t')\frac{1}{M}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\sigma_{\rm b}(\omega)}{\omega}e^{-i\omega(t-t')}, \tag{1.13}$$

and write

$$\Theta(t-t')C_{\rm b}(t,t') = i\hbar M\dot{\gamma}(t-t') + i\hbar M\Delta\omega^2\delta(t-t') \tag{1.14}$$

with

$$\Delta\omega^2 \equiv -\frac{1}{M}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{\sigma_{\rm b}(\omega')}{\omega'} = -\frac{1}{M}\sum_i \frac{c_i^2}{M_i\Omega_i^2}. \tag{1.15}$$

Inserting the first term in the decomposition (1.14) into (1.11), the dissipative part of the influence functional can be integrated by parts in $t'$ and becomes

$$\begin{aligned}\mathcal{A}_D^{\rm FV}[x_+,x_-] = &-\frac{M}{2}\int_{t_a}^{t_b} dt\int_{t_a}^{t_b} dt'\,(x_+-x_-)(t)\gamma(t-t')(\dot{x}_++\dot{x}_-)(t')\\ &+\frac{M}{2}\int_{t_a}^{t_b} dt(x_+-x_-)(t)\gamma(t-t_b)(x_++x_-)(t_a).\end{aligned} \tag{1.16}$$

The $\delta$-function in (1.14) contributes to $\mathcal{A}_D^{\rm FV}[x_+,x_-]$ a term

$$\Delta\mathcal{A}_{\rm loc}[x_+,x_-] = \frac{M}{2}\int_{t_a}^{t_b} dt\Delta\omega^2(x_+^2-x_-^2)(t), \tag{1.17}$$

which may simply be absorbed into the potential terms in the path integral (1.9) renormalizing them to

$$-\frac{i}{\hbar}\int_{t_a}^{t_b} dt\,\left[V_{\rm ren}(x_+)-V_{\rm ren}(x_-)\right], \tag{1.18}$$

The odd bath function $\sigma_{\rm b}(\omega')$ can be expanded in a power series with only odd powers of $\omega'$. The lowest approximation

$$\sigma_{\rm b}(\omega') \approx 2M\gamma\omega', \tag{1.19}$$

describes Ohmic dissipation with some friction constant $\gamma$. For frequencies much larger than the atomic relaxation rates, the friction goes to zero. This behavior is modeled by the Drude form of the spectral function

$$\sigma(\omega') \approx 2M\gamma\omega'\frac{\omega_D^2}{\omega_D^2+\omega'^2}. \tag{1.20}$$

Inserting this into Eq. (1.13), we obtain the Drude form of the function $\gamma(t)$:

$$\gamma_D^R(t) \equiv \Theta(t)\,\gamma\omega_D e^{-\omega_D|t|}. \tag{1.21}$$

The superscript emphasizes the retarded nature. This can also be written as a Fourier integral

$$\gamma_D(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\gamma_D(\omega)e^{-i\omega t}, \tag{1.22}$$

with the Fourier components

$$\gamma_D^R(\omega') = \gamma\frac{i\omega_D}{\omega'+i\omega_D}. \tag{1.23}$$

The position of the pole in the lower half-plane ensures the retarded nature of the friction term by producing the Heaviside function in (1.21).

In the Ohmic limit (1.19), the dissipative part of the influence functional simplifies. Then $\gamma(t)$ becomes narrowly peaked at positive $t$ may be expressed in terms of a *right-sided* $\delta$-function as

$$\gamma(t) \to \gamma\,\delta^R(t), \tag{1.24}$$

whose superscript $R$ indicates the retarded asymmetry of the $\delta$-function, which has the property that

$$\int dt\, \Theta(t)\delta^R(t) = 1. \tag{1.25}$$

With this, (1.16) becomes a local action

$$\mathcal{A}_D^{\rm FV}[x_+, x_-] = -\frac{M}{2}\gamma \int_{t_a}^{t_b} dt(x_+ - x_-)(\dot{x}_+ + \dot{x}_-)^R - \frac{M}{2}\gamma(x_+^2 - x_-^2)(t_a). \tag{1.26}$$

The right-sided nature of the $\delta$-function causes an infinitesimal *negative* shift in the time argument of the velocities $(\dot{x}_+ + \dot{x}_-)(t)$ with respect to the factor $(x_+ - x_-)(t)$, indicated by the superscript $R$. It expresses the *causality* of the friction forces and will be seen to be crucial in producing a probability conserving time evolution of the probability distribution.

The second term changes only the curvature of the effective potential at the initial time, and can be ignored. In the first term it is important to observe that the retarded nature of the dissipative term and of the function $\gamma(t)$ in (1.13) ensures that the velocity term $(\dot{x}_+ + \dot{x}_-)(t)$ lies *before* $(x_+ - x_-)(t)$ in a time-sliced path integral. This ensures the *causality* of the friction forces.

It is useful to incorporate the slope information (1.19) also into the bath correlation function $A_{\rm b}(t,t')$ in (1.11), and factorize it as

$$A_{\rm b}(t,t') = 2M\gamma k_B T K(t,t'), \tag{1.27}$$

where

$$\begin{aligned} K(t,t') = K(t-t') &\equiv \frac{1}{2M\gamma k_B T} \sum_i c_i^2 \langle \{\hat{X}_i(t), \hat{X}_i(t')\}\rangle_T \\ &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} K(\omega') e^{-i\omega'(t-t')}, \end{aligned} \tag{1.28}$$

with Fourier transform

$$K(\omega') \equiv \frac{1}{2M\gamma} \frac{\sigma(\omega')}{\omega'} \frac{\hbar\omega'}{2k_B T} \coth \frac{\hbar\omega'}{2k_B T}, \tag{1.29}$$

which in the limit of a purely Ohmic dissipation simplifies to

$$K(\omega') = K^{\rm Ohm}(\omega') \equiv \frac{\hbar\omega'}{2k_B T} \coth \frac{\hbar\omega'}{2k_B T}. \tag{1.30}$$

The function $K(\omega')$ has the normalization $K(0) = 1$, giving $K(t-t')$ a unit temporal area:

$$\int_{-\infty}^{\infty} dt\, K(t-t') = 1. \tag{1.31}$$

In the classical limit $\hbar \to 0$,

$$K(\omega') = \frac{\omega_D^2}{\omega'^2 + \omega_D^2}, \tag{1.32}$$

and

$$K(t-t') = \frac{1}{2\omega_D} e^{-\omega_D(t-t')}. \tag{1.33}$$

In the limit of Ohmic dissipation, this becomes a $\delta$-function. Thus $K(t-t')$ may be viewed as a $\delta$-function broadened by quantum fluctuations and relaxation effects.

With the function $K(t,t')$, the fluctuation part of the influence functional in (1.10), (1.11), (1.9) becomes

$$\mathcal{A}_F^{\rm FV}[x_+, x_-] = i\frac{M\gamma k_B T}{\hbar} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, (x_+ - x_-)(t)\, K(t,t')\, (x_+ - x_-)(t'). \tag{1.34}$$

Here we have used the symmetry of the function $K(t,t')$ to remove the Heaviside function $\Theta(t-t')$ from the integrand, extending the range of $t'$-integration to the entire interval $(t_a, t_b)$.

In the Ohmic limit, the probability of the particle to move from $x_a t_a$ to $x_b\, t_b$ is given by the path integral

$$
\begin{aligned}
|(x_b t_b|x_a t_a)|^2 = & \int \mathcal{D}x_+(t) \int \mathcal{D}x_-(t) \\
& \times \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \left[\frac{M}{2}(\dot{x}_+^2-\dot{x}_-^2)-(V(x_+)-V(x_-))\right]\right\} \\
& \times \exp\left\{-i\int_{t_a}^{t_b} dt\, \frac{M\gamma}{2\hbar}(x_+-x_-)(t)(\dot{x}_++\dot{x}_-)^R(t)\right. \\
& \qquad \left. -\frac{M\gamma k_BT}{\hbar^2}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\,(x_+-x_-)(t)\,K(t,t')\,(x_+-x_-)(t')\right\}.
\end{aligned} \tag{1.35}
$$

This is the *closed-time path integral* of a particle in contact with a thermal reservoir.

The paths $x_+(t), x_-(t)$ may also be associated with a forward and a backward movement of the particle in time. For this reason, (1.35) is also called a *forward–backward path integral*. The hyphen is pronounced as *minus*, to emphasize the opposite signs in the partial actions.

It is now convenient to change integration variables and go over to average and relative coordinates of the two paths $x_+$, $x_-$:

$$
\begin{aligned}
x &\equiv (x_+ + x_-)/2, \\
y &\equiv x_+ - x_-.
\end{aligned} \tag{1.36}
$$

Then (1.35) becomes

$$
\begin{aligned}
|(x_b t_b|x_a t_a)|^2 = \int \mathcal{D}x(t) \int \mathcal{D}y(t)\, \exp\Bigg\{ & -\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \left[M\left(-\dot{y}\dot{x}+\gamma y\dot{x}^R\right)+V\left(x+\frac{y}{2}\right)-V\left(x-\frac{y}{2}\right)\right] \\
& -\frac{M\gamma k_BT}{\hbar^2}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, y(t)K(t,t')y(t')\Bigg\}.
\end{aligned} \tag{1.37}
$$

## II. FOKKER-PLANCK EQUATION

At high-temperatures, the Fourier transform of the Kernel $K(t,t')$ in Eq. (1.30) tends to unity such that $K(t,t')$ becomes a $\delta$-function, and the bath correlation function (1.27) becomes approximately

$$
A_{\mathrm{b}}(t,t') \approx w\,\delta(t-t'), \tag{2.1}
$$

where we have introduced the constant proportional to the temperature:

$$
w \equiv 2M\gamma k_BT, \tag{2.2}
$$

which is related to the so-called *diffusion constant*

$$
D \equiv k_BT/M\gamma \tag{2.3}
$$

by

$$
w = 2\gamma^2M^2D/T. \tag{2.4}
$$

Then the path integral (1.37) for the probability distribution of a particle coupled to a thermal bath simplifies to

$$
\begin{aligned}
P(x_bt_b|x_at_a) \equiv |(x_bt_b|x_at_a)|^2 = & \int \mathcal{D}x(t) \int \mathcal{D}y(t) \\
\times \exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, y[M\ddot{x}+M\gamma\dot{x}^R+V'(x)] - \frac{w}{2\hbar^2}\int_{t_a}^{t_b} dt\, y^2\right\}.
\end{aligned} \tag{2.5}
$$

The superscript $R$ records the infinitesimal backward shift of the time argument as in Eq. (1.26). The $y$-variable can be integrated out, and we obtain

$$P(x_bt_b|x_at_a) = \int \mathcal{D}x(t) \, \exp\left\{-\frac{1}{2w}\int_{t_a}^{t_b} dt \, [M\ddot{x} + M\gamma\dot{x}^R + V'(x)]^2\right\}. \tag{2.6}$$

This looks like a euclidean path integral associated with the Lagrangian

$$L_{\rm e} = \frac{1}{2w}[M\ddot{x} + M\gamma\dot{x} + V'(x)]^2. \tag{2.7}$$

The solution of such path integrals with squares of second time derivatives in the Lagrangian is given in Ref. [9]. The result will, however, be different, due to time-ordering of the $\dot{x}^R$-term.

Apart from this, the Lagrangian is not of the conventional type since it involves a second time derivative. The action principle $\delta\mathcal{A} = 0$ now yields the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0. \tag{2.8}$$

This equation can also be derived via the usual Lagrange formalism by considering $x$ and $\dot{x}$ as independent generalized coordinates $x$, $v$.

## III. CANONICAL PATH INTEGRAL FOR PROBABILITY DISTRIBUTION

It is well-known that a path integral satisfies a Schrödinger type of equation. For the path integral (2.6) this is known as a *Fokker-Planck equation*. The relation is established (see the textbook [6]) by rewriting the path integral in canonical form. Treating $v = \dot{x}$ as an independent dynamical variable, the canonical momenta of $x$ and $v$ are (see Section 17.3 of the textbook Ref. [10])

$$\begin{aligned} p = i\frac{\partial L}{\partial \dot{x}} &= i\frac{M\gamma}{w}[M\ddot{x} + M\gamma\dot{x} + V'(x)] \\ &= i\frac{M\gamma}{w}[M\dot{v} + M\gamma v + V'(x)], \\ p_v = i\frac{\partial L}{\partial \ddot{x}} &= \frac{1}{\gamma}p. \end{aligned} \tag{3.1}$$

The Hamiltonian is given by the Legendre transform

$$H(p,p_v,x,v) = L_{\rm e}(\dot{x},\ddot{x}) - \sum_{i=1}^{2}\frac{\partial L_{\rm e}}{\partial \dot{x}_i}\dot{x}_i \tag{3.2}$$

$$= L_{\rm e}(v,\dot{v}) + ipv + ip_v\dot{v}, \tag{3.3}$$

where $\dot{v}$ has to be eliminated in favor of $p_v$ using (3.1). This leads to

$$H(p,p_v,x,v) = \frac{w}{2M^2}p_v^2 - ip_v[\gamma v + \frac{1}{M}V'(x)] + ipv. \tag{3.4}$$

The the canonical path integral representation for the probability reads therefore

$$\begin{aligned} P(x_bt_b|x_at_a) = &\int \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi} \int \mathcal{D}v \int \frac{\mathcal{D}p_v}{2\pi} \\ &\times \exp\left\{\int_{t_a}^{t_b} dt \, \left[i(p\dot{x} + p_v\dot{v}) - H(p,p_v,x,v)\right]\right\}. \end{aligned} \tag{3.5}$$

It is easy to verify that the path integral over $p$ enforces $v \equiv \dot{x}$, after which the path integral over $p_v$ leads back to the initial expression (2.6). We may keep the auxiliary variable $v(t)$ as an independent fluctuating quantity in all formulas and decompose the probability $P(x_bt_b|x_at_a)$ with respect to the content of $v$ as an integral

$$P(x_bt_b|x_at_a) = \int_{-\infty}^{\infty} dv_b \int_{-\infty}^{\infty} dv_a \, P(x_bv_bt_b|x_av_at_a). \tag{3.6}$$

The more detailed probability on the right-hand side has the path integral representation

$$P(x_b v_b t_b | x_a v_a t_a) = |(x_b v_b t_b | x_a v_a t_a)|^2 = \int \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi} \int \mathcal{D}v \int \frac{\mathcal{D}p_v}{2\pi} \times \exp\left\{ \int_{t_a}^{t_b} dt \left[ i(p\dot{x} + p_v \dot{v}) - H(p, p_v, x, v) \right] \right\}, \tag{3.7}$$

where the end points of $v$ are now kept fixed at $v_b = v(t_b)$, $v_a = v(t_a)$.

We now use the relation between a canonical path integral and the Schrödinger equation to conclude that the probability distribution (3.7) satisfies the Schrödinger-like differential equation:[1]

$$H(\hat{p}, \hat{p}_v, x, v) P(x\, v\, t_b | x_a v_a t_a) = -\partial_t P(x\, v\, t | x_a v_a t_a). \tag{3.8}$$

This is the *Fokker-Planck equation in the presence of inertial forces*.

At this place we note that when going over from the classical Hamiltonian (3.4) to the Hamiltonian operator in the differential equation (3.8) there is an operator ordering problem. When writing down Eq. (3.8) we do not know in which order the momentum $p_v$ must stand with respect to $v$. If we were dealing with an ordinary functional integral in (2.6) we would know the order. It would be found as in the case of the electromagnetic interaction to by symmetric: $-(\hat{p}_v \hat{v} + \hat{v} \hat{p}_v)/2$.

On physical grounds, it is easy to guess the correct order The differential equation (3.8) has to conserve the total probability $\int dx\, dv P(x\, v\, t_b | x_a v_a t_a)$ for all times $t$. This is guaranteed if all momenta stand to the left of all coordinates in the Hamiltonian operator. Indeed, integrating the Fokker-Planck equation (3.8) over $x$ and $v$, only a left-hand position of the momentum operators leads to a vanishing integral, and thus to a time independent total probability. We suspect that this order must be derivable from the retarded nature of the velocity in the term $y\dot{x}^R$ in (2.5). The proof that this is so is the essential point of this paper, by which it goes beyond an earlier treatment of this subject in Ref. [11].

## IV. SOLVING THE OPERATOR ORDERING PROBLEM

Since the ordering problem in the Hamiltonian operator associated with (3.4) does not involve the potential $V(x)$, we study this problem most simply by considering the free Hamiltonian

$$H_0(p, p_v, x, v) = \frac{w}{2M^2} p_v^2 - i\gamma p_v v + ipv. \tag{4.1}$$

which is associated with the Lagrangian path integral

$$P_0(x_b t_b | x_a t_a) \propto \int \mathcal{D}x(t)\, \exp\left\{ -\frac{1}{2w} \int_{t_a}^{t_b} dt \left[ M\ddot{x} + M\gamma \dot{x}^R \right]^2 \right\}. \tag{4.2}$$

We furthermore may concentrate on the probability with $x_b = x_a = 0$, and assume $t_b - t_a$ to be very large. Then the frequencies of all Fourier decompositions are continuous.

Forgetting for a moment the retarded nature of the velocity $\dot{x}$, the Gaussian path integral can immediately be done and yields

$$P_0(0\, t_b | 0\, t_a) \propto \mathrm{Det}^{-1}(-\partial_t^2 - \gamma \partial_t) \propto \exp\left[ -(t_b - t_a) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \log(\omega'^2 - i\gamma\omega') \right]. \tag{4.3}$$

The integral on the right-hand side diverges. This is a consequence of the fact that we have not used Feynman's original time slicing procedure for defining the path integral on a temporal lattice of spacing $a$. As in the case of an ordinary harmonic oscillator discussed in detail in [2,6] this would lead to a finite integral in which $\omega'$ is replaced by $\tilde{\omega}' \equiv (2 - 2\cos a\omega')/a^2$:

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[1] See the review paper by S. Chandrasekhar, Rev. Mod. Phys. *15*, 1 (1943).

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\log[\tilde{\omega}'^4+\gamma^2\tilde{\omega}'^2]=\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\log\tilde{\omega}'^2+\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\log[\tilde{\omega}'^2+\gamma^2]=0+\epsilon(\gamma)\frac{\gamma}{2}, \tag{4.4}$$

where $\epsilon(\gamma)\equiv\gamma/|\gamma|$ is the sign of $\gamma$. For a derivation see Eqs. (2.319) and (2.346) in Ref. [6]. This finite result can equally well be obtained without time slicing by regularizing the divergent integral in (4.3) by analytic regularization according to the rule

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\log(\omega'^2+\omega^2)=\epsilon(\omega)\frac{\omega}{2}, \tag{4.5}$$

Analytic regularization has been introduced by t'Hooft and Veltman [7] into quantum field theory as the only way to regularize nonabelian gauge theories. Recently it has been shown to be the only way of defining path integrals perturbatively in such a way that they are invariant under coordinate transformations [8]. It is therefore suggestive to apply the same procedure also to the path integrals under discussion.

In the present context we need a generalization of the integral (4.5):

$$\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\log(\omega'+i\omega)=\epsilon(\omega_R)\frac{\omega}{2}, \tag{4.6}$$

where $\omega_R$ denotes the real part of $\omega$. This integral follows directly from a splitting of the logarithm in (4.5) into $\log(\omega'+i\omega)+\log(\omega'-i\omega)$ and assuming an equal result for each integral. This symmetric assignment is dictated by the requirement of invariance under canonical transformations. The partition function of the euclidean harmonic oscillator action $\int d\tau\,(\dot{x}^2+\omega^2x^2)/2$ is given by $1/\sqrt{\det(-\partial_t^2+\omega^2)}$, and must be equal to the partition function derived from the creation-annihilation representation $\int d\tau\,(-a^\dagger\dot{a}-\omega a^\dagger a)$ which is $1/\det(-\partial_\tau-\omega)$.

Since $\partial_t+\omega(t)$ is a first-order differential operator whose eigenfunctions are simple exponentials (a so-called *integrating factor*), Formula (4.6) can easily be generalized to positive time-dependent frequencies:

$$\text{Det}\;[\partial_t+\omega(t)]=\exp\left[\frac{1}{2}\int_{t_a}^{t_b}dt\,\omega(t)\right]. \tag{4.7}$$

The derivation of Eq. (4.6) goes by performing the sum over the limit $t_b-t_a\to\infty$ of a proper expression for a finite time interval $t_b-t_a$, which is Matsubara-like frequencies $\omega_n\equiv 2\pi n/\beta$

$$\frac{1}{t_b-t_a}\sum_{n=-\infty}^{\infty}\log(\omega_n+i\omega)=\frac{1}{i(t_b-t_a)}\log\{2\sin[i(t_b-t_a)\omega]\}\underset{t_b-t_a\to\infty}{\longrightarrow}\epsilon(\omega_R)\frac{\omega}{2}. \tag{4.8}$$

As long as $t_b-t_a$ is finite, this equation is invariant under $\gamma\to\gamma+2\pi i/(t_b-t_a)$. In the limit of infinite $t_b-t_a$, this property is lost. In what follows, we therefore assume tacitly that the integral (4.6) is always evaluated at a large but finite $t_b-t_a$ as in (4.8), although we shall sloppily write the result in the limiting form, for brevity.

With this convention we obtain for the functional determinant in (4.3):

$$\begin{aligned}\text{Det}\;(-\partial_t^2-\gamma\partial_t)&=\text{Det}\;(i\partial_t)\text{Det}\;(i\partial_t-i\gamma)=\exp\left[\text{Tr}\;\log(i\partial_t)+\text{Tr}\;\log(i\partial_t-i\gamma)\right]\\&=\exp\left[(t_b-t_a)\frac{\gamma}{2}\right]\end{aligned} \tag{4.9}$$

and thus

$$P_0(0\,t_b|0\,t_a)\propto\exp\left[-(t_b-t_a)\frac{\gamma}{2}\right], \tag{4.10}$$

This corresponds to an energy $\gamma/2$ and an ordering $-i\gamma(\hat{p}_vv+v\hat{p}_v)/2$ in the Hamilton operator.

We now take the retarded time argument of $\dot{x}^R$ into account. Specifically, we replace the term $\gamma y\dot{x}^R$ in (4.2) by $\int dtdt'\,y\,\gamma_D^R(t-t')\,x(t)$ containing the retarded Drude function (1.21) of the friction. The the frequency integral in (4.3) becomes

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\log\left(\omega'^2-\gamma\frac{\omega'\omega_D}{\omega'+i\omega_D}\right)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[-\log(\omega'+i\omega_D)+\log\left(\omega'^2+i\omega_D-\gamma\omega'\omega_D\right)\right], \tag{4.11}$$

where we have omitted a vanishing integral over $\log\omega'$ on account of (4.6). We now use (4.6) to find

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ -\log(\omega' + i\omega_D) + \log\left(\omega'^2 + i\omega'\omega_D - \gamma\omega_D\right) \right] = -\frac{\omega_D}{2} + \epsilon(\omega_{1R})\frac{\omega_1}{2} + \ \epsilon(\omega_{2R})\frac{\omega_2}{2}, \tag{4.12}$$

where $-i\omega_{1,2}$ are the solutions of the quadratic equation

$$\omega'^2 + i\omega'\omega_D - \gamma\omega_D = 0. \tag{4.13}$$

For a large Drude frequency $\omega_D$, they are given by

$$\omega_1 = \omega_D - \gamma, \qquad \omega_2 = \gamma. \tag{4.14}$$

Inserting these into (4.12) we find a vanishing integral rather than $\gamma$ in (4.10), and thus a functional determinant

$$\text{Det}\,(-\partial_t^2 - \gamma\partial_t^R) = \exp\left[\text{Tr}\,\log(-\partial_t^2 - \gamma\partial_t^R)\right] = 1, \tag{4.15}$$

instead of (4.9). The notation $\gamma\partial_t^R$ symbolizes now the specific retarded functional matrix a la Drude with a large $\omega_D$:

$$\gamma\partial_t^R(t,t') \equiv \int dt''\gamma_D^R(t-t'')\partial_{t''}\delta(t''-t'). \tag{4.16}$$

With the determinant (4.15), the probability becomes a constant

$$P_0(0\,t_b|0\,t_a) = \text{const}\,, \tag{4.17}$$

This shows that the retarded nature of the friction force has *subtracted* an energy $\gamma/2$ from the energy in (4.10). With the ordinary path integral corresponding to a Hamilton operator with a symmetrized term $-i(\hat{p}_v\hat{v} + \hat{v}\hat{p}_v)/2$, the subtraction of $\gamma/2$ has changed this to $-i\gamma\hat{p}_v\hat{v}$.

Note that the opposite case of an advanced velocity term $\dot{x}^A$ in (4.2) would be approximated by a Drude function $\gamma_D^A(t)$ which looks just like $\gamma_D^R(t)$ in (1.23), but with *negative* $\omega_D$. The right-hand side of (4.12) becomes now $2\gamma$ rather than zero, The corresponding formula for the functional determinant is

$$\text{Det}\,(-\partial_t^2 - \gamma\partial_t^A) = \exp\left[\text{Tr}\,\log(-\partial_t^2 - \gamma\partial_t^A)\right] = \exp\left[(t_b - t_a)\gamma\right], \tag{4.18}$$

where $\gamma\partial_t^A$ stands for the advanced version of the functional matrix (4.16) in which $\omega_D$ is replaced by $-\omega_D$. Thus we find

$$P_0(0\,t_b|0\,t_a) \propto \exp\left[-(t_b - t_a)\gamma\right], \tag{4.19}$$

with an *additional* energy $\gamma/2$ with respect to the ordinary formula (4.10). This corresponds to the opposite (unphysical) operator order $-i\gamma v\hat{p}_v$ in $\hat{H}_0$, which would violate the probability conservation of time evolution twice as much as the symmetric order.

The above formulas for the functional determinants can easily be extended to the slightly more general case where $V(x)$ is the potential of a harmonic oscillator $V(x) = M\omega_0^2x^2/2$. Then the path integral (2.6) for the probability becomes

$$P_0(x_bt_b|x_at_a) \propto \int \mathcal{D}x(t)\, \exp\left\{-\frac{1}{2w}\int_{t_a}^{t_b} dt\,[M\ddot{x} + M\gamma\dot{x}^R + \omega_0^2 x]^2\right\}, \tag{4.20}$$

which we evaluate at $x_b = x_a = 0$, where it is given by the properly retarded expression

$$\begin{aligned} P_0(0\,t_b|0\,t_a) &\propto \text{Det}^{-1}(-\partial_t^2 - \gamma\partial_t + \omega_0^2) \\ &\propto \exp\left[-(t_b - t_a)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\log(\omega'^2 - i\gamma\omega' - \omega_0^2)\right]. \end{aligned} \tag{4.21}$$

The logarithm is decomposed into $\log(\omega' + i\omega_1) + \log(\omega' + i\omega_2)$ with

$$\omega_{1,2} = \frac{\gamma}{2}\left(1 \pm \sqrt{1 - 4\omega_0^2/\gamma^2}\right). \tag{4.22}$$

Using the analytically regularized formula (4.6), we find

$$\begin{aligned}\text{Det}\,(-\partial_t^2-\gamma\partial_t-\omega_0^2) &= \text{Det}\,(i\partial_t+i\omega_1)\text{Det}\,(i\partial_t+i\omega_2)=\exp\left[\text{Tr}\,\log(i\partial_t+i\omega_1)+\text{Tr}\,\log(i\partial_t+i\omega_2)\right]\\ &= \exp\left[(t_b-t_a)\frac{\omega_1+\omega_2}{2}\right]=\exp\left[(t_b-t_a)\frac{\gamma}{2}\right].\end{aligned} \tag{4.23}$$

The $\omega_0$-independence can be understood immediately by forming the derivative of the logarithm of the functional determinant in (4.9) with respect to $\omega_0^2$, which yields the trace of the associated Green function:

$$\frac{\partial}{\partial\omega_0^2}\log\text{Det}\,(-\partial_t^2-\gamma\partial_t-\omega_0^2)=\frac{\partial}{\partial\omega_0^2}\text{Tr}\,\log(-\partial_t^2-\gamma\partial_t-\omega_0^2)=-\int dt\,\partial_t(-\partial_t^2-\gamma\partial_t-\omega_0^2)^{-1}(t,t). \tag{4.24}$$

In Fourier space, the right-hand side turns into the frequency integral

$$-\int\frac{d\omega'}{2\pi}\frac{1}{(\omega'+i\omega_1)(\omega'+i\omega_1)}. \tag{4.25}$$

Since the two poles lie below the contour of integration, we may close it in the upper half-plane and obtain zero. Closing it in the lower half plane would initially lead to two nonzero contributions from the residues of the two poles which, however, cancel each other.

The derivative with respect to $\gamma$,

$$\frac{\partial}{\partial\gamma}\text{Tr}\,\log\partial_t(-\partial_t^2-\gamma\partial_t-\omega_0^2)=-\int dt\,[\partial_t(-\partial_t^2-\gamma\partial_t-\omega_0^2)^{-1}](t,t). \tag{4.26}$$

can also be calculated by performing the corresponding integral in momentum space:

$$i\int\frac{d\omega'}{2\pi}\frac{\omega'}{(\omega'+i\omega_1)(\omega'+i\omega_1)}. \tag{4.27}$$

If we now close the contour of integration with an infinite semicircle in the upper half plane to obtain a vanishing integral from the residue theorem, we must subtract the integral over the semicircle $i\int d\omega'/2\pi\omega'$ and obtain $1/2$, in agreement with (4.23).

Formula (4.23) can be generalized further to time-dependent coefficients

$$\text{Det}\,\left[-\partial_t^2-\gamma(t)\partial_t-\Omega^2(t)\right]=\exp\left\{\text{Tr}\,\log\left[-\partial_t^2-\gamma\partial_t-\Omega^2(t)\right]\right\}=\exp\left[\int_{t_a}^{t_b}dt\frac{\gamma(t)}{2}\right]. \tag{4.28}$$

This follows from the factorization

$$\text{Det}\,\left[-\partial_t^2-\gamma(t)\partial_t-\Omega^2(t)\right]=\text{Det}\,\left[\partial_t+\Omega_1(t)\right]\text{Det}\,\left[\partial_t+\Omega_2(t)\right] \tag{4.29}$$

with

$$\Omega_1(t)+\Omega_2(t)=\gamma(t),\qquad \partial_t\Omega_2(t)+\Omega_1(t)\Omega_2(t)=\Omega^2(t), \tag{4.30}$$

using formula (4.7).

The probability of the path integral (2.6) without retardation of the velocity term is therefore

$$P_0(0\,t_b|0\,t_a)\propto\exp\left[-\frac{1}{2}(t_b-t_a)\gamma\right], \tag{4.31}$$

as in (4.10).

Let us now introduce retardation of the velocity term by using the $\omega'$-dependent Drude expression (1.23) for the friction coefficient. First we consider again the harmonic path integral (4.20), for which (4.21) becomes

$$P_0(0\,t_b|0\,t_a)\propto\exp\left\{-(t_b-t_a)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\log\left[\omega'^2-i\gamma_D^R(\omega)\omega'-\omega_0^2\right]\right\}. \tag{4.32}$$

For a large Drude frequency $\omega_D\gg\gamma$ the roots are now

$$\omega_{1,2}=\frac{\gamma}{2}\left(1\pm\sqrt{1-4\omega_0^2/\gamma}\right),\qquad \omega_3=\omega_D-\gamma. \tag{4.33}$$

Using once more formula (4.6), we see that $\gamma$ and the $\omega_0$-terms disappear, and we remain with

$$P_0(0\,t_b|0\,t_a) = \text{const}\,. \tag{4.34}$$

This implies a unit functional determinant

$$\text{Det}\,(\partial_t^2 + i\gamma\partial_t^R + \omega_0^2) = 1, \tag{4.35}$$

in contrast to the unretarded determinant (4.23).

The $\gamma$-independence of (4.35) can also be deduced from a simple heuristic argument by forming the derivative with respect to $\gamma$:

$$\frac{\partial}{\partial\gamma}\log\text{Det}\,(-\partial_t^2 - \gamma\partial_t^R - \omega_0^2) = \frac{\partial}{\partial\gamma}\text{Tr}\,\log(-\partial_t^2 - \gamma\partial_t^R - \omega_0^2) = -\int dt\,[\partial_t^R(\partial_t^2 - \gamma\partial_t - \omega_0^2)^{-1}(t,t). \tag{4.36}$$

Since the retarded derivative carries a retarded Heaviside factor $\Theta(t-t')$ which is zero for $t=t'$, the derivative with respect to $\gamma$ vanishes identically.

By analogy with (4.29), the general retarded determinant is also independent of $\gamma(t)$ and $\Omega(t)$.

$$\text{Det}\,\left[-\partial_t^2 - \gamma(t)\partial_t^R - \Omega^2(t)\right] = 1. \tag{4.37}$$

An advanced time derivative in the determinant (4.35) would, of course, have produced the result

$$\text{Det}(\partial_t^2 + i\gamma\partial_t^A + \omega_0^2) = \exp\left[(t_b - t_a)\gamma\right]. \tag{4.38}$$

which can be understood as being due to the advanced version of the Heaviside function which vanishes for $t > t'$ and is unity for $t > t'$.

In the advanced case, the general formula would be

$$\text{Det}\,\left[-\partial_t^2 - \gamma(t)\partial_t^A - \Omega^2(t)\right] = \exp\left[\int dt\,\gamma(t)\right]. \tag{4.39}$$

All three determinants are correct also for finite time intervals, due to the solvability of the first-order differential equation by means of an integrating factor.

By comparing the functional determinants (4.23) and (4.35) we see that the retardation prescription can be avoided by a trivial additive change of the Lagrangian (2.7) to

$$L_{\rm e}(x,\dot x) = \frac{1}{2w}\left[\ddot x + M\gamma\dot x + V'(x)\right]^2 - \frac{\gamma}{2}. \tag{4.40}$$

From this Lagrangian, the path integral can be calculated in any standard way [6].

## V. STRONG DAMPING

For $\gamma \gg V''(x)/M$, the dynamics is dominated by dissipation, and the Lagrangian (2.7) takes a more conventional form in which only $x$ and $\dot x$ appear:

$$L_{\rm e}(x,\dot x) = \frac{1}{2w}\left[M\gamma\dot x^R + V'(x)\right]^2 = \frac{1}{4D}\left[\dot x^R + \frac{1}{M\gamma}V'(x)\right]^2, \tag{5.1}$$

where $\dot x^R$ lies slightly *before* $V'(x(t))$. The probability

$$P(x_b t_b|x_a t_a) = \int\mathcal{D}x\,\exp\left[-\int_{t_a}^{t_b} dt\,L_{\rm e}(x,\dot x^R)\right] \tag{5.2}$$

looks like an ordinary euclidean path integral for the density matrix of a particle of mass $M = 1/2D$. As such it obeys a differential equation of the Schrödinger type. Forgetting a moment about the subtleties of the retardation, we introduce an auxiliary momentum integration and go over to the canonical representation of (5.2):

$$P(x_bt_b|x_at_a) = \int \mathcal{D}x \int \frac{\mathcal{D}p}{2\pi} \exp\left\{\int_{t_a}^{t_b} dt \left[ip\dot{x} - 2D\frac{p^2}{2} + ip\frac{1}{M\gamma}V'(x)\right]\right\}. \tag{5.3}$$

This probability distribution satisfies therefore the Schrödinger type of equation

$$H(\hat{p}_b, x_b)P(x_bt_b|x_at_a) = -\partial_{t_b}P(x_bt_b|x_at_a) \tag{5.4}$$

with the Hamiltonian operator

$$H(\hat{p}, x) \equiv 2D\frac{\hat{p}^2}{2} - i\hat{p}\frac{1}{M\gamma}V'(x). \tag{5.5}$$

In order to conserve probability, the momentum operator has to stand to the left of the potential term. Only then does the integral over $x_b$ of Eq. (5.4) vanish. Equation (5.4) is the overdamped or ordinary .

Without the retardation on $\dot{x}$ in (5.2), the path integral would certainly give a symmetrized operator $-i[\hat{p}V'(x) + V'(x)\hat{p}]/2$ in $\hat{H}$. This follows from the fact that the coupling $(1/2DM\gamma)\dot{x}V'(x)$ looks precisely like the coupling of a particle to a magnetic field with a "vector potential" $A(x) = (1/2DM\gamma)V'(x)$. In this case we can also perform immediately the path integral (5.2)

$$P_0(x_bt_b|x_at_a) \propto \int \mathcal{D}x(t) \exp\left\{-\frac{1}{2w}\int_{t_a}^{t_b} dt\, [M\gamma\dot{x} + V'(x)]^2\right\} \tag{5.6}$$

at $x_b = x_a = 0$, which is given by

$$P_0(0\,t_b|0\,t_a) \propto \text{Det}^{-1}\left[\partial_t + V'(x)/M\gamma\right], \tag{5.7}$$

where from formula (4.7)

$$\text{Det}\,\left[\partial_t + V''(x)/M\gamma\right] = \exp\left[\int dt\, V''(x)/2M\gamma\right]. \tag{5.8}$$

The effect of retardation of the velocity in (5.1) is obvious since the trivial retarded determinant (4.37) is independent of the strength of the damping:

$$\text{Det}\,\left[\partial_t^R + V''(x)/M\gamma\right] = 1. \tag{5.9}$$

In the advanced case one would have

$$\text{Det}\,\left[\partial_t^A + V''(x)/M\gamma\right] = \exp\left[\int dt\, V''(x)/M\gamma\right]. \tag{5.10}$$

For the differential equation (5.4), the difference between the ordinary and the retarded results (5.8) and (5.9) implies that the initially symmetric operator order $-i[\hat{p}V'(x)+V'(x)\hat{p}]/2$ in $\hat{H}$ is changes into $-i[\hat{p}V'(x)+V'(x)\hat{p}]/2 - V''(x)/2 - i\hat{p}V'(x)$, as necessary for conservation of probability.

As in Eq. (4.40) we can avoid the retardation of the velocity by adding to the Lagrangian (5.1) a term containing the second derivative of the potential:

$$L_e(x, \dot{x}) = \frac{1}{4D}\left[\dot{x} + \frac{1}{M\gamma}V'(x)\right]^2 - \frac{1}{2M\gamma}V''(x). \tag{5.11}$$

From this the path integral can be calculated with the same slicing as for the gauge-invariant coupling of a magnetic vector potential (see Sections 10.6 and 11.3 in the textbook [6])

$$P_0(x_bt_b|x_at_a) \propto \int \mathcal{D}x(t) \exp\left[-\frac{1}{4D}\int_{t_a}^{t_b} dt \left\{\frac{1}{4D}\left[\dot{x} + \frac{V'(x)}{M\gamma}\right]^2 - \frac{V''(x)}{2M\gamma}\right\}\right]. \tag{5.12}$$

## VI. LANGEVIN EQUATIONS

For for high $\gamma T$. the forward–backward path integral (1.37) has only small fluctuations of $y$, and $K(t,t')$ becomes a $\delta$-function. Then we can expand

$$V\left(x+\frac{y}{2}\right)-V\left(x-\frac{y}{2}\right)\sim yV'(x)+\frac{y^3}{24}V'''(x)+\dots\ , \tag{6.1}$$

keeping only the first term. We further introduce an auxiliary quantity $\eta(t)$ by

$$\eta(t)\equiv M\ddot{x}(t)+M\gamma\dot{x}^R(t)+V'(x(t)). \tag{6.2}$$

With this, the exponential function in (1.37) becomes after a partial integration of the first term using the endpoint properties $y(t_b)=y(t_a)=0$:

$$\exp\left\{-\frac{i}{\hbar}\int_{t_a}^{t_b}dt\, y\eta-\frac{w}{2\hbar^2}\int_{t_a}^{t_b}dt\, y^2(t)\right\}, \tag{6.3}$$

where $w$ is the constant (2.2). The variable $y$ can obviously be integrated out and we find a probability distribution

$$P[\eta]\propto\exp\left\{-\frac{1}{2w}\int_{t_a}^{t_b}dt\,\eta^2(t)\right\}. \tag{6.4}$$

The defining equation (6.2) for $\eta(t)$ may be viewed as a *stochastic differential equation* to be solved for arbitrary initial positions $x(t_a)=x_a$ and velocities $\dot{x}(t_a)=v_a$. The differential equation is driven by a Gaussian random *noise* variable $\eta(t)$ with a correlation function

$$\langle\eta(t)\eta(t')\rangle_T=w\,K(t-t'), \tag{6.5}$$

For each noise function $\eta(t)$, the the solution of the differential equation yields a path $x_\eta(x_a,x_b,t_a)$ with a final position $x_b=x_\eta(x_a,x_b,t_b)$ and velocity $v_b=\dot{x}_\eta(x_a,x_b,t_b)$, all being functionals of $\eta(t)$. From this we can calculate the distribution $P(x_bv_bt_b|x_av_at_a)$ of the final $x_b$ and $v_b$ by summing over all paths resulting from the noise functions $\eta(t)$ with the probability distribution (6.4). The result is of course the same as the distribution (3.7) obtained previously from the canonical path integral.

It is useful to exhibit clearly the dependence on initial and final velocities by separating the stochastic differential equation (6.2) into two first-order equations:

$$M\dot{v}(t)+M\gamma v^R(t)+V'(x(t))=\eta(t), \tag{6.6}$$

$$\dot{x}(t)=v(t), \tag{6.7}$$

to be solved for initial values $x(t_a)=x_a$ and $\dot{x}(t_a)=v_a$. For a given noise function $\eta(t)$, the final positions and velocities have the probability distribution

$$P_\eta(x\,v\,t|x_av_at_a)=\delta(x_\eta(t)-x_a)\delta(\dot{x}_\eta(t)-v_a). \tag{6.8}$$

Given these distributions for all possible noise functions $\eta(t)$, we find the final probability $P(x_av_bt_b|x_av_at_a)$ from the path integral over all $\eta(t)$ calculated with the noise distribution (6.4). We shall write this in the form

$$P(x\,v\,t|x_av_at_a)=\langle P_\eta(x\,v\,t|x_av_at_a)\rangle_\eta, \tag{6.9}$$

where the expectation value of an arbitrary functional of $F[x]$ is defined by the path integral

$$\langle F[x]\rangle_\eta\equiv\mathcal{N}\int\mathcal{D}x\,P[\eta]F[x]. \tag{6.10}$$

The normalization factor $\mathcal{N}$ is fixed by the condition $\langle\,1\,\rangle=1$, to preserve total probability.

By a change of integration variables from $x(t)$ to $\eta(t)$, the expectation value (6.10) can be rewritten as a functional integral

$$\langle F[x]\rangle_\eta\equiv\mathcal{N}\int\mathcal{D}\eta\,P[\eta]\ F[x], \tag{6.11}$$

In principle, the integrand contains a factor $J^{-1}[x]$, where $J[x]$ is the functional Jacobian

$$J[x] \equiv \mathrm{Det}\,[\delta\eta(t)/\delta x(t')] = \det[M\partial_t^2 + M\gamma\partial_t^R + V''(x(t))]. \tag{6.12}$$

However, in Eq. (4.37) we have seen that the retarded determinant is unity, thus justifying its omission in (6.11).

From the probability distribution $P(x_b v_b t_b|x_a v_a t_a)$ we find the pure position probability $P(x_b t_b|x_a t_a)$ by integrating over all initial and final velocities as in Eq. (3.6). Thus we have shown that a solution of the forward–backward path integral at high temperature (2.6) can be obtained from a solution of the stochastic differential equations (6.2), or more specifically, from the pair of stochastic differential equations (6.6) and (6.7).

## VII. SUPERSYMMETRY

Recalling the origin (5.8) of the extra last term in the exponent of the path integral (5.12), this can be rewritten in a slightly more implicit but useful way as

$$P_0(x_b t_b|x_a t_a) \propto \int \mathcal{D}x(t)\,\mathrm{Det}\left[\partial_t + \frac{V''(x)}{M\gamma}\right]\exp\left\{-\frac{1}{4D}\int_{t_a}^{t_b} dt\,\frac{1}{4D}\left[\dot{x} + \frac{V'(x)}{M\gamma}\right]^2\right\} \tag{7.1}$$

In the form (7.1), the time ordering of the velocity term is arbitrary. It may be quantum mechanical, but equally well retarded or advanced, as long as it appears in the same way in both the Lagrangian and determinant. An interesting structural observation is possible by generating the determinant with the help of an auxiliary fermion field $c(t)$ from a path integral over $c(t)$:

$$\det[\partial_t + V''(x(t))/M\gamma] \propto \int \mathcal{D}c\mathcal{D}\bar{c}\, e^{-\int dt\bar{c}(t)\left[M\gamma\partial_t + V''(x(t))\right]c(t)}. \tag{7.2}$$

In quantum field theory, such auxiliary fermionic fields are referred to as *ghost fields*. With these we can rewrite the path integral (5.2) for the probability as an ordinary path integral

$$P(x_b t_b|x_a t_a) = \int \mathcal{D}x \int \mathcal{D}c\mathcal{D}\bar{c}\,\exp\left\{-\mathcal{A}_{\mathrm{PS}}[x,c,\bar{c}]\right\}. \tag{7.3}$$

where $\mathcal{A}_{\mathrm{PS}}$ is the euclidean action

$$\mathcal{A}_{\mathrm{PS}} = \frac{1}{2DM^2\gamma^2}\int_{t_a}^{t_b} dt\left\{\frac{1}{2}\left[M\gamma\dot{x} + V'(x)\right]^2 + \bar{c}(t)\left[M\gamma\partial_t + V''(x(t))\right]c(t)\right\}, \tag{7.4}$$

first written down by Parisi and Sourlas.[2] This action has a particular property: If we denote the expression in the first brackets by

$$U_x \equiv M\gamma\partial_t x + V'(x), \tag{7.5}$$

the operator between the Grassmann variables in (7.4) is simply the functional derivative of $U_x$:

$$U_{xy} \equiv \frac{\delta U_x}{\delta y} = M\gamma\partial_t + V''(x). \tag{7.6}$$

Thus we may write

$$\mathcal{A}_{\mathrm{PS}} = \frac{1}{2D}\int_{t_a}^{t_b} dt\left[\frac{1}{2}U_x^2 + \bar{c}(t)\,U_{xy}c(t)\right], \tag{7.7}$$

where $U_{xy}c(t)$ is the usual short notation for the functional matrix multiplication $\int dt' U_{xy}(t,t')c(t')$. The relation between the two terms makes this action *supersymmetric*. It is invariant under transformations which mix the Fermi

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[2] G. Parisi and N. Sourlas, Phys. Rev. Lett. *43*, 744 (1979); Nucl. Phys. B *206*, 321 (1982).

and Bose degrees of freedom. Denoting by $\varepsilon$ a small anticommuting Grassmann variable, the action is invariant under the field transformations

$$\delta x(t) = \bar{\varepsilon} c(t) + \bar{c}(t)\varepsilon, \tag{7.8}$$

$$\delta \bar{c}(t) = -\bar{\varepsilon} U_x, \tag{7.9}$$

$$\delta c(t) = U_x \varepsilon. \tag{7.10}$$

The invariance follows immediately after observing that

$$\delta U_x = \bar{\varepsilon} U_{xy} c(t) + \bar{c}(t) U_{xy} \varepsilon. \tag{7.11}$$

Formally, a similar construction is also possible for a particle with inertia in the path integral (2.6), which is an ordinary path integral involving the Lagrangian (4.40). Here we can write

$$P(x_b t_b | x_a t_a) = \mathcal{N} \int \mathcal{D}x(t)\, J[x] \exp\left\{ -\frac{1}{2w} \int_{t_a}^{t_b} dt\, [M\ddot{x} + M\gamma\dot{x} + V'(x)]^2 \right\}. \tag{7.12}$$

where $J[x]$ abbreviates the determinant

$$J[x] = \det[M\partial_t^2 + M\gamma\partial_t + V''(x(t))] \tag{7.13}$$

which is known from formula (4.28). The path integral (7.12) is valid for *any* ordering of the velocity term, as long as it is the same in the exponent and the functional determinant.

We may now express the functional determinant as a path integral over fermionic ghost fields

$$J[x] = \det[M\partial_t^2 + M\gamma\partial_t + V''(x(t))] \propto \int \mathcal{D}c\mathcal{D}\bar{c}\, e^{-\int dt\, \bar{c}(t)\left[M\partial_t^2 + M\gamma\partial_t + V''(x(t))\right]c(t)}, \tag{7.14}$$

and rewrite the probability $P(x_b t_b | x_a t_a)$ as an ordinary path integral

$$P(x_b t_b | x_a t_a) \propto \int \mathcal{D}x(t) \int \mathcal{D}c\mathcal{D}\bar{c} \exp\{-\mathcal{A}^{\rm KS}[x, \bar{c}]\}, \tag{7.15}$$

where $\mathcal{A}[x, \bar{c}]$ is the euclidean action

$$\mathcal{A}^{\rm KS}[x, \bar{c}] \equiv \int_{t_a}^{t_b} dt \left\{ \frac{1}{2w} [M\ddot{x} + M\gamma\dot{x} + V'(x)]^2 + \bar{c}(t) \left[ M\partial_t^2 + M\gamma\partial_t + V''(x(t)) \right] c(t) \right\}. \tag{7.16}$$

This formal expression contains subtleties arising from the boundary conditions when calculating the Jacobian (7.14) from the functional integral on the right-hand side. It is necessary to factorize the second-order operator in the functional determinant and express each factor as a determinant as in (7.14). At the end, the action is again supersymmetric, but there are twice as many auxiliary Fermi fields [12].

## VIII. STOCHASTIC QUANTUM LIOUVILLE EQUATION

At lower temperatures, where quantum fluctuations become important, the forward–backward path integral (1.37) does not allow us to derive a Schrödinger-like differential equation for the probability distribution $P(x\, v\, t | x_a v_a t_t)$. To see the obstacle, we go over to the canonical representation of (1.37):

$$|(x_b t_b | x_a t_a)|^2 = \int \mathcal{D}x(t)\, \mathcal{D}y(t) \int \frac{\mathcal{D}p(t)}{2\pi} \frac{\mathcal{D}p_y(t)}{2\pi} \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt\, [p\dot{x} + p_y\dot{y} - H_T] \right\}, \tag{8.1}$$

where

$$H_T = \frac{1}{M} p_y p_x + \gamma p_y y + V(x + y/2) - V(x - y/2) - i\frac{w}{2\hbar} y\hat{K}y \tag{8.2}$$

plays the role of a Hamiltonian. Here $\hat{K}y(t)$ denotes the product of the functional matrix $\hat{K}(t,t')$ with the functional vector $y(t')$ defined by $\hat{K}y(t) \equiv \int dt' K(t,t')y(t')$. After omitting the $y$-integrations at the endpoints, we obtain a path integral representation for the product of amplitudes

$$U(x_b y_b t_b | x_a y_a t_a) \equiv (x_b + y_b/2\; t_b | x_a + y_a/2\; t_a)(x_b - y_b/2\; t_b | x_a - y_a/2\; t_a)^*. \tag{8.3}$$

This plays the role of a time-evolution amplitude for the quantum statistical density matrix $\rho(x,y;t)$, satisfying

$$\rho(x,y;t) = \int dx_a\, dy_a U(x_b y_b t_b | x_a y_a t_a) \rho(x_a, y_a; t_a). \tag{8.4}$$

The Fourier transform of $\rho(x,y;t)$ this with respect to $y$ is the Wigner function

$$W(x,p,t) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ipy_b/\hbar} \rho(x\, y; t). \tag{8.5}$$

When considering the change of $U(x\, y\, t | x_a y_a t_a)$ over a small time interval $\epsilon$, the momentum variables $p$ and $p_y$ have the same effect as differential operators $-i\partial_{x_b}$ and $-i\partial_{y_b}$, respectively. The last term in $H_T$, however, is nonlocal in time, thus preventing a derivation of a Schrödinger-like differential equation.

The locality problem can be removed by introducing a noise variable $\eta(t)$ with the correlations function

$$\langle \eta(t)\eta(t') \rangle_\eta = \frac{w}{2} K^{-1}(t,t'). \tag{8.6}$$

Then we can define a temporally local $\eta$-dependent Hamilton operator

$$\hat{H}_\eta \equiv \frac{1}{M} (\hat{p}_x + \gamma y) \hat{p}_y + V(x+y/2) - V(x-y/2) - y\eta \tag{8.7}$$

which governs the evolution of $\eta$-dependent versions of the amplitude products (8.3) via the *stochastic Schrödinger equation*

$$i\hbar\partial_t U_\eta(x\, y\, t | x_a y_a t_a) = \hat{H}_\eta\, U_\eta(x\, y\, t | x_a y_a t_a). \tag{8.8}$$

Averaging this equation over $\eta$ using (8.6) yields for $y_a = y_b = 0$ the same probability distribution as the forward–backward path integral (1.37):

$$|(x_b t_b | x_a t_a)|^2 = U(x_b\, 0\, t_b | x_a\, 0\, t_a) \equiv \langle U(x_b\, 0\, t_b | x_a\, y_a\, t_a) \rangle_\eta \tag{8.9}$$

At high temperatures, the noise averaged stochastic Schrödinger equation (8.8) takes the form

$$i\hbar\partial_t U(x\, y\, t | x_a\, y_a\, t_a) = \hat{\bar{H}} U(x\, y\, t | x_a\, y_a\, t_a), \tag{8.10}$$

where $\hat{\bar{H}}$ is the Hamiltonian associated with the Lagrangian in the forward–backward path integral (1.37):

$$\hat{\bar{H}} \equiv \frac{1}{M} \hat{p}_y \hat{p}_x + \gamma y \hat{p}_y + V(x+y/2) - V(x-y/2) - i\frac{w}{2\hbar} y^2. \tag{8.11}$$

In terms of the separate path positions $x_\pm = x \pm y/2$ where $p_x = \partial_+ + \partial_-$ and $p_y = (\partial_+ - \partial_-)/2$, this takes the more familiar form

$$\hat{\bar{H}} \equiv \frac{1}{2M} \left(\hat{p}_+^2 - \hat{p}_-^2\right) + V(x_+) - V(x_-) + \frac{\gamma}{2}(x_+ - x_-)(\hat{p}_+ - \hat{p}_-) - i\hbar\Lambda(x_+ - x_-)^2. \tag{8.12}$$

In the last term we have introduced a useful quantity, the so-called *decoherence rate per square distance*

$$\Lambda \equiv \frac{w}{2\hbar^2} = \frac{M\gamma k_B T}{\hbar^2}. \tag{8.13}$$

It is composed of the damping rate $\gamma$ and the squared thermal length

$$l_{\rm e}(\hbar\beta) \equiv \sqrt{2\pi\hbar^2\beta/M} \tag{8.14}$$

as

$$\Lambda = \frac{2\pi\gamma}{l_{\rm e}^2(\hbar\beta)}, \tag{8.15}$$

and controls the rate of decay of interference peaks.[3]

Note that the order of the operators in the mixed term of the form $y\hat{p}_y$ in Eq. (8.11) is opposite to the mixed term $-i\hat{p}_v v$ in the differential operator (3.4) of the Fokker-Planck equation. This order is necessary to guarantee the conservation of probability. Indeed, multiplying the time evolution equation (8.10) by $\delta(y)$, and integrating both sides over $x$ and $y$, the left-hand side vanishes.

The correctness of this order can be verified by calculating the fluctuation determinant of the path integral for the product of amplitudes (8.3) in the Lagrangian form, which looks just like (1.37), except that the difference between forward and backward trajectories $y(t) = x_+(t) - x_-(t)$ is nonzero at the endpoints. For the fluctuation which vanish at the end points, this is irrelevant. As explained before, the order is a short-time issue, and we can take $t_b - t_a \to \infty$. Moreover, since the order is independent of the potential, we may consider only the free case $V(x \pm y/2) \equiv 0$. The relevant fluctuation determinant was calculated in formula (4.9). In the Hamiltonian operator (8.11), this implies an additional energy $-i\gamma/2$ with respect to the symmetrically ordered term $\gamma\{y, \hat{p}_y\}/2$, which brings it to $\gamma y\hat{p}_y$, and thus the order in (8.12).

## IX. CONCLUSION

With the help of analytic regularization we have shown that the forward–backward path integral of a point particle in a thermal bath of harmonic oscillators yields, at large temperature, a probability distribution obeying a Fokker-Planck equation with the correct operator ordering which ensures probability conservation. By the same token, they yield the correct Langevin equations with and without inertia.

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[1] R.P. Feynman and F.L. Vernon, Ann. Phys. (N. Y.) *24*, 118 (1963); also Sections 12.8 and 12.9 of Ref. [2].

[2] R.P. Feynman A.R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.

[3] H. Dekker, Phys. Rep. **80**, 1 (1981)

[4] A.G. Caldeira and A.J. Leggett, Phys. Rev. Lett. **46**, 211 (1981). Ann. Phys. **149**, 374 (1983); **153**, 445(E); Physica A **121**, 587 (1983); A **130** 374(E) (1985); Phys. Rev. A **31**, 1059 (1985).

[5] U. Weiss, *Quantum Dissipative Systems*, World Scientific, 1993.

[6] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics,* World Scientific, Singapore 1995, (http://www.physik.fu-berlin.de/~kleinert/re0.html#b5)

[7] G. 't Hooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972).

[8] H. Kleinert, A. Chervyakov, Phys. Lett. B **464** , 257 (1999) (hep-th/9906156); Phys. Lett. B (in press) (quant-ph/0002067); Phys. Lett. A **273**, 1 (2000) (quant-ph/0003095)

[9] H. Kleinert, J. Math. Phys. *27*, 3003 (1986).

[10] H. Kleinert, *Gauge Fields in Condensed Matter*, op. cit., Vol. II, World Scientific, Singapore (1989)

[11] A. Schmid, J. Low Temp. Phys. *49*, 609 (1982).
To solve the operator ordering problem, Schmid assumes that a time-sliced derivation of the forward–backwards path integral would yield a sliced version of the stochastic differential equation (6.2) $\eta_n \equiv (M/\epsilon)(x_n - 2x_{n-1} + x_{n-2}) + +(M\gamma/2)(x_n - x_{n-2}) + \epsilon V'(x_{n-1}$. The matrix $\partial\eta/\partial x$ has a constant determinant $(M/\epsilon)^N(1+\epsilon\gamma/2)^N$. His argument [cited also in the textbook by
U. Weiss, *Quantum Dissipative Systems*, World Scientific, 1993,
in the discussion following Eq. (5.93)] is unacceptable for two reasons: First, his slicing is not derived. Second, the resulting determinant has the wrong continuum limit proportional to $\exp\left[\int dt\,\gamma/2\right]$ for $\epsilon \to 0$, $N = (t_b - t_a)/\epsilon \to \infty$, corresponding to the unretarded functional determinant (4.28), whereas the correct limit should be $\gamma$-independent, by Eq. (4.35).
The above textbook by U. Weiss contains many applications of nonequilibrium path integrals.


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[3] See the collection of articles D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.O. Stamatescu, H.D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory*, Springer, Berlin, 1996.

[12] H. Kleinert and S. Shabanov, Phys. Lett. A *235*, 105 (1997) (quant-ph/9705042).