

Three-loop critical exponents, amplitude functions and amplitude ratios from variational perturbation theory

H. Kleinert, B. Van den Bossche*[†]

Institut für Theoretische Physik, Arnimallee 14 D-14195 Berlin, Germany

We use variational perturbation theory to calculate various universal amplitude ratios above and below T_c in minimally subtracted ϕ^4 -theory with N components in three dimensions. In order to best exhibit the method as a powerful alternative to Borel resummation techniques, we consider only to two- and three-loops expressions where our results are analytic expressions. For the critical exponents, we also extend existing analytic expressions for two loops to three loops.

I. INTRODUCTION

Recently, quantum mechanical variational perturbation theory [1] has been successfully extended to quantum field theory, where it has proven to be a powerful tool for determining critical exponents in three [2–4] as well as in $4 - \epsilon$ dimensions [5,6]. The purpose of this paper is to apply this theory to amplitude ratios which can be measured experimentally. Their perturbation expansions suffer from the same asymptotic nature as those of the critical exponents, thus requiring delicate resummation procedures. These have been the subject of numerous studies, of which we can only mention a few, by various groups. There are two main approaches followed by various authors which we shall divide according to their method into a Paris school and a Parisi school.

The Paris school follows Wilson's ideas [7,8] by considering epsilon expansions in $D = 4 - \epsilon$ dimensions, making use of the fact that in the upper critical dimension $D_{\text{up}} = 4$ the theory is scale invariant. The results are at first power series in the renormalized coupling constant g . For small ϵ , the coupling constant goes, in the critical limit of vanishing mass, to a stable infrared fixed point $g \rightarrow g^*$, where scaling laws are found [9]. The position of the fixed point is found as a power series in ϵ which makes critical exponents and amplitude functions likewise power series in ϵ . These series diverge. The large-order behavior [10,11] suggests that these series are Borel summable [12,13]. The exact ϵ -expansions of the critical exponents are known up to five loops [14,15]. They have been resummed with the

*Alexander von Humboldt Fellow

[†]Collaborateur scientifique du FNRS, On leave from absence of Physique Nucléaire Théorique, B5, Université de Liège Sart-Tilman, 4000 Liège, Belgium

help of Borel transformations and analytic mapping methods in Refs. [16–18].

The Parisi school follows Ref. [19] in studying perturbation expansions directly in $D = 3$ dimensions [20–25]. In the original works, renormalization conditions are used according to which renormalized correlation functions should behave for small momenta like $G(p) \approx (p^2 + m^2)^{-1}$. Recently, these normalization conditions have been replaced by dimensional regularization near $D = 3$ to remove divergences (see for instance [23] which uses a regularization in $D = 3 - \varepsilon$ dimensions). Contrary to the ε -expansions around $D_{\text{up}} = 4$, the system is not treated near the dimension of naive scale-invariance, and the scaling properties are no longer obvious order by order in g . In addition, singular terms violating Griffith's analyticity are introduced which show up by amplitudes having unpleasant logarithmic dependences on the coupling constant.

The universal amplitude ratios were first discussed in [26] in the context of Wilson's renormalization group approach, and by Bervillier [27] within the field theoretic approach developed in [9]. The experimentally most easily accessible amplitude ratios are formed from the amplitudes of the leading power behaviors of various physical quantities in $T - T_c$ above and below the critical temperature T_c . A typical example, and one of the best measured amplitude ratios, is for the specific heat of superfluid helium above and below T_c . It was obtained in a zero-gravity experiment by Lipa et al. [28], who parameterized the specific heat as follows (we use the second of the references quoted in [28]):

$$C^\pm = \frac{A^\pm}{\alpha} |t|^{-\alpha} (1 + D|t|^\Delta + E|t|^{2\Delta}) + B, \quad t = T/T_c - 1, \quad (1)$$

with $\alpha = -0.01056 \pm 0.0004$, $\Delta = 0.5$, $A^+/A^- = 1.0442 \pm 0.001$, $A^-/\alpha = -525.03$, $D = -0.00687$, $E = 0.2152$ and $B = 538.55$ (J/mol K). This parametrization is an approximation to the Wegner expansion form

$$F = F_\pm |t|^\chi \left(1 + a_{0,1}|t|^{\Delta_0} + a_{0,2}|t|^{2\Delta_0} + a_{0,3}|t|^{3\Delta_0} + \dots \right. \\ \left. + a_{1,1}|t|^{\Delta_1} + a_{1,2}|t|^{2\Delta_1} + a_{1,3}|t|^{3\Delta_1} + \dots \right) \quad (2)$$

with χ a combination of critical exponents and F_\pm denoting the leading amplitude above and below T_c , respectively. Compared to this general Wegner expansion, higher powers in $\Delta_0 \equiv \Delta$ have been neglected in (1), as well as daughter powers Δ_i , $i \geq 1$. This will be also the case in the present work, where we shall take into account only one exponent Δ , related to ω by the relation $\Delta = \omega\nu$.

Further amplitude ratios are formed from the amplitudes $a_{i,j}$ of the nonleading power behaviors in $T - T_c$, the so-called confluent terms. Amplitude ratios of confluent terms are also universal quantities [25,29–31]. They are known up to five loops below T_c [25] ($N = 1$), or even up to six loops [30] ($N = 1, 2, 3$) if only the $T > T_c$ regime is considered. (N is the number of components of the field.) None of them will be examined here.

Apart from critical exponents and amplitude ratios, experimental observations show that the equation of state and the free energy have a simple scaling form of the Widom type, whose field-theoretic explanation can be found in various textbooks [9,12,13]. For example, the free energy of a system with magnetization M_B may be represented near T_c by $F(t, M_B) = |t|^{2-\alpha} f(|t|/M_B^{1/\beta})$, with α and β being critical exponents and t is the relative distance to the critical temperature. The scaling equation of state has been calculated in ϵ -expansions to order ϵ^2 for general $O(N)$ -symmetry [32] and to order ϵ^3 for the Ising model ($N = 1$) [33].

A. Perturbative calculation of amplitude ratios

Amplitude ratios relate the properties of the disordered phase, which are easy to calculate, to those of the ordered phase, which are much harder to derive. Several methods have been proposed to connect the two phases. One of them is due to Bagnuls and Bervillier [24], and was applied further in [25,34]. A similar procedure was followed in [22,23] for the amplitude ratio of correlation lengths, which had been omitted by Bagnuls and Bervillier. Calculations in three dimensions are usually numerical [20,24,25], although low orders can be treated analytically (see [22,23,34] for analytic three-loop results). Such analytic studies are important since they offer insight into the nonanalyticity with respect to the coupling constant. The amplitude ratio found in [22,23] is restricted to the Ising case $N = 1$. The same is true for [35], which includes all diagrams up to three loops, as is the case of [34]. The latter work also incorporates a cubic anisotropic term, but for which the replica limit $N \rightarrow 0$ is considered, allowing to probe the critical behavior of the weakly dilute Ising system.

All power series are divergent and require resummation. Numerically, this has been done for the Ising model in [21] to five loops for the critical exponents, various amplitude ratios, and the equation of state. Reference [21] also contains comparisons between the results of different groups (both for $D = 3$ and $D = 4 - \epsilon$), with experiments and with high-temperature series. For the most up-to-date work, see [18], which besides the critical exponents and amplitude ratios for the Ising model gives also the critical exponents for general $O(N)$ symmetry.

An other approach has been followed by Dohm and collaborators in Aachen [36–38] who proposed to use an analytic renormalization scheme in the form of minimal subtraction when working in $D = 3$ dimensions. The use of the minimal subtraction scheme in field theories at fixed dimensions $2 < D < 4$ has one important advantage: the renormalization constants are the same in both the symmetric phase with $T > T_c$ and the ordered phase with $T < T_c$. The renormalization constants are power series in the renormalized coupling constant with coefficients which are poles in ϵ up to the given order of the perturbative series

$$Z = 1 + \sum_{i=1}^L a_i g^i, \quad a_i = \sum_{j=1}^i b_j \epsilon^{-j}. \quad (3)$$

The most important property of this scheme is that the mass does not enter explicitly the expansions, which can therefore be used on both side of T_c . Since the critical exponents are related to the renormalization constants, the mass independence of the Z_i implies a clear decomposition of the correlation functions into amplitude functions and power parts. Working in three dimensions, there is a prize to pay: logarithmic singularities in the coupling constant. They can be removed using suitable length scales. This may be the physical length scale ξ^+ above T_c , and an other length scale ξ_- related, in the critical regime, to the longitudinal mass below T_c . Since they are not exactly equals, the Aachen group call the length scale ξ_- a pseudolength. A precise definition of ξ_- has been given in [36]. With different collaborators, Dohm has applied this scheme to derive various critical exponents and renormalization-group functions above T_c [37], to calculate the heat capacity, the order parameter and the superfluid density (both above and below T_c), as well as some useful universal combination of observable quantities [38]. So far, these works have been limited to low orders. The Ising model is the simplest system, since it contains no massless Goldstone modes which cause extra infrared singularities at intermediate stages of perturbative calculations of the thermodynamical quantities on the coexistence curve where the external magnetic field vanishes. The infrared singularities are the reason why the analytical equation of state and amplitude functions below T_c have been restricted [27,38,39] to two loops for general N . These extra infrared singularities, which cancel at the end of the calculations, are caused by the physical singularities of the transverse susceptibility. Being physical, they remain at the end. Due to these difficulties, numerical studies up to five loops below T_c , with accurate Borel resummation are available only for the Ising case [21,25,40,41]. Only analytic three-loop calculations for the thermodynamic quantities below T_c have become recently available for the general $O(N)$ -system [42]. Based on these, calculations in which some contributions were evaluated up to five loops were done for amplitude ratios at $N = 2$ and $N = 3$ [41], proceeding as follow: Amplitude functions for the heat capacity were calculated using the three-loop result of [42] and five-loop results for the vacuum renormalization constant [41,43] and the critical exponent α . For α , use has been made of the values given in [17] for $N = 1$, of the value given in the first of Ref. [28] for $N = 2$ (this being the initial result of the space shuttle experiment, which was subsequently corrected), and of the value given in Ref. [44] for $N = 3$. Since then, the works [2-6,13,18] have appeared and seem to be the best available references concerning resummed data. Although this is not the main subject of this paper, it is interesting to see in which way the new values of α affect the amplitude ratios of the heat capacity given in [41]. This will be done in Section IV.

In the following, we shall calculate amplitude ratios with the help of Kleinert's variational perturbation theory [1–6,13]. To exhibit the method most clearly, we shall base our study on analytical results only. This will restrict us to the level of three loops. Working at such low orders, the accuracy of our resummed values cannot compete with some existing five-loop calculations. For this reason, we shall not include nor discuss error bars in the final results.

To illustrate the method of variational perturbation theory, we shall first show how to obtain analytic expressions for the critical exponents, thus extending an earlier two-loop analytic calculation in Ref. [5]. After this, we apply the procedure to amplitude ratios of various experimental quantities. The critical exponents are computed directly from the renormalization constants of the theory. In the minimal subtraction scheme, the renormalization constants have only pole terms in ϵ . For the amplitude functions, this is no longer true: in a $D = 4 - \epsilon$ approach, they have to be expanded in ϵ . For this reason it is not a priori clear at which level the variational method has to be applied. For the purpose of showing the power of the method to resum amplitude ratios, it is then better to calculate amplitude ratios in three dimensions. A resummation of amplitude ratios within the ϵ -expansion method is postponed to a later publication [45]. We shall also consider only the expansions of the Aachen group, especially their analytical two-loop [46] and three-loop [42] expansions. As a bonus, since the renormalization constants are the same (apart for trivial coefficients coming from the respective conventions) in the minimal subtraction scheme in $D = 4 - \epsilon$ dimensions and in fixed $D = 3$ dimensions, the critical exponents will be the same in variational perturbation theory. This will be shown explicitly below.

The paper is organized as follows. In Section II, we define the model and the conventions. In Section III, we briefly review the strong-coupling approach and apply it to the evaluation of the critical exponents at the level of two and three loops, extending the results of Ref. [5]. Section IV is the main part of this paper, where we show how the strong-coupling limit of various amplitudes and amplitude ratios are determined. In Section IVF, we use the latest available value for the exponents α and ν [2–6,13,18] to calculate the amplitude ratio of the heat capacity and the universal combination R_C (constructed from the leading amplitudes of the heat capacity, the order parameter and the susceptibility above T_c), for $N = 0, \dots, 4$, and to calculate the amplitude ratio of the susceptibilities in the Ising model ($N = 1$). Finally, we draw our conclusion in Section V. For completeness, we have added an Appendix containing all formulas taken from other publications, and calculations related to them.

II. MODEL AND CONVENTIONS

The critical behavior of many different physical systems can be described by an $O(N)$ -symmetric ϕ^4 -theory. In particular, the case $N = 0$ describes polymers, $N = 1$ the Ising transition (a universality class which comprises binary fluids, liquid-vapor transitions and antiferromagnets), $N = 2$ the superfluid Helium transition, $N = 3$ isotropic magnets (transition of the Heisenberg type), and $N = 4$ phase transition of Higgs fields at finite temperature. In the presence of an external field h_B , the field energy is given by the Ginzburg-Landau functional

$$\mathcal{H} = \int d^D x \left[\frac{1}{2}(\nabla\phi_B)^2 + \frac{1}{2}r_0\phi_B^2 + u_B(\phi_B^2)^2 - h_B\cdot\phi_B \right]. \quad (4)$$

To facilitate comparisons with the results of the Aachen group [42,46], we use the same normalizations. The fields ϕ_B and the external magnetic field h_B have N components, u_B is the bare coupling constant, and r_0 a bare mass term, to be specified later. The integrals are evaluated in dimensional regularization. In dimension $D = 3$, ϕ^4 -theory is superrenormalizable. This means that only a finite number of counterterms have to be added in order to make observables finite. More economically, the divergencies can be removed by a shift of the mass term and reexpanding in $r_0 - r_{0c}$, where r_{0c} is the critical value of r_0 . In ϵ -expansions, r_{0c} vanishes. Near the critical temperature, r_0 behaves like $r_{0c} + a_0 t$, where t is the reduced temperature $(T - T_c)/T_c$. When working near $D = 3$ dimensions, it is possible to use a simplified shift δr_0 that only contains the $D = 3$ pole of r_{0c} (and not the poles at $D_l > 3$ with $l = 3, 4, 5, \dots$, where $D_l \equiv 4 - 2/l$). For convenience, we write the differences as a new mass term: $r_0 - r_{0c} = m_B^2$ and $r_0 - \delta r_0 = m_B'^2$. In this way, we arrive to a new bare theory, with a mass term $m_B'^2$ which may be considered as the physical square mass of the theory. The introduction of the mass m_B' makes the theory finite. It has however to be distinguished from the mass, field and coupling constant renormalization which still has to be performed: this latter renormalization, related to the introduction of the renormalization constants Z_i , is nothing else than a change of variables reflecting the fundamental scale-invariance hypothesis of the renormalization group approach. The distinction between the two steps – making the theory finite and renormalizing – is irrelevant in $D = 4 - \epsilon$ dimensions because $r_{0c} = 0$ at $\epsilon = 0$: Finiteness of the theory and the renormalization program are more intimately related than in $D = 3$ dimensions. For a thorough discussion of the difference between the renormalization in $D = 4 - \epsilon$ and fixed $D = 3$ dimensions, see [24,25], in particular p. 7215 in [24].

Within the minimal renormalization scheme, the renormalization constants Z , which are introduced to remove the poles at $D = 4$, are given by

$$m_B^2 = m^2 \frac{Z_{m^2}}{Z_\phi}, \quad (5)$$

$$A_D u_B = \mu^\epsilon \frac{Z_u}{Z_\phi^2} u, \quad (6)$$

$$\phi_B = Z_\phi^{1/2} \phi, \quad (7)$$

the quantities on the right-hand-side being the renormalized ones. In Eq. (6), μ is an arbitrary reference mass scale and

$$A_D = \Gamma(1 + \epsilon/2)\Gamma(1 - \epsilon/2)\bar{S}_D, \quad \text{with } \bar{S}_D = \frac{2\pi^{D/2}}{\Gamma(D/2)(2\pi)^D} \quad (8)$$

is a convenient geometric factor. The number \bar{S}_D is equal to $S_D/(2\pi)^D$ where S_D is the surface of a sphere in D -dimensions. Since A_D goes to \bar{S}_D when $D \rightarrow 4$, the renormalization constants have the same form [47] in $D = 3$ as in $D = 4 - \epsilon$, and the resummation for the critical exponents is identical for the two approaches. This will be made clear below. For the amplitude calculations, however, things are different: If the expansions are truncated at some order, they turn out to depend on the difference between A_D and \bar{S}_D . Rather than saying that the normalization of A_D is a matter of convenience to simplify the D -dependence of lower order results [36,37], we shall see that the use of the geometric factor (8) improves low-order results: For example, the one-loop expansion of the amplitude function for the order parameter is identical to the zero-loop order [38].

With these conventions and notations, the renormalization constants in minimal subtraction are given up to three loops by [13–15]

$$Z_{m^2} = 1 + \frac{4(N+2)}{\epsilon} u + 8(N+2) \left[\frac{2(N+5)}{\epsilon^2} - \frac{3}{\epsilon} \right] u^2 + 8(N+2) \left[\frac{8(N+5)(N+6)}{\epsilon^3} - \frac{4(11N+50)}{\epsilon^2} + \frac{31N+230}{\epsilon} \right] u^3, \quad (9)$$

$$Z_u = 1 + \frac{4(N+8)}{\epsilon} u + 16 \left[\frac{(N+8)^2}{\epsilon^2} - \frac{5N+22}{\epsilon} \right] u^2 + \frac{8}{3} \left[\frac{24(N+8)^3}{\epsilon^3} - \frac{16(N+8)(17N+76)}{\epsilon^2} + \frac{96\zeta(3)(5N+22) + 35N^2 + 942N + 2992}{\epsilon} \right] u^3, \quad (10)$$

$$Z_\phi = 1 - \frac{4(N+2)}{\epsilon} u^2 - \frac{8}{3}(N+2)(N+8) \left(\frac{4}{\epsilon^2} - \frac{1}{\epsilon} \right) u^3. \quad (11)$$

They are related to that in Ref. [13] by the replacement $u \rightarrow g/12$. This factor comes from the different coefficient of the coupling term $u \rightarrow g/4!$ in (4) and the fact that a factor $1/(4\pi)^2$ is absorbed in the definition of g in [13], whereas a factor $A_{D=4} = 1/(8\pi^2)$ is included here.

These renormalization constants serve to calculate all critical exponents including the exponent ω which characterizes the approach to scaling. This is the subject of Section III in which we illustrate the working of variational perturbation theory.

III. EXACT CRITICAL EXPONENTS UP TO THREE LOOPS

Variational perturbation theory has been developed for the calculation of critical exponents in [2] and [5] in $D = 3$ and $D = 4 - \epsilon$ dimensions, respectively. A review can be found in the textbook [13]. So we need to recall here only the main steps of the procedure.

Let $f_L(\bar{u}_B)$ be the partial sum of order L of a power series

$$f \approx f_L(\bar{u}_B) = \sum_{i=0}^L f_i \bar{u}_B^i. \quad (12)$$

In the present context,

$$\bar{u}_B = u_B \mu^{-\epsilon} A_D \quad (13)$$

with $D = 3$ and $\epsilon = 1$, i.e., $\bar{u}_B = u_B/(4\pi\mu)$. The mass scale μ will be specified later. As seen from Eq. (6), this scale leads to a dimensionless coupling constant \bar{u}_B . We assume that in Eq. (12), the ultraviolet divergencies have been removed. In $D = 3$ dimensions, this is achieved by working with m_B^2 instead of r_0 . However, r_{0c} is a nonperturbative quantity in three dimensions, and working with m_B^2 or $m_B'^2$ generates nonanalyticities due to the presence of logarithms of the coupling constant. These will be removed by the introduction of the correlation length ζ_+ above T_c and of the length ζ_- below T_c , see [38]. The mass scale μ will be identified with the inverse of these correlation lengths ζ_{\pm}^{-1} in the two phases. Since the correlations lengths go to infinity like $|t|^{-\nu}$ as the critical point is approached, the series have to be evaluated in the limit of an infinite dimensionless bare coupling constant \bar{u}_B . In the renormalization group approach, this regime is studied by mapping the expressions into a regime of finite renormalized quantities using the renormalization constants (5)–(7). If we can find directly the strong-coupling limit, this renormalization is avoidable. To understand this, consider the relation between the renormalized and the bare coupling constant at the one-loop order $u = u_B \mu^{-\epsilon} - c/\epsilon(u_B \mu^{\epsilon})^2$, where c is a constant. At the critical point, $\mu \rightarrow 0$, or $\bar{u}_B \rightarrow \infty$, and the series expansion breaks down. If we sum a ladder of loop diagrams, we obtain $1/u = 1/(u_B \mu^{-\epsilon}) + c/\epsilon$. Now critical theory can easily be reached to give a renormalized $u^* = \epsilon/c$. A strong-coupling expansion in the bare coupling will turn out to give the same result. From our point of view, the renormalization group approach is simply a specific procedure of evaluating power series in the strong-coupling limit.

In $D = 4 - \epsilon$ dimensions, the situation is slightly more involved since renormalization is also necessary to obtain UV-finite quantities, the mass shift $r_0 - r_{0c}$ not being sufficient for this goal as in the superrenormalizable case $D = 3$, since $r_{0c} = 0$ as $\epsilon \rightarrow 0$. As far as this paper is concerned, we shall make use of the fact that $D = 3$ and $D = 4 - \epsilon$

dimensions series expansions in term of renormalized quantities are available in the literature. These will be converted back to bare expansion, using the inverse of Eqs. (5)–(7). For $D = 3$ dimensions, this expresses all physical quantities in powers of u_B/μ . The mass scale μ is identified with ζ_{\pm}^{-1} in the disordered or ordered phase, respectively. In $D = 4 - \epsilon$ dimensions, the critical theory is obtained by identifying $\mu \rightarrow m$ with the renormalized mass m in the disordered phase. In a subsequent publication [48], we will show how to perform directly a calculation in term of UV-finite bare quantities in $D = 4 - \epsilon$. In this way, the renormalization procedure is superfluous, our sole problem being the evaluation of the expansions in the limit of infinite coupling constant.

Inverting Eq. (6), we have the expansion

$$u = \bar{u}_B \left\{ 1 - \frac{4(N+8)}{\epsilon} \bar{u}_B + 8 \left[\frac{2(N+8)^2}{\epsilon^2} + \frac{3(3N+14)}{\epsilon} \right] \bar{u}_B^2 - 8 \left[\frac{8(N+8)^3}{\epsilon^3} + \frac{32(N+8)(3N+14)}{\epsilon^2} + \frac{96\zeta(3)(5N+22) + 33N^2 + 922N + 2960}{3\epsilon} \right] \bar{u}_B^3 \right\}. \quad (14)$$

The expansion (14) has the same strong-coupling limit in $D = 3$ and $D = 4 - \epsilon$ dimensions, and it does not matter that $\mu = \zeta_{\pm}^{-1}$ for $D = 3$ or $\mu = m$ for $D = 4 - \epsilon$ since both quantities go to zero in the critical limit with the same power $|t|^{-\nu}$. With relation (14) between u and \bar{u}_B , we obtain the bare coupling expansion of the renormalized square mass and fields:

$$m^2 \equiv Z_r^{-1} m_B^2 = m_B^2 \left\{ 1 - \frac{4(N+2)}{\epsilon} \bar{u}_B + 4(N+2) \left[\frac{4(N+5)}{\epsilon^2} + \frac{5}{\epsilon} \right] \bar{u}_B^2 - 16(N+2) \left[\frac{4(N+5)(N+6)}{\epsilon^3} + \frac{53N+274}{3\epsilon^2} + \frac{(5N+37)}{\epsilon} \right] \bar{u}_B^3 \right\}, \quad (15)$$

$$\phi \equiv Z_{\phi}^{-1/2} \phi_B = \phi_B \left[1 + \frac{2(N+2)}{\epsilon} \bar{u}_B^2 - \frac{4}{3}(N+2)(N+8) \left(\frac{8}{\epsilon^2} + \frac{1}{\epsilon} \right) \bar{u}_B^3 \right]. \quad (16)$$

These two expressions are sufficient to calculate the critical exponents ν and γ and, via scaling relations, all other exponents. Note that the value of the renormalized coupling constant at the critical point u^* is not needed to obtain ν and γ . The expansion (14) is however useful for obtaining an accurate exponent ω of the approach to scaling. It was pointed out in [49] that ω can also be deduced from the expansions of ν and γ . However, to reach the same accuracy, this requires always one more loop compared to the loop order we are interested in. For this reason, we shall take the advantage of Eq. (14), whose three-loop order contains all necessary information to get ω to that given order.

A. Method

Starting from Eq. (12), we follow [2,5,13] to write its strong-coupling limit as

$$f_L(\bar{u}_B \rightarrow \infty) = \text{opt}_{\hat{u}_B} \left[\sum_{i=0}^L f_i \hat{u}_B^i \sum_{j=0}^{L-i} \binom{-i\epsilon/\omega}{j} (-1)^j \right]. \quad (17)$$

The symbol $\text{opt}_{\hat{u}_B}$ denotes optimization with respect of \hat{u}_B . This expression holds provided it yields a nonzero constant. This limit will be denoted by f^* :

$$f(\bar{u}_B \rightarrow \infty) = f^* + c_0 \bar{u}_B^{-\omega/\epsilon} + O(\bar{u}_B^{-2\epsilon/\omega}), \quad (18)$$

where c_0 is a constant. The optimization is supposed to make f depend minimally on \hat{u}_B . In practice, this amounts to taking the first derivative to zero (odd orders) or, when it yields complex results, to taking the second derivative to zero and selecting turning points.

After having determined the optimum at various order L , it is still necessary to extrapolate the result to infinite order $L \rightarrow \infty$. The general large- L behavior of the strong-coupling limit has been derived from an analysis in the complex plane in [2,13]:

$$f_L^* \approx f^* + c_1 \exp(-c_2 L^{1-\omega}), \quad (19)$$

with constants c_1 and $c_2 > 0$. Knowing this behavior, a graphical extrapolation procedure may be used to find $f_\infty^* = f^*$.

To apply the above algorithm to critical exponents, we proceed as follows: Let W_L be a function obtained from perturbation theory. It has an expansion

$$W_L(\bar{u}_B) = \sum_{i=0}^L W_i \bar{u}_B^i. \quad (20)$$

Suppose that we also know this function has a leading power behavior $\bar{u}_B^{p/q}$ for large \bar{u}_B . The power p/q is given by a logarithmic derivative

$$\frac{p}{q} = \frac{d \log W_L}{d \log \bar{u}_B}. \quad (21)$$

The right-hand-side is a power series representation of a function of the type (12), with p/q being f^* and the approach to f^* in the form of powers $\bar{u}_B^{-\omega/\epsilon}$. Equation (21) will be used later for the determination of the critical exponents. If the series (20) goes to a constant in the strong-coupling limit, the exponent p is vanishing, and we are left with

$$\frac{d \log W_L}{d \log \bar{u}_B} = 0. \quad (22)$$

This equation can be solved for q , i.e., for ω . Note that (21) strictly holds for $p \neq 0$. However, it can be shown that this equation may be used also for $p = 0$, i.e., that (22) is a consistent equation for functions which go to a constant

in the strong-coupling limit. This is explained in Appendix A. In the following, we shall directly use (21) and (22) for two- and three-loop expansions where everything can be calculated analytically. We give below the associated formulas resulting from Eq. (17). Setting $\rho = 1 + \epsilon/\omega$, we find for $L = 2$:

$$f_{L=2}^* = \text{opt}_{\hat{u}_B} (f_0 + f_1 \rho \hat{u}_B + f_2 \hat{u}_B^2) = f_0 - \frac{\rho^2}{4} \frac{f_1^2}{f_2}, \quad (23)$$

while the three-loop results $L = 3$ leads to

$$f_{L=3}^* = \text{opt}_{\hat{u}_B} (f_0 + \bar{f}_1 \hat{u}_B + \bar{f}_2 \hat{u}_B^2 + f_3 \hat{u}_B^3) = f_0 - \frac{1}{3} \frac{\bar{f}_1 \bar{f}_2}{f_3} \left(1 - \frac{2}{3} r\right) + \frac{2}{27} \frac{\bar{f}_2^3}{f_3^2} (1 - r), \quad (24)$$

where $\bar{f}_1 = f_1 \rho(\rho + 1)/2$, $\bar{f}_2 = f_2(2\rho - 1)$, $r = \sqrt{1 - 3f_1 f_3 / f_2^2}$. If the square root is imaginary, the optimal value is given by the unique turning point. Practically, and this is a virtue of the analytic result, this square root is always imaginary for $D = 3$, at least as for the exponent ω . The turning point condition leads to

$$f_{L=3}^* = f_0 - \frac{1}{3} \frac{\bar{f}_1 \bar{f}_2}{f_3} + \frac{2}{27} \frac{\bar{f}_2^3}{f_3^2}, \quad (25)$$

i.e., to same expression as Eq. (24), but with $r = 0$. In the case $D = 4 - \epsilon$ with $\epsilon \rightarrow 0$, r is real. However, for ω the ϵ -expansion of r produces higher orders in ϵ than the three-loop approximation admits. Then, in both $D = 3$ and the ϵ -expansion, (25) is the relevant equation. A word of caution is nevertheless necessary: The positive root r of

$$\hat{u}_B^* = \frac{f_2}{3f_3} (-1 \pm r) \quad (26)$$

has to be chosen in order to match the three-loop result with the two-loop one in the limit $f_3 \rightarrow 0$. Doing so, it must be assumed that f_2 and f_1 are nonvanishing. When optimizing with $f_2 = 0$, it is immediate to show that if $f_1 f_3 > 0$, then the optimum corresponds to $\hat{u}_B^*(f_2 \rightarrow 0) = 0$ and $f_{L=3}^* = f_0$. The other possibility, $f_1 = 0$, is also interesting since it occurs in the determination of the exponent η . It can be verified that $f_1 = 0$ implies taking the negative root $r = -1$, so that $\hat{u}_B^*(f_1 \rightarrow 0) = -2\bar{f}_2/(3f_3)$ and $f_3^* = f_0 + 4\bar{f}_2^3/(27f_3^2)$. This possibility has not been discussed in the previous works [2,5,13].

B. Critical exponents

After the introduction to the resummation method to be used in this work, we can now turn to the actual determination of the critical exponents. We start from the definitions within the conventional renormalization formalism of the functions

$$\gamma_m = \left. \frac{\mu}{m^2} \frac{\partial m^2}{\mu} \right|_B, \quad (27)$$

$$\gamma_\phi = \left. \mu \frac{\partial}{\partial \mu} \log Z_\phi \right|_B, \quad (28)$$

$$\beta_u = \left. \mu \frac{u}{\partial \mu} \right|_B, \quad (29)$$

which, in the critical regime $m_B^2 \rightarrow 0$, render the critical exponents $\eta_m = \gamma_m^*$ and $\eta = \gamma_\phi^*$ if the first two equations are calculated at the fixed point u^* determined by the zero of the third function β_u . The derivative of β_u at u^* is the critical exponent of the approach to scaling $\omega = \partial \beta_u / \partial u|_{u^*}$.

Using the relation between the bare coupling constant u_B and the reduced one \bar{u}_B given in Eq. (13), Eqs. (27) and (28) become

$$\eta_m = -\epsilon \frac{d}{d \log \bar{u}_B} \log \frac{m^2}{m_B^2} = -\epsilon \frac{d}{d \log \bar{u}_B} \log Z_r^{-1}, \quad (30)$$

$$\eta = \epsilon \frac{d}{d \log \bar{u}_B} \log \frac{\phi^2}{\phi_B^2} = 2\epsilon \frac{d}{d \log \bar{u}_B} \log Z_\phi^{-1/2}, \quad (31)$$

where the renormalization constants Z_r^{-1} and $Z_\phi^{-1/2}$ have been explicitly given up to three loops in Eqs. (15) and (16), respectively. The associated power series expansion in \bar{u}_B of the exponents η_m and η will now be treated with the help of the formalism described in the previous section, up to two and three loops.

C. Critical exponents from two-loop expansions

In order to calculate the two-loop expansions in the critical strong-coupling limit, we need to know ω to this order. This will be calculated from Eq. (14). Dividing this series by \bar{u}_B , we know that the leading power behaviour as $\bar{u}_B \rightarrow \infty$ is -1 since u is supposed to go to the constant value u^* : $u \bar{u}_B^{-1}|_{\bar{u} \rightarrow \infty} = u^* \bar{u}_B^{-1}$. Calculating the logarithmic derivative of (14) and expanding up to second order in \bar{u}_B , we have

$$\frac{d}{d \log \bar{u}_B} \log \frac{u}{\bar{u}_B} = \frac{-4(N+8)}{\epsilon} \bar{u}_B + 16 \left[\frac{(N+8)^2}{\epsilon^2} + \frac{3(3N+14)}{\epsilon} \right] \bar{u}_B^2. \quad (32)$$

We now apply formula (A4). Combining with (23), we identify

$$-1 = -\frac{\rho^2}{4} \frac{[-4(N+8)/\epsilon]^2}{16 [(N+8)^2/\epsilon^2 + 3(3N+14)\epsilon]}, \quad (33)$$

i.e.,

$$\frac{\rho^2}{4} = 1 + 3\epsilon \frac{3N+14}{(N+8)^2}, \quad (34)$$

from which we can deduce ω :

$$\omega = \frac{\epsilon}{\rho - 1} = \frac{\epsilon}{-1 + 2\sqrt{1 + 3\epsilon(3N + 14)/(N + 8)^2}}. \quad (35)$$

It is identical to the result obtained in [5]. As a check of (35), we verify that it reproduces the well-known ϵ -expansion

$$\omega_\epsilon = \epsilon - 3\epsilon^2(3N + 14)/(N + 8)^2. \quad (36)$$

We refer the reader to [5,13] for plots of the function (35) as ϵ goes from 0 to 1, and for a comparison with the unresummed ϵ -expansion. The strong-coupling limit of ω may also be calculated from (A1) with an analytic expression different from (35), although numerically they are practically the same, and they certainly have the same ϵ -expansion [49].

This determination of ω illustrates what we said in Section II, that in the minimal renormalization scheme the critical exponents lead to identical results in $D = 3$ and $D = 4 - \epsilon$ dimensions. This will also be true for the critical exponents to be calculated in the sequel [50]. For this reason, we shall always keep trace of ϵ to facilitate the comparison, although our work is in $D = 3$ dimensions. Only for amplitude ratios to be calculated later will such a comparison be impossible and ϵ be set equal to 1 everywhere.

Knowing ω , we can now determine the exponents η and η_m . According to (30) and (31), we take the logarithmic derivative of (15) and (16), reexpand the results up to the second order in \bar{u}_B^2 , and obtain

$$\eta_m = 4(N + 2)\bar{u}_B - 8(N + 2) \left[\frac{2(N + 8)}{\epsilon} + 5 \right] \bar{u}_B^2, \quad (37)$$

$$\eta = 8(N + 2)\bar{u}_B^2. \quad (38)$$

Evaluating η_m in the strong-coupling limit in the same way as ω , i.e., following the algorithm (17), we find

$$\eta_m = \frac{\rho^2}{4} \frac{[4(N + 2)]^2}{8(N + 2)[2(N + 8)/\epsilon + 5]} = \frac{(N + 2)}{(N + 8) + 5\epsilon/2} \left[\epsilon + \epsilon^2 \frac{3(3N + 14)}{(N + 8)^2} \right]. \quad (39)$$

For η , the situation is less clear. In [2,5], it was argued that the two-loop result cannot be computed from Eq. (38) since no linear term in \bar{u}_B is present. A direct application of the resummation algorithm would give an optimum $\hat{u}_B^* = 0$, then a value $\eta = 0$ at two-loop order. This does not lead to the correct ϵ -expansion, according to which the exponent start with ϵ^2 , i.e., with a non-vanishing two-loop contribution. To apply variational perturbation theory, it is necessary to modify the procedure. In Ref. [5], this was done by considering a different critical exponent

$$\gamma = \nu(2 - \eta), \quad (40)$$

with

$$\nu = \frac{1}{2 - \eta_m}. \quad (41)$$

To obtain their strong coupling limit, we insert for η_m and η their perturbative expansions (37) and (38), respectively, and reexpand the resulting ratios in power of \bar{u}_B up to the second order. This gives

$$\gamma = 1 + 2(N+2)\bar{u}_B - 4(N+2) \left[\frac{2(N+8)}{\epsilon} - (N-4) \right] \bar{u}_B^2. \quad (42)$$

The critical exponent ν itself has the expansion

$$\nu = \frac{1}{2} + (N+2)\bar{u}_B - 2(N+2) \left[\frac{2(N+8)}{\epsilon} - (N-3) \right] \bar{u}_B^2. \quad (43)$$

The strong-coupling limits are, using $\rho^2/4$ from Eq. (34),

$$\gamma = 1 + \frac{(N+2)}{2(N+8) - \epsilon(N-4)} \left[\epsilon + \epsilon^2 \frac{3(3N+14)}{(N+8)^2} \right], \quad (44)$$

$$\nu = \frac{1}{2} \left\{ 1 + \frac{(N+2)}{2(N+8) - \epsilon(N-3)} \left[\epsilon + \epsilon^2 \frac{3(3N+14)}{(N+8)^2} \right] \right\}. \quad (45)$$

Their ϵ -expansion are in agreement with $D = 4 - \epsilon$ results [5,13]. From these expressions we can recover η using the relation $\eta = 2 - \gamma/\nu$. The result has now the correct ϵ -expansion:

$$\eta = \frac{N+2}{2(N+8)^2} \epsilon^2. \quad (46)$$

This calculation of η via ν and γ was made in [2,5] to compensate the lack of a linear term in (38). Let us point out that, even if the ϵ -expansion is not recovered, it is nevertheless hidden in a direct resummation of (38) to $\eta = 0$. To see this, we add a small dummy linear term ζu , to the defining equation (31), leading to the expansion

$$\eta = \zeta \bar{u}_B + \left[8(N+2) - \zeta \frac{4(N+8)}{\epsilon} \right] \bar{u}_B^2. \quad (47)$$

Using (23) and (34), this leads to the strong-coupling value

$$\eta = -\frac{\rho^2}{4} \frac{\zeta^2}{8(N+2) - 4(N+8)\zeta/\epsilon}, \quad (48)$$

which is zero for $\zeta = 0$. Consider however the ϵ -expansion of the right-hand-side performed at a finite ζ :

$$\eta = \frac{\rho^2}{4} \frac{\zeta \epsilon}{4(N+8)} \left[1 + \frac{2(N+2)}{N+8} \frac{\epsilon}{\zeta} \right]. \quad (49)$$

If we now take the limit $\zeta \rightarrow 0$, the right-hand-side starts directly like ϵ^2 . Together with the lowest-order value 1 of $\rho^2/4$, we obtain correctly (46).

For consistency, the different two-loop results for η , once from (44) and (45), and once $\eta = 0$ from (48) should not be too far from each other. This can indeed be verified by plotting the curves $\eta = 2 - \gamma/\nu$ against a few values of N . The curves are all close to the $\eta = 0$ axis for all N , approaching it for $N \rightarrow \infty$.

Also for higher-loop orders, η could be obtained from the strong-coupling limit γ and ν , or by taking a direct strong-coupling limit. Variational perturbation theory does not know which of these approaches should be better. Ultimately, if we know enough term in the series expansion, the extrapolation to infinite order L should certainly become insensible to which function is resummed.

One may wonder if it is possible to set up a unique optimal function of the critical exponents from which to derive the strong-coupling limit. The answer to this question would improve the theory considerably.

Collecting the different results of this section, we have the $D = 3$ results

$$\omega = \frac{1}{-1 + 2\sqrt{1 + 3(3N + 14)/(N + 8)^2}}, \quad (50)$$

$$\gamma = \frac{2N^3 + 63N^2 + 540N + 1492}{(N + 8)^2(N + 20)}, \quad (51)$$

$$\nu = \frac{N^3 + 31N^2 + 262N + 714}{(N + 8)^2(N + 19)}, \quad (52)$$

$$\eta_m = \frac{2(N + 2)}{2N + 21} \left[1 + \frac{3(3N + 14)}{(N + 8)^2} \right], \quad (53)$$

$$\eta = \frac{2(N + 2)}{N + 20} \frac{(N + 8)^2 + 3(3N + 14)}{2(N + 8)^3 + 5(N + 8)^2 + 3(N + 2)(3N + 14)}, \quad (54)$$

$$u^* = \frac{1}{4(N + 8)} + \frac{3}{4} \frac{3N + 14}{(N + 8)^3}, \quad (55)$$

where we also included the value of the renormalized coupling constant at the IR-fixed point. It is obtained from the one-loop series in u of the expansion (14):

$$u = \bar{u}_B - \frac{4(N + 8)}{\epsilon} \bar{u}_B^2. \quad (56)$$

We can restrict ourselves to one loop since it corresponds to a power \bar{u}_B^2 . The two-loop calculation was however needed to get ω correctly, which itself enters (56). With the help of (23), we obtain

$$u^* = \frac{\rho^2}{4} \frac{\epsilon}{4(N + 8)} = \frac{\epsilon}{4(N + 8)} + \frac{3}{4} \frac{3N + 14}{(N + 8)^3} \epsilon^2 \quad (57)$$

Since only two critical exponents are independent [12,13], all other can be derived from Eqs. (50)–(54). These two loop expressions are only a lowest approximation to the exact results. In the next section, we evaluate analytically the strong-coupling limit of the exponents at the three-loop level.

D. Critical exponents from three-loop expansions

The three-loop calculations are algebraically more involved. Moreover, as far as the critical exponents are concerned (we will see later that this is not necessarily true for the amplitude functions) the optimum of the function (17) is not given by the vanishing of the first derivative, but by a turning point, i.e., by the vanishing of the second derivative. At the three-loop order, this implies that the parameter r in (24) is zero, leading to the three-loop strong-coupling limit result (25). It is this feature which renders the calculation analytically manageable, involving only a cubic equation for the determination of ρ (without $r = 0$, we would have had to solve an eight-order equation). In order to obtain ω to three loop, we generalize (32) to the same order, and find

$$-1 \equiv \frac{d}{d \log \bar{u}_B} \log \frac{u}{\bar{u}_B} = \frac{-4(N+8)}{\epsilon} \bar{u}_B + 16 \left[\frac{(N+8)^2}{\epsilon^2} + \frac{3(3N+14)}{\epsilon} \right] \bar{u}_B^2 - 8 \left[\frac{8(N+8)^3}{\epsilon^3} + \frac{60(N+8)(3N+14)}{\epsilon^2} + \frac{96\zeta(3)(5N+22) + 33N^2 + 922N + 2960}{\epsilon} \right] \bar{u}_B^3. \quad (58)$$

From this we extract the coefficients f_i ($i = 0, \dots, 3$) of (24). The argument of the square root r turns then out to be negative, and the equation to be solved is (25). This is true not only for $\epsilon = 1$, but also for all $\epsilon \in [0, 1]$. Since (25) is a cubic equation for ρ , there are three solutions, one of which is always negative, which we discard as unphysical, leaving us with two solutions. Only one of them is connected smoothly to the two-loop result. The purely algebraic form of the solution, generalization of the square root coming from solving (34), is somewhat too lengthily to be written down here. As a check, we have derived its epsilon expansion which reads

$$\rho_\epsilon = 2 + \frac{3(3N+14)}{(N+8)^2} \epsilon - \frac{96\zeta(3)(5N+22)(N+8) + 33N^3 + 214N^2 + 1264N + 2512}{4(N+8)^4} \epsilon^2 \quad (59)$$

and leads to the correct ϵ -expansion for $\omega = \epsilon/(\rho - 1)$:

$$\omega_\epsilon = \epsilon - \frac{3(3N+14)}{(N+8)^2} \epsilon^2 + \frac{96\zeta(3)(5N+22)(N+8) + 33N^3 + 538N^2 + 4288N + 9568}{4(N+8)^4} \epsilon^3 \quad (60)$$

which is the extension of (36) to the order ϵ^3 . This is to be compared with Eq. (17.15) of the textbook [13].

The trigonometric representation is however compact enough to be written down here explicitly, at least for $\epsilon = 1$. Introducing an angle θ and two coefficient a_0, b_0 defined by

$$\theta = \arccos \left(\frac{[13776 + 4738N + N^2(8N + 405) + 96(5N + 22)\zeta(3)]^2}{2[106 + N(N + 28)] \{(N + 8)[13776 + 4738N + N^2(8N + 405) + 96(5N + 22)\zeta(3)]\}^{3/2}} \right) \\ \times \frac{1}{[2209664 + 1040160N + 162982N^2 + 9683N^3 + 184N^4 + 672(N + 8)(5N + 22)\zeta(3)]^{3/2}} \\ \times \left\{ 67181166592 + 64001040384N + 25893312000N^2 + 5641828480N^3 + 713027988N^4 + 54733044N^5 \right.$$

$$\begin{aligned}
& + 2760157N^6 + 88332N^7 + 1440N^8 \\
& - 192(N+8)(5N+22) [4084864 + 1952480N + 323706N^2 + 20021N^3 + 514N^4] \zeta(3) \\
& + 746496 [(N+8)(5N+22)\zeta(3)]^2 \left. \vphantom{+ 2760157N^6} \right\}, \tag{61}
\end{aligned}$$

$$a_0 = \frac{1}{446336 + 213280N + 35334N^2 + 2179N^3 + 56N^4 - 864(N+8)(5N+22)\zeta(3)} \tag{62}$$

$$\begin{aligned}
b_0 &= 3\sqrt{(N+8)[13776 + 4738N + N^2(8N+405) + 96(5N+22)\zeta(3)]} \\
&\times \sqrt{[2209664 + 1040160N + 162982N^2 + 9683N^3 + 184N^4 + 672(N+8)(5N+22)\zeta(3)]}, \tag{63}
\end{aligned}$$

the relevant root of (25) can be written as

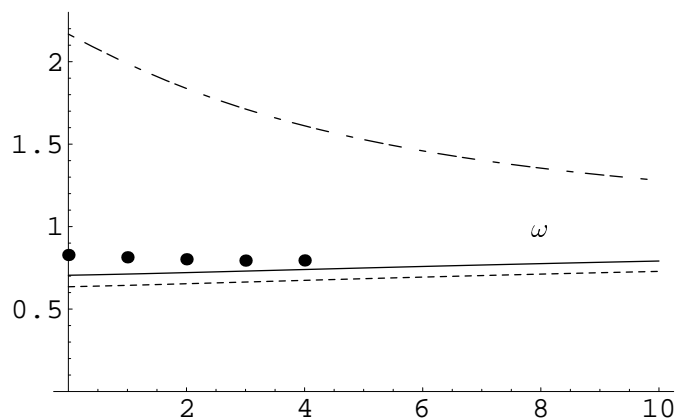
$$\rho = -\frac{1}{6} + \frac{256}{3}a_0 [106 + N(N+25)]^2 - a_0b_0 \cos\left(\frac{-2\pi + \theta}{3}\right). \tag{64}$$

For the physically interesting cases $N = 0, \dots, 4$, we obtain the values for $D = 3$ dimensions

N	0	1	2	3	4
ρ	2.41829	2.40384	2.38683	2.36910	2.35157
ω	0.705073	0.712332	0.721069	0.730405	0.73988
ω (Ref. [13])	0.8035	0.7998	0.7948	0.7908	
ω (Ref. [18])	0.812	0.799	0.789	0.782	0.774

where we also indicated the theoretical values given in Refs. [13,18].

Figure 1 illustrates the two- and three-loop critical exponent of the approach to scaling $\omega = \epsilon/(\rho - 1)$ as a function of N calculated from (34) and (64), respectively. For comparison, we also give the three-loop unresummed result (60), evaluated at $\epsilon = 1$ and the theoretical values given in Tables 1 and 3 of [18]. The latter are based on a five-loop analysis supplemented by a large loop order analysis.



N

FIG. 1. Two-loop (short-dashed) and three-loop (solid) critical exponent ω for different $O(N)$ -symmetries. For comparison, the ϵ -expansion (mixed-dashed) and the theoretical values of [18] (dots) are also given.

Once ω is known to three loops, the other exponents and the strong-coupling limit u^* of the renormalized coupling constant can be determined to the same order. To obtain u^* , the two-loop expansion of u in powers of \bar{u}_B is enough since it is of order $\mathcal{O}(\bar{u}_B^3)$. Recall that the three-loop expansion of $u(\bar{u}_B)$ is needed only to calculate ω . From (14) we identify f_1, f_2, f_3 and use (25) (since the argument of the corresponding r in (24) is negative) to obtain the critical value

$$u^* = \frac{\epsilon(N+8)\rho(\rho+1)(2\rho-1)}{12[2(N+8)^2+3\epsilon(3N+14)]} - \frac{2\epsilon(N+8)^3(2\rho-1)^3}{27[2(N+8)^2+3\epsilon(3N+14)]^2} \quad (65)$$

with ρ from (64). If we use instead the ϵ -expansion of ρ given in (59), we obtain

$$u^* = \frac{\epsilon}{4(N+8)} + \frac{3}{4} \frac{3N+14}{(N+8)^3} \epsilon^2 + \frac{4544+1760N+110N^2-33N^3-96(N+8)(5N+22)\zeta(3)}{32(N+8)^5} \epsilon^3. \quad (66)$$

In the same way, we find the strong-coupling limit of the critical exponents γ and ν , as defined in (40) and (41) together with (30) and (31), the latter two exponents being obtained from the the mass (15) and wave function (16) renormalization, respectively. The three-loop perturbative expansions are

$$\begin{aligned} \gamma &= 1 + 2(N+2)\bar{u}_B - 4(N+2) \left[\frac{2(N+8)}{\epsilon} - (N-4) \right] \bar{u}_B^2 \\ &\quad + 4(N+2) \left[\frac{8(N+8)^2}{\epsilon^2} - \frac{4(2N^2-N-106)}{\epsilon} + 194 + N(2N+17) \right] \bar{u}_B^3, \end{aligned} \quad (67)$$

$$\begin{aligned} \nu &= \frac{1}{2} + (N+2)\bar{u}_B - 2(N+2) \left[\frac{2(N+8)}{\epsilon} - (N-3) \right] \bar{u}_B^2 \\ &\quad + 4(N+2) \left[\frac{4(N+8)^2}{\epsilon^2} - \frac{2(2N^2+N-90)}{\epsilon} + 95 + N(N+9) \right] \bar{u}_B^3, \end{aligned} \quad (68)$$

from which it is immediate to identify the expansion coefficients f_0, \dots, f_3 which enter (25), to obtain

$$\begin{aligned} \gamma &= 1 - \frac{\epsilon(N+2)[\epsilon(N-4)-2(N+8)]\rho(\rho+1)(2\rho-1)}{3[8(N+8)^2-4\epsilon(2N^2-N-106)+\epsilon^2(2N^2+17N+194)]} \\ &\quad + \frac{8\epsilon(N+2)[\epsilon(N-4)-2(N+8)]^3(2\rho-1)^3}{27[8(N+8)^2-4\epsilon(2N^2-N-106)+\epsilon^2(2N^2+17N+194)]^2}, \end{aligned} \quad (69)$$

$$\begin{aligned} \nu &= \frac{1}{2} - \frac{\epsilon(N+2)[\epsilon(N-3)-2(N+8)]\rho(\rho+1)(2\rho-1)}{12[4(N+8)^2-2\epsilon(2N^2+N-90)+\epsilon^2(N^2+9N+95)]} \\ &\quad + \frac{\epsilon(N+2)[\epsilon(N-3)-2(N+8)]^3(2\rho-1)^3}{27[4(N+8)^2-2\epsilon(2N^2+N-90)+\epsilon^2(N^2+9N+95)]^2}, \end{aligned} \quad (70)$$

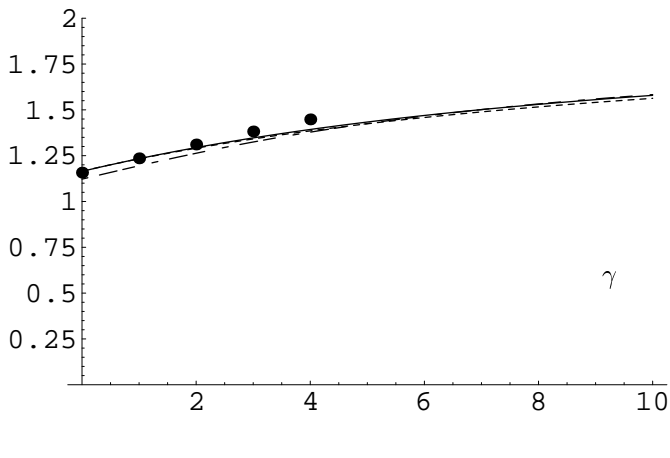
where ρ for $\epsilon = 1$ can be obtained from (64). The associated ϵ -expansions can be obtained using (59). They read

$$\gamma = 1 + \frac{N+2}{2(N+8)} \epsilon + \frac{(N+2)(N^2+22N+52)}{4(N+8)^3} \epsilon^2$$

$$+ \frac{(N+2) [3104 + 2496N + 664N^2 + 44N^3 + N^4 - 48(N+8)(5N+22)\zeta(3)]}{8(N+8)^5} \epsilon^3, \quad (71)$$

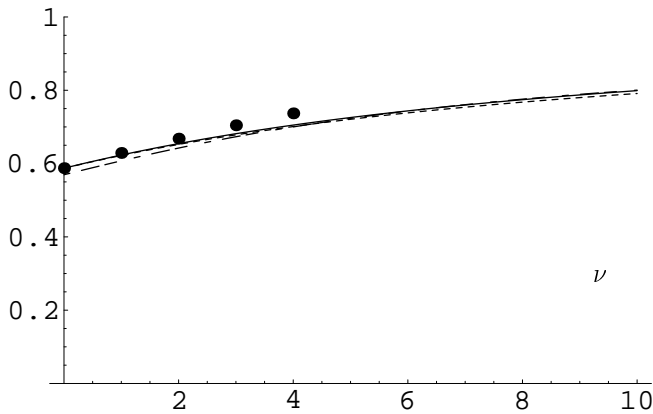
$$\nu = \frac{1}{2} + \frac{N+2}{4(N+8)} \epsilon + \frac{(N+2)(N+3)(N+20)}{8(N+8)^3} \epsilon^2 + \frac{(N+2) [8640 + 5904N + 1412N^2 + 89N^3 + 2N^4 - 96(N+8)(5N+22)\zeta(3)]}{32(N+8)^5} \epsilon^3. \quad (72)$$

Figures 2 and 3 illustrate the two- and three-loop critical exponents γ and ν , respectively, as a function of N . They are given by Eqs. (51) and (69) for γ and by Eqs. (52) and (70) for ν . For completeness, we also plot the ϵ -expansion (71) and (72) of the exponents, as well as the theoretical values quoted in Tables 1 and 3 of [18]. Contrary to the case of the critical exponent ω , we see that the two- and three-loop critical exponents are very close together. This is a virtue of working self-consistently with ω obtained at the same loop order. In [2,5], the extrapolated ω to infinite loop order was used instead. This implies that each loop order result for γ and ν was not very close to its asymptotic limit (contrary to what we get here). However, the extrapolation formula (19) works precisely for this case, and very precise extrapolated results for γ and ν could be obtained. In our present work, the critical exponents are not very far from their asymptotic limit, already at the two- and three-loop level. However, the extrapolation formula (19) cannot be used. It is not yet clear to the authors how it will be possible to extrapolate the five-loop results obtained using the present formalism. This question is left aside for a future work. We also note in passing that the ϵ -expansion result is not too far from the values obtained in the strong-coupling limit.



N

FIG. 2. Two-loop (short-dashed) and three-loop (solid) critical exponent γ . For comparison, the ϵ -expansion (short- and long-dashed) and the theoretical values of [18] (dots) are also given.



N

FIG. 3. Two-loop (short-dashed) and three-loop (solid) critical exponent ν . For comparison, the ϵ -expansion (short- and long-dashed) and the theoretical values of [18] (dots) are also given.

The critical exponent η is obtained using $2 - \gamma/\nu$, with γ and ν from (69) and (70), respectively. It has the ϵ -expansion

$$\eta = \frac{(N+2)}{2(N+8)^2} \epsilon^2 - \frac{(N+2)(N^2 - 56N - 272)}{8(N+8)^4} \epsilon^3. \quad (73)$$

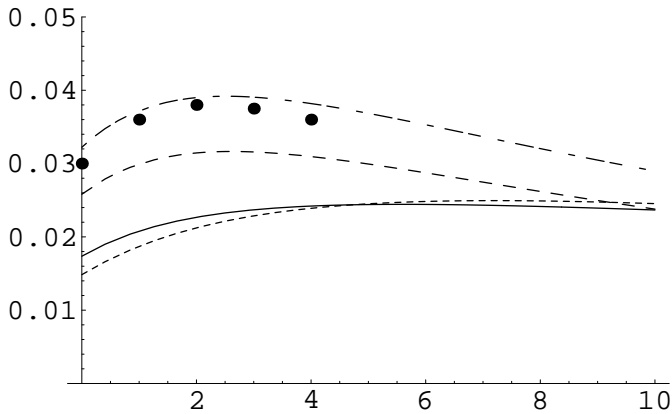
Let us also calculate directly the strong coupling limit of η from its definition (31):

$$\eta = 8(N+2)\bar{u}_B^2 - 8(N+2)(N+8) \left(\frac{8}{\epsilon} + 1 \right) \bar{u}_B^3. \quad (74)$$

At the two-loop level, the result was zero. At the three-loop level, the calculation is different from that of γ and ν because there is no linear term in \bar{u}_B . This has already been discussed after Eq. (26): although we are working at the three-loop level, the optimum of the variational perturbation theory is not governed by a turning point but by an extremum for which the sign of the root r is the opposite to the usual case. The solution corresponds to $r = -1$ and the optimum is $\hat{u}_B = -2\bar{f}_2/(3f_3)$, so that

$$\eta = \frac{4}{27} \frac{\bar{f}_2^3}{f_3^2} = \frac{32}{27} \frac{(2\rho - 1)^3 (N+2)}{(N+8)^2 (8+\epsilon)^2} \epsilon^2. \quad (75)$$

With (59), this leads again to the correct ϵ -expansion (73). The difference between $\eta = 2 - \gamma/\nu$ and (75) at $\epsilon = 1$ is illustrated on Figure 4 which also shows the direct evaluation of the ϵ -expansion series (73) as well as the theoretical values quoted in Tables 2 and 3 of [18].



.32N

FIG. 4. Two-loop (short-dashed) and three-loop (solid) critical exponent η from the definition $2 - \gamma/\nu$. For comparison, the ϵ -expansion (short- and long-dashed), η from the strong-coupling limit of the direct (medium-dashed) series (75) and the theoretical values of [18] (dots) are also given.

It is amusing to see that the ϵ -expansion is the best approximation, followed by the strong-coupling limit of the direct series (75). Comparing the different results, we see that they differ by about 30%. This is due to the absolute smallness of η . The error is small compared to unity.

To end up this section we also give the critical exponent η_m . Up to three loops, the bare perturbation expansion reads, from (15) and (30),

$$\eta_m = 4(N+2)\bar{u}_B - 8(N+2) \left[\frac{2(N+8)}{\epsilon} + 5 \right] \bar{u}_B^2 + 16(N+2) \left[\frac{4(N+8)^2}{\epsilon^2} + \frac{2(19N+122)}{\epsilon} + 3(5N+37) \right] \bar{u}_B^3, \quad (76)$$

from which we deduce the strong-coupling limit with ρ from (64)

$$\eta_m = \frac{\epsilon(N+2)(2N+16+5\epsilon)\rho(\rho+1)(2\rho-1)}{3[4(N+8)^2 + \epsilon(38N+244) + 3\epsilon^2(5N+37)]} - \frac{4\epsilon(N+2)(2N+16+5\epsilon)^3(2\rho-1)^3}{27[4(N+8)^2 + \epsilon(38N+244) + 3\epsilon^2(5N+37)]^2}. \quad (77)$$

Its ϵ -expansion is

$$\eta_m = \frac{N+2}{N+8}\epsilon + \frac{(N+2)(13N+44)}{2(N+8)^3}\epsilon^2 + \frac{(N+2)[5312 + 2672N + 452N^2 - 3N^3 - 96(N+8)(5N+22)\zeta(3)]}{8(N+8)^5}\epsilon^3. \quad (78)$$

The result (77) is analytically different but numerically close to that obtained via the scaling relation (41), implying $\eta_m = 2 - \nu^{-1}$, as illustrated in Figure 5. For completeness, the figure also shows the ϵ -expansion (78) and the theoretical values quoted in Tables 2 and 3 of [18].

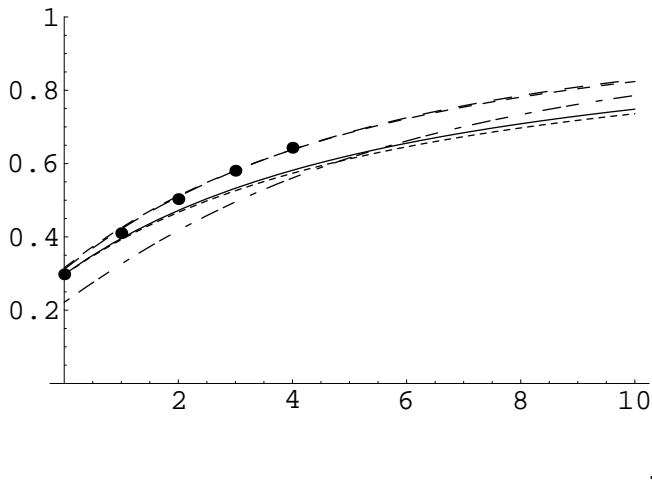


FIG. 5. Two-loop (short-dashed) and three-loop (solid) critical exponent η_m from the definition $2 - \nu^{-1}$. For comparison, the ϵ -expansion (short- and long-dashed), η_m from the strong-coupling limit of the direct (medium-dashed) two-loop (39) and three-loop (long-dashed) series (77) and the theoretical values of [18] (dots) are also given.

We see a better agreement with the theoretical values quoted from [18] when the exponent is evaluated in the strong-coupling limit of the direct series (39) and (77). This was also the same for the exponent η .

Collecting the different results of this section, we have the analytical form of the $D = 3$ -dimensions critical exponents in the three-loop order

$$\omega = \frac{1}{\rho - 1}, \quad (79)$$

$$\gamma = 1 + \frac{(N+2)(N+20)\rho(\rho+1)(2\rho-1)}{3(2N^2+149N+1130)} - \frac{8(N+2)(N+20)^3(2\rho-1)^3}{27(2N^2+149N+1130)^2}, \quad (80)$$

$$\nu = \frac{1}{2} + \frac{(N+2)(N+19)\rho(\rho+1)(2\rho-1)}{12(N^2+71N+531)} - \frac{(N+2)(N+19)^3(2\rho-1)^3}{27(N^2+71N+531)^2}, \quad (81)$$

$$\eta_m = \frac{(N+2)(2N+21)\rho(\rho+1)(2\rho-1)}{3(4N^2+117N+611)} - \frac{4(N+2)(2N+21)^3(2\rho-1)^3}{27(4N^2+117N+611)^2}, \quad (82)$$

$$\eta = \frac{32}{2187} \frac{(2\rho-1)^3(N+2)}{(N+8)^2}, \quad (83)$$

$$u^* = \frac{(N+8)\rho(\rho+1)(2\rho-1)}{12[2(N+8)^2+3(3N+14)]} - \frac{2(N+8)^3(2\rho-1)^3}{27[2(N+8)^2+3(3N+14)]^2}, \quad (84)$$

where ρ is given in Eq. (64). For η and η_m , we took the strong-coupling limit of the direct expansions: Eq. (75) for η and Eq. (77) for η_m . These results have to be compared with the two-loop ones given in (50)–(54).

For completeness, we give below the table of the critical exponents to three loops and the comparison with Refs. [13,18]:

N	0	1	2	3	4
γ	1.16455	1.2338	1.29426	1.34697	1.39307
γ (Ref. [13])	1.1576	1.2349	1.3105	1.3830	
γ (Ref. [18])	1.1596	1.2396	1.3169	1.3895	1.456
ν	0.587376	0.623381	0.654552	0.681561	0.705071
ν (Ref. [13])	0.5874	0.6292	0.6697	0.7081	
ν (Ref. [18])	0.5882	0.6304	0.6703	0.7073	0.741
η_m	0.311607	0.421796	0.509799	0.580684	0.638337
η_m (Ref. [13])	0.2976	0.4107	0.5068	0.5878	
η_m (Ref. [18])	0.2999	0.4137	0.5081	0.5862	0.6505
η	0.0258218	0.029917	0.031452	0.0315846	0.03096
η (Ref. [13])	0.0316	0.0373	0.0396	0.0367	
η (Ref. [18])	0.0284	0.0335	0.0354	0.0355	0.0350

They cannot compete with the five-loop calculation of [2–6,13,18]. However, our results are analytical, and already close to the asymptotic limit although we made no assumption about the large order behaviour of the theory. We consider this as promising. In a subsequent publication, we will present a numerical calculation up to five loops, with large-order behavior information included, of our self-consistent formalism.

IV. CALCULATION OF AMPLITUDE FUNCTIONS AND RATIOS

From now on, we shall focus entirely upon the $D = 3$ -dimensions model. As we mentioned in the introduction, it is only for the critical exponents that the minimal subtraction scheme leads to the same resummed values both for $D = 3$ and $D = 4 - \epsilon$. For this reason, it made sense to study the ϵ -expansions of the critical exponents, which was also useful for comparing with calculations in $4 - \epsilon$ dimensions. The reason for this equality is the mass independence of the renormalization constants in this MS scheme. The mass independence implies a decomposition of the correlation functions into amplitude functions and power parts, for which the latter can be evaluated in the symmetric phase. The amplitude functions, however, depend on being in the ordered or disordered phase. Moreover, the situation is complicated for $N > 1$ by the presence of Goldstone singularities, most of which have to cancel at the end of the calculations: only the physical singularities, for example those occurring in the transverse susceptibilities, should stay

at the end of the calculations.

For this reason, apart from the three-loop work [41], no three- or higher-loop calculation has been done for $N > 1$ below T_c , even numerically. The only relatively easy case is $N = 1$ for which extensive numerical work has been done below T_c up to five-loop order [21,25,40]. Above T_c , all N can be treated in the same way [20,24,51]. In the latter reference the critical exponents η and η_m have even been obtained to seven loops, with resummation performed in [3,13,21,52].

We have explained in detail in the first part of this paper that it is unnecessary to go to the renormalized theory since all results can be obtained from the strong-coupling limit of the bare theory. In the literature, the effective potential is given in terms of the renormalized quantities [42,41,46]. To apply our theory, we shall rewrite the expressions back in the bare form, using (14).

A. Available expansions

Let us list the most important available amplitude functions derived from the minimally renormalized model at $D = 3$ at vanishing external magnetic field h_B . Up to two loops, they can be found in Ref. [46]:

- the square of the order parameter $M_B^2 = \langle \phi_B^2 \rangle$ below T_c :

$$f_\phi = \frac{1}{32\pi u} + \left[\frac{1}{27\pi}(160 - 82N) + \frac{2}{\pi}(N - 1) \ln 3 \right] u, \quad (85)$$

- the stiffness of phase fluctuations below T_c (some authors call this helicity modulus [53]) Υ :

$$f_\Upsilon = \frac{1}{8u} + \frac{1}{3} + \left[\frac{1}{54}(2378 - 683N) + 8(N - 3) \ln 3 \right] u, \quad (86)$$

- the q^2 part of the transverse susceptibility χ_T :

$$f_{\chi_T} = 1 + \frac{8}{3}u + \left[\frac{488}{3} - 4N - 128 \ln 3 \right] u^2, \quad (87)$$

- the specific heat C^\pm above and below T_c :

$$F_+ = -N - 2N(N + 2)u, \quad (88)$$

$$F_- = \frac{1}{2u} - 4 + 8(10 - N)u, \quad (89)$$

- the isotropic susceptibility above T_c [55]

$$f_{\chi_+} = 1 - \frac{92}{27}(N+2)u^2, \quad (90)$$

- the amplitude function of the susceptibility below T_c , which we obtain taking the inverse of the two loop numerical expansion given (up to five loops) in [40]:

$$f_{\chi_-} = 1 + 18u + 164.44u^2. \quad (91)$$

The latter quantity is restricted to $N = 1$.

From the series expansion of f_{χ_T} and f_{Υ} , one sees that the relation

$$f_{\Upsilon} = 4\pi f_{\phi} f_{\chi_T} \quad (92)$$

is satisfied to two loops. This is not a surprise: the bare helicity modulus, defined as $\Upsilon = 2\partial\Gamma_B/\partial q^2|_{q=0}$ where Γ_B is the free energy, can be shown (at least to two loops [46]) to be identical to $M_B^2(\partial\chi_T^{-1}/\partial q^2)|_{q=0}$. This is a consequence of a Ward identity for the broken $O(N)$ -symmetry below T_c .

In Ref. [42], the perturbation expansions of the amplitude functions for the order parameter and for the specific heat have been carried to three loops. The additional terms are (we use the notation $f_j = \sum_i f_j^{(i)} u^i$):

$$f_{\phi}^{(3)} = -\frac{1}{1080\pi} \left\{ 2500N^2 + 65104N + 29056 + 8640(5N+2)\zeta(3) + 58320c_1 - 15\pi^2(19N^2 + 643N + 499) \right. \\ \left. - 180(64N^2 + 640N + 457) \text{Li}_2\left(-\frac{1}{3}\right) - 80(194N^2 + 1616N - 1675) \ln 3 + 16(860N^2 + 8357N - 7867) \ln 2 \right. \\ \left. + 270(N-1) \left[-8c_2 + 32 \text{Li}_2\left(-\frac{1}{2}\right) + 42 \text{Li}_2\left(\frac{1}{3}\right) - 64 \text{Li}_2(-2) + 21(\ln 3)^2 + 16(\ln 2)^2 - 96(\ln 2)(\ln 3) \right] \right\}, \quad (93)$$

$$F_+^{(3)} = -4N(N+2) \left(N - \frac{7}{27} + 4 \ln \frac{4}{3} \right), \quad (94)$$

$$F_-^{(3)} = -\frac{1}{27}(1080N^2 + 3464N + 31120) - 128(5N+22)\zeta(3) - 864c_1 + \frac{2}{3}\pi^2(9N^2 + N + 17) \\ + 216 \text{Li}_2\left(-\frac{1}{3}\right) - 32(4N+17) \ln 3 + \frac{32}{3}(31N+95) \ln 2 \\ + 4(N-1) \left[-8c_2 + 16 \text{Li}_2\left(-\frac{1}{2}\right) + 6 \text{Li}_2\left(\frac{1}{3}\right) - 32 \text{Li}_2(-2) + 3(\ln 3)^2 + 8(\ln 2)^2 - 48(\ln 2)(\ln 3) \right], \quad (95)$$

where $\text{Li}_2(x) = \sum_{n=0}^{\infty} x^n/n^2$ is the dilogarithmic function [54], and c_1 and c_2 are two numerical constants given by a single variable integration over elementary functions [35,42]:

$$c_1 = \int_0^1 \frac{dx}{\sqrt{6-2x^2}} \left[\ln \frac{3}{4} + \ln \frac{3+x}{2+x} + \frac{x}{2+x} \left(\ln \frac{3+x}{3} + \frac{x}{2-x} \ln \frac{2+x}{4} \right) \right] \approx 0.021737576333, \quad (96)$$

$$c_2 = \frac{\pi^2}{4\sqrt{2}} + \sqrt{2} \int_0^1 \frac{dx}{\sqrt{1+x^2}} \left[\ln \frac{x}{1+x} + \frac{\ln(1+x)}{x} \right] \approx 0.973771427. \quad (97)$$

For completeness, we give in Appendix B some hints on how to obtain these amplitude functions. For the details, see Refs. [41,46]. Our own contribution concerns the susceptibilities above and below T_c : Using the three-loop integrals available in the literature [42,35], we have been able to calculate analytically the three-loop extension of the amplitude of the isotropic susceptibility f_{χ_+} :

$$f_{\chi_+}^{(3)} = -\frac{8}{27}(N+2)(N+8) \left[-21 + 12\pi^2 + 128 \ln \frac{3}{4} + 144 \text{Li}_2 \left(-\frac{1}{3} \right) \right], \quad (98)$$

as well as the three-loop amplitude function of the susceptibility below T_c , for $N = 1$:

$$f_{\chi_-} = 1 + 18u + \frac{1480}{9}u^2 + \left[1072 - 11664c_1 + 3\pi^2 + 10480 \ln \frac{4}{3} + 36 \text{Li}_2 \left(-\frac{1}{3} \right) \right] u^3. \quad (99)$$

Our analytical two-loop coefficient $1480/9$ agrees with the numerical coefficient given in (91). We shall comment on the three-loop one later. The details of the calculation are given in Appendix C and D.

B. Amplitude ratios

Besides the amplitude functions, we shall also evaluate three important ratios: the amplitude ratio of the heat capacity, the universal combination R_C , and the amplitude ratio of the susceptibilities for $N = 1$. For a review of amplitude ratios, see [56]. The relevant equations for their determination is given in Appendix E. One of the best measured amplitude ratios was mentioned in the introduction: it is the amplitude ratio of the specific heat of superfluid helium above and below T_c , corresponding to $N = 2$. It can however be defined for all N and, using our notation, can be written as [38,42]

$$\frac{A^+}{A^-} = \left(\frac{b^+}{b^-} \right)^\alpha \left(\frac{4\nu B^* + \alpha F_+^*}{4\nu B^* + \alpha F_-^*} \right), \quad (100)$$

where α and ν are critical exponents and B^* is the vacuum renormalization group function associated with the additive renormalization constant of the vacuum, evaluated at the critical point. It is known to five loops in the minimal subtraction scheme [42,43] and reads, up to three loops

$$uB = \frac{N}{2}u + 3N(N+2)u^3. \quad (101)$$

The ratio b^+/b^- is equal to [41]: $b^+/b^- = 2\nu P_+^*/[(3/2) - 2\nu P_+^*]$, where P_+ is a polynomial in u , related to the scale above T_c . Its analytical derivation is given in Appendix F and reads, up to three-loops:

$$P_+ = 1 - 2(N+2)u + 4(N+2)u^2 + \frac{8}{27}(N+2) \left[-3(63N+572) + 24(N+8)\pi^2 + 4(43N+182) \ln \frac{3}{4} + 288(N+8) \text{Li}_2 \left(-\frac{1}{3} \right) \right] u^3. \quad (102)$$

The experimental test for the validity of the strong-coupling expansion is to match (100) with (1) for $N = 2$. We shall see in the next subsection if this can be done.

The ratio R_C is defined by the universal combination of amplitudes [56] $R_C = \Gamma^+ A^+ / A_M^2$ where Γ^+ and A_M are the leading amplitudes of the isotropic susceptibility above T_c and of the order parameter below T_c , respectively. This ratio has been written in Ref. [42] as

$$R_C = \frac{(2\nu P_+^*)^{2-2\beta}}{(3/2 - 2\nu P_+^*)^{-2\beta}} \frac{4\nu B^* + \alpha F_+^*}{16\pi} \frac{1}{f_\phi^* f_{\chi_+}^*}. \quad (103)$$

All the quantities have been defined previously, but for β which may be taken from the hyperscaling relation $\beta = \nu(D - 2 + \eta)/2 = \nu(1 + \eta)/2$ in $D = 3$ dimensions. However, our own calculation for R_C gives a correction to (103):

$$R_C = \frac{(2\nu P_+^*)^{2-\nu(D-2)}}{(3/2 - 2\nu P_+^*)^{-\nu(D-2)}} \frac{4\nu B^* + \alpha F_+^*}{16\pi} \frac{1}{f_\phi^* f_{\chi_+}^*} = \frac{(2\nu P_+^*)^{2-2\beta}}{(3/2 - 2\nu P_+^*)^{-2\beta}} \left(\frac{b^+}{b^-}\right)^{\nu\eta} \frac{4\nu B^* + \alpha F_+^*}{16\pi} \frac{1}{f_\phi^* f_{\chi_+}^*}. \quad (104)$$

Since this disagrees with (103), we give our derivation of this result in Appendix E. We have verified that the numerical values coming from (103) and (104) do agree within 1%. This is traced back to the small value of the exponent η . In the following we shall however consider (104). We hope that the analytical discrepancy between (103) and (104) will soon be resolved.

The third ratio to be investigated is the amplitude ratio of the susceptibilities for $N = 1$. Such a ratio can also be defined for the longitudinal susceptibilities for $N > 1$. This is a nontrivial task requiring an appropriate description [57] due to Goldstone singularities and this will not be investigated here. Using the notation of [40,55], the amplitude ratio can be written as

$$\frac{\Gamma^+}{\Gamma^-} = \frac{f_{\chi_-}^*}{f_{\chi_+}^*} \left(\frac{\xi_+}{\xi_-}\right)^2 = \frac{f_{\chi_-}^*}{f_{\chi_+}^*} \left(\frac{b^+}{b^-}\right)^{2\nu}, \quad (105)$$

where the ratio b^+/b^- has been defined below Eq. (100), and where the quantities are restricted to $N = 1$.

The question arises now to calculate the amplitude functions and ratios. As for the case of the critical exponents, we shall proceed also by order, starting with two loops.

C. Amplitude functions from two-loop expansions

In order to apply strong-coupling theory to the amplitude functions (85)–(88), we must reexpand them in powers of the bare coupling \bar{u}_B using (14) up to two loops. The strong-coupling limit is then given by the general expression (23), with $\rho^2/4$ given by Eq. (34) at $\epsilon = 1$.

We start considering f_ϕ . To deal with a Taylor series, as assumed in the general theory in Section III A, we consider uf_ϕ :

$$uf_\phi = \frac{1}{32\pi} + \left[\frac{1}{27\pi}(160 - 82N) + \frac{2}{\pi}(N - 1) \ln 3 \right] \bar{u}_B^2. \quad (106)$$

This series is special because the linear term in \bar{u}_B , is absent: the optimal value (23) is therefore given by $\hat{u}_B^* = 0$, and the two-loop value of uf_ϕ in the strong-coupling limit is the same as the lowest-order value, which is independent of N :

$$u^* f_\phi^* = \frac{1}{32\pi}. \quad (107)$$

It is worth pointing out here the effect of the special choice for A_D in (8). We mentioned there that this coefficient did not have any influence upon the critical exponent. This is because the factor A_D can be absorbed in u_B to give \bar{u}_B , implying the same strong-coupling limit. However, amplitude functions are A_D -dependent. In particular, for uf_ϕ , the chosen value $A_3 = 1/(4\pi)$ has made the linear term disappear. One sees that this choice correspond to an optimization: the zero order, the one-loop and the two-loop optimum values coincide. One expect then that the third-loop order contributes only to a small deviation from it. This is indeed the case, as will be shown in the next section, and confirm previous expectations [38,41].

The same situation holds for the amplitude function (90) of the susceptibility above T_c . The linear term in u being absent, the optimal value to two loops is independent of N and is equal to

$$f_{\chi_+}^* = 1. \quad (108)$$

The strong-coupling limit of the amplitude function of the stiffness of phase fluctuations and of the q^2 -dependent part of the transverse susceptibility can also be easily determined. The bare expansion is obtained combining (86), (87) and (14) to two loops:

$$uf_\Upsilon = \frac{1}{8} + \frac{\bar{u}_B}{3} + \left[\frac{1}{54}(1802 - 755N) + 8(N - 3) \ln 3 \right] \bar{u}_B^2, \quad (109)$$

$$f_{\chi_T} = 1 + \frac{8}{3}\bar{u}_B - \frac{4}{3}(11N - 58 + 96 \ln 3)\bar{u}_B^2. \quad (110)$$

The corresponding optima are given by (23) with (34) from which we obtain

$$u^* f_\Upsilon^* = \frac{1}{8} + \frac{6(N^2 + 25N + 106)}{(N + 8)^2 [755N - 1802 - (432N - 1296) \ln 3]}, \quad (111)$$

$$f_{\chi_T}^* = 1 + \frac{16(N^2 + 25N + 106)}{3(N + 8)^2 [11N - 58 + 96 \ln 3]}. \quad (112)$$

The result (111) has a pole for $N = 2(648 \ln 3 - 901)/(432 \ln 3 - 755) \approx 1.349$, indicating that the strong-coupling result is unreliable. We expect the pole to be an artifact of the limitation to two loops which disappears at the three-loop level. Since f_Υ is not known to three-loops, we can only give plausible arguments for this expectation, suggested by the calculation of $u^*(F_-^* - F_+^*)$ up to three loops in Eq. (136), where a similar pole arises at the two-loop level but disappears for three loops due to the interplay of the coefficients of the loop expansion. The trouble with (111) derives from the fact that the term of order \bar{u}_B^2 in (109) change sign for the mentioned value of $N \approx 1.349$, and at the two-loop level nothing can compensate this. This is in contrast with critical exponents which were observed to be alternating series in powers of \bar{u}_B . The result (112) for f_{χ_T} is smooth for all positive N . A more reliable result for f_Υ^* than the singular (111) can therefore be obtained by combining (107) with (112) via relation (92), leading to

$$f_\Upsilon^* = f_{\chi_T}^*/8. \quad (113)$$

Note that for $N \geq 4$, far away from the pole, the two results (111) and (113) agree within 2%.

It is worth pointing out that an evaluation of the renormalized expression (87) at the critical point u^* given by (55) leads to a result compatible with (112) within less than 1%. This is due to the fact that higher-order correction to the zero order result $f_{\chi_T} = 1$ are small for all N . This is in contrast to f_ϕ and f_Υ , where two-loop corrections are important.

We now turn to the amplitude functions F_\pm which enter the heat capacity above and below T_c . At the two-loop level, they are given by Eqs. (88) and (89), respectively. With the relation between the renormalized and bare coupling constant (14) to two loops, we have the expansions

$$uF_+ = -N\bar{u}_B + 2N(N+14)\bar{u}_B^2, \quad (114)$$

$$uF_- = \frac{1}{2} - 4\bar{u}_B + 8(N+26)\bar{u}_B^2. \quad (115)$$

With the help of (23) and (34), we obtain

$$u^*F_+^* = -\frac{N}{2} \frac{N^2 + 25N + 106}{(N+8)^2(N+14)}, \quad (116)$$

$$u^*F_-^* = \frac{1}{2} - 2 \frac{N^2 + 25N + 106}{(N+8)^2(N+26)}. \quad (117)$$

In [41,42], uF_+ was not a good candidate for Borel resummation because its u -expansion (88) lacking alternating signs of its coefficients. This problem is absent in variational perturbation theory since the expansion (114) in term of the bare coupling constant \bar{u}_B does have alternating sign. The latter is then expected to lead to a reliable result (116). This will be confirmed by the three-loop result of the next section.

To apply the usual Borel resummation at the level of the renormalized quantities, Refs. [41,42] wrote the amplitude ratio of the heat capacity as

$$\frac{A^+}{A^-} = \left(\frac{b^+}{b^-}\right)^\alpha \left(1 - \alpha \frac{F_-^* - F_+^*}{4\nu B^* + \alpha F_-^*}\right), \quad (118)$$

instead of (100), and resummed $u(F_- - F_+)$ and uF_- , avoiding the direct resummation of uF_+ . For comparison, we give below the optimal value of the difference $u(F_- - F_+)$. It is determined from the expansions (88) and (89). Using (14), it yields

$$u(F_- - F_+) = \frac{1}{2} + (N-4)\bar{u}_B - 2(N^2 + 10N - 104)\bar{u}_B^2. \quad (119)$$

Its strong-coupling limit is, from (23) and (34):

$$u^*(F_-^* - F_+^*) = \frac{1}{2} + \frac{(N-4)^2(N^2 + 25N + 106)}{2(N+8)^2(N^2 + 10N - 104)}. \quad (120)$$

The latter expression diverge for a positive value of $N = -5 + \sqrt{129} \approx 6.358$. Then, the difference $u(F_+ - F_-)$ is not the good quantity for the strong-coupling limit at the two-loop level. We should rather evaluate uF_+ and uF_- separately in the amplitude ratio (100), instead of using the equivalent expression (118). We shall see in the next section that the pole of $u^*(F_+^* - F_-^*)$ is an artifact of the two-loop calculation. A similar conclusion was also obtained for the strong-coupling limit of f_Υ , see Eqs. (111) and (113). For $N \ll 4$ and for $N \gg -5 + \sqrt{129}$, the two-loop expansion (119) is alternating, and we expect that the strong-coupling result (120) is reliable. As an indication for this, we compare (120) with the difference of the optimized $u^*F_\pm^*$ values given in Eqs. (116) and (117)

$$u^*\Delta F_\pm^* = \frac{1}{2} - \frac{N^2 + 25N + 106}{(N+8)^2} \left(\frac{2}{N+26} - \frac{N}{2(N+14)} \right). \quad (121)$$

In Figure 6, we compare the two curves (120) and (121).

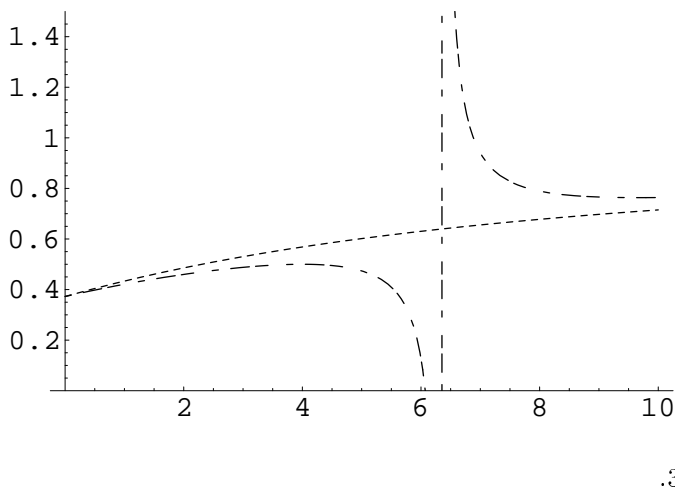


FIG. 6. Comparison between the two-loop strong-coupling limit of $u^*(F_-^* - F_+^*)$ and $u^*\Delta F_\pm^*$.

As far as the amplitude ratio of the heat capacity (100), or (118), is concerned, we still need to determine the strong-coupling limit of the renormalization group function $B(u)$ of the vacuum (101) and of the polynomial P_+ defined in (102). Because there is no contribution of the two-loop order to (101), its strong-coupling limit is

$$u^* B^* = u^* \frac{N}{2}. \quad (122)$$

Since the optimal two-loop result is identical to the one-loop result, it is clear that we may expect the large order limit $L \rightarrow \infty$ to differ only little from $N/2$. This has been confirmed in the five-loop resummation performed in [41], and will be also seen in our three-loop calculation in the next section.

The polynomial P_+ given in (102) is evaluated in the strong-coupling limit using the same lines. The starting point is the expansion in powers of the bare coupling constant given in Eq. (F4) of Appendix F. Its two-loop part combined with Eqs. (23) and (34) leads to

$$P_+^* = 1 - \frac{(N+2)(N^2+25N+106)}{(N+8)^2(2N+17)}. \quad (123)$$

The last amplitude we shall calculate using strong-coupling theory is f_{χ_-} . From (23), (34) at $N=1$ and from the two-loop part of (D3), we find, with Eqs. (23) and (34),

$$f_{\chi_-}^* = 1 + 9 \frac{\rho^2}{4} \frac{18^2}{4532} = 211/103. \quad (124)$$

Combining with the unit value (108) of $f_{\chi_+}^*$, we have a ratio $f_{\chi_-}^*/f_{\chi_+}^*$ identical to (124). However, this ratio might as well be determined as the strong-coupling limit of its perturbative expansion, instead of evaluating independently the strong-coupling limit of the numerator and the denominator. The relevant equation is given in Appendix D: Using the two-loop expansion of (D6), we have, with Eqs. (23) and (34),

$$\left(\frac{f_{\chi_-}}{f_{\chi_+}} \right)^* = 1 + \frac{\rho^2}{4} \frac{3 \times 18^2}{1420} = \frac{751}{355}. \quad (125)$$

D. Amplitude functions from three-loop expansions

Some of the amplitude functions have been obtained to three loops. We now turn to their strong-coupling limit. This is done by applying Eqs. (24), (25) and (64) to the different amplitude expansions.

We start with the amplitude function of the square of the order parameter. Combining the two-loop expansion (85) with the three-loop term $f_\phi^{(3)}$ (93), and using also the relation between the bare and renormalized coupling constant (14), we have the three-loop expansion

$$uf_\phi = \frac{1}{32\pi} + \left[\frac{1}{27\pi}(160 - 82N) + \frac{2}{\pi}(N - 1) \ln 3 \right] \bar{u}_B^2 + \left\{ f_\phi^{(3)} - 8(N + 8) \left[\frac{1}{27\pi}(160 - 82N) + \frac{2}{\pi}(N - 1) \ln 3 \right] \right\} \bar{u}_B^3. \quad (126)$$

From this, we read off the expansion coefficients f_0, f_1, f_2, f_3 entering Eqs. (24), (25) and (64). Since the linear term f_1 vanishes, we have to follow the development below Eq. (26), adapting it to the present case. This development was done assuming a series with alternating sign since the expansions of the critical exponents had this property. Here, this is no longer true. Consider once more the derivation of the strong-coupling limit following from the optimal value of $f = f_0 + \bar{f}_2 \hat{u}_B^2 + f_3 \hat{u}_B^3$: $\hat{u}_B^*(2\bar{f}_2 + 3f_3 \hat{u}_B^*) = 0$. Two solutions are possible: $\hat{u}_B^* = 0$ and $\hat{u}_B^* = -2\bar{f}_2/(3f_3)$. The latter was relevant for the critical exponent η . This does not mean that the other solution has to be rejected. In fact, looking at the nature of the extremum (minimum or maximum), we see directly that the first solution corresponds to

$$\left. \frac{\partial^2 f}{\partial \hat{u}_B^2} \right|_{\hat{u}_B = \hat{u}_B^* = 0} = 2\bar{f}_2, \quad (127)$$

while the other leads to

$$\left. \frac{\partial^2 f}{\partial \hat{u}_B^2} \right|_{\hat{u}_B = \hat{u}_B^* = -2\bar{f}_2/(3f_3)} = -2\bar{f}_2. \quad (128)$$

If one solution is a maximum, the other one is a minimum. Looking for the sign of \bar{f}_2 in Eq. (126), we see that it is positive for $N < N_\phi$, with $N_\phi = (80 - 27 \ln 3)/(41 - 27 \ln 3) \approx 4.43992$, corresponding to a maximum, and negative for N greater, corresponding to a minimum. Variational perturbation theory at loop order $L > 1$ says nothing about the nature of the extremum. It might be a minimum or a maximum. In quantum mechanics, this has been explained in the book [1]. In quantum field theory, the exponent η illustrates this: we had chosen the maximum (recall Eq. (75)). In this way, the ϵ -expansion was obtained. Taking the solution $\hat{u}_B^* = 0$, corresponding to the minimum, we would have obtained the three-loop result $\eta = 0$. The lack of reproducing the ϵ -expansion gives a hint that the maximum solution has to be chosen. In the case of f_ϕ we can also argue that the maximum solution has to be chosen, although there is here no ϵ -expansion available, by definition of the model. However, at the point where $f_3 = 0$, we have to recover an optimization problem of a quadratic equation in \hat{u}_B , see Eq. (126). We know for this function that, because no linear term is present, the strong-coupling limit is $u^* f_\phi = 1/(32\pi)$. This implies that $\hat{u}_B^* = 0$ at this point, i.e., that the maximum solution has to be chosen. By continuity, this remains true in a neighborhood. The nature of the solution can only be changed when both solutions are equal, i.e., for N smaller than its value N_ϕ making f_2 vanish. Below N_ϕ , we can imagine that we have an interchange of solutions, and that the minimum has to be chosen. In this case, we would have $u^* f_\phi^* = 1/(32\pi)$ for all N . If we decide to keep the maximum for all N , which we could prove to be true only for $N \geq N_\phi$, this would imply that $\hat{u}_B^* = -2\bar{f}_2/(3f_3)$ has to be chosen below N_ϕ and $\hat{u}_B^* = 0$ above.

Below N_ϕ we have $f^* = f_0 + 4\bar{f}_2^3/(27\bar{f}_3^2)$, as was the case for the critical exponent η , while above N_ϕ , the solution is $f^* = f_0$. The strong-coupling limit of f_ϕ to three loops is then

$$u^* f_\phi^* = \frac{1}{32\pi} + \frac{4}{27}(2\rho - 1)^3 \frac{[(160 - 82N)/(27\pi) + 2(N - 1)(\ln 3)/\pi]^3}{\left\{f_\phi^{(3)} - 8(N + 8)[(160 - 82N)/(27\pi) + 2(N - 1)(\ln 3)/\pi]\right\}^2} \Theta\left(\frac{80 - 27 \ln 3}{41 - 27 \ln 3} - N\right) \quad (129)$$

with ρ given by Eq. (64), and where $\Theta(x)$ is the step function of Heaviside, being equal to 1 for $x > 0$ and being vanishing for $x < 0$. As mentioned, we cannot be assured that for $N < N_\phi$ the maximum has still to be chosen. The possibility that the three-loop result is identical to the two-loop result remains. Would the above analysis not be performed, i.e., choosing the minimum solution everywhere, we would have obtained (129) without the step function, meaning the presence of a pole at the vanishing of the coefficient of the cubic term in (126), i.e., for $N \approx 4.92915$. We have checked that the solution is sharply peaked near this value so that it would appear that the optimal value is valid almost everywhere. In fact, since the pole gives a very peaked contribution, a calculation at fixed integer value of N would have missed it completely, making one to believe that the resummation was correct. But this would not be true, the true solution being (129) everywhere. We give in Figure 7 the comparison between our two- and three-loop results. Our values for $N < N_\phi$ lie above the two-loop result $1/(32\pi)$ obtained in (107).

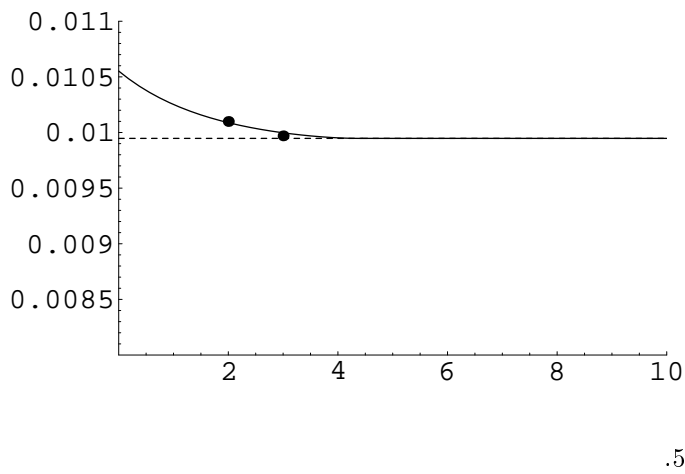


FIG. 7. Comparison between the two-loop (short-dashed) and three-loop (solid) amplitude function of the order parameter.

The resummed values [42] obtained using a Borel resummation are indicated by the dots for values of N available.

This is also the case for the resummed values given in [42] for $N = 2, 3$ as can be seen in the following table:

N	0	1	2	3	4	$N > N_\phi = (80 - 27 \ln 3)/(41 - 27 \ln 3)$
$u^* f_\phi^*$ (2 loops)	$1/(32\pi)$	$1/(32\pi)$	$1/(32\pi)$	$1/(32\pi)$	$1/(32\pi)$	$1/(32\pi) = 0.00994718$
$u^* f_\phi^*$ (3 loops)	0.0105523	0.0102518	0.0100884	0.00999735	0.00995195	$1/(32\pi)$
$u^* f_\phi^*$ (Ref. [42])			0.010099	0.00997		

The agreement between our two- and three-loop order, and between our work and [42], is excellent. It is due to the fact that the term of order zero contains almost all information on this amplitude.

The three-loop amplitude functions uF_+ and uF_- are given by Eqs. (88), (94), (89) and (95). As in the case of the previous amplitudes, the present expansions may not be alternating. This may make the argument of the parameter r in Eq. (24) positive, so that (24) has to be used to obtain the strong-coupling limit rather than (25). However, (25) remains correct for all N for uF_+ , while the alternating property is lost for uF_- for $N \gtrsim 40$. Since the physical cases corresponds to $N = 0, 1, 2, 3, 4$, we can ignore the alternative (24) and (25) is used throughout. Using the relation (14) between the bare and renormalized coupling constant, the three-loop bare extension of Eqs. (114) and (115) are

$$uF_+ = -N\bar{u}_B + 2N(N+14)\bar{u}_B^2 + \left[F_+^{(3)} - 24N(7N+46)\right]\bar{u}_B^3, \quad (130)$$

$$uF_- = \frac{1}{2} - 4\bar{u}_B + 8(N+26)\bar{u}_B^2 + \left[F_-^{(3)} - 480(3N+22)\right]\bar{u}_B^3. \quad (131)$$

This allows to identify the appropriate f_0, f_1, f_2, f_3 functions to enter Eq. (25). In the strong-coupling limit, we obtain

$$u^*F_+^* = \frac{2N^2(N+14)\rho(\rho+1)(2\rho-1)}{6\left[F_+^{(3)} - 24N(7N+46)\right]} + \frac{2\left[2N(N+14)\right]^3(2\rho-1)^3}{27\left[F_+^{(3)} - 24N(7N+46)\right]^2}, \quad (132)$$

$$u^*F_-^* = \frac{1}{2} + \frac{32(N+26)\rho(\rho+1)(2\rho-1)}{6\left[F_-^{(3)} - 480(3N+22)\right]} + \frac{2\left[8(N+26)\right]^3(2\rho-1)^3}{27\left[F_-^{(3)} - 480(3N+22)\right]^2}, \quad (133)$$

with ρ from Eq. (64).

Figures 8 and 9 show the comparison between the two-loop (116) and (117) results of the previous section and the corresponding three-loop (132) and (133) results, as well as a comparison with values given in [41], when available.

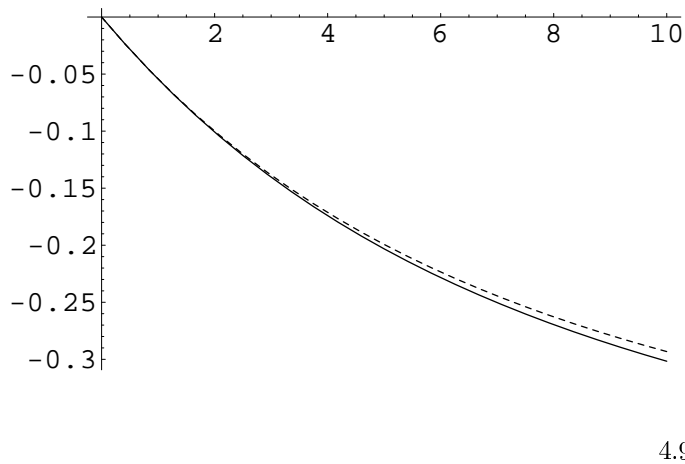
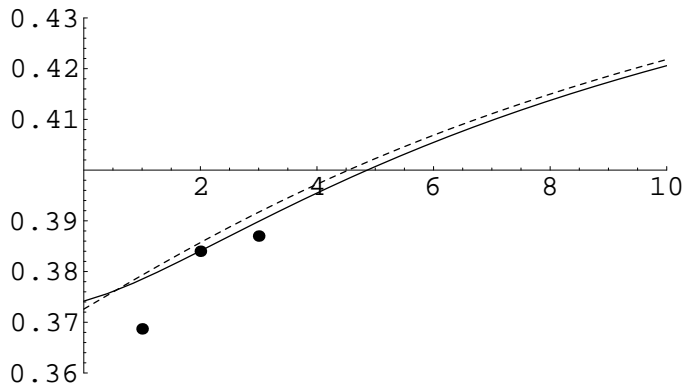


FIG. 8. Comparison between the strong-coupling limit of the two-loop (shot-dashed) and three-loop amplitude function (solid) $u^*F_+^*$.



$2.99N^*F_*$

FIG. 9. Comparison between the strong-coupling limit of the two-loop (shot-dashed) and three-loop amplitude function (solid) $u^*F_-^*$. The resummed values [42] obtained using a Borel resummation are indicated by the dots for values of N available.

To be more precise concerning the comparison with [41], we give the appropriate values of $u^*F_-^*$ in the next table:

N	0	1	2	3	4
$u^*F_-^*$ (2 loops)	0.372596	0.379287	0.385714	0.391707	0.397222
$u^*F_-^*$ (3 loops)	0.374166	0.378474	0.384065	0.389883	0.395484
$u^*F_-^*$ (Ref. [41])		0.3687	0.384	0.387	

From this table, we see that that the strong-coupling limit results for $u^*F_-^+$ at the three-loop level differ only a little from their two-loop counterpart. This was also see on Figure 9 and on Figure 8 for $u^*F_+^*$. For $N = 1$, we can also infer from the table that the results coming from variational perturbation theory and from a Borel resummation [41] are not in excellent agreement, not even within the error-bars of the latter: $u^*F_-^*(N = 1) = 0.3687 \pm 0.0040$. The agreement is however recovered for the values $N = 2, 3$.

For $u^*F_+^*$, there is no available comparison between our work and others. The authors of [41] could not performed a reliable Borel resummation, presumably because of the lack of an alternating series. A comparison is however possible for the difference $u^*(F_-^* - F_+^*)$. We have seen in the previous section that the two-loop evaluation of this difference in the strong-coupling limit did not work well in our case because the second-order term in the bare expansion changes sign for some value of N . Let us see how the situation changes at the three-loop level, which has the expansion (see Eqs. (130) and (131))

$$u(F_- - F_+) = \frac{1}{2} + (N - 4)\bar{u}_B - 2(N^2 + 10N - 104)\bar{u}_B^2 + \left[F_-^{(3)} - F_+^{(3)} + 24(7N^2 - 14N - 440) \right] \bar{u}_B^3. \quad (134)$$

The coefficient of the second-order term vanishes for $N = -5 + \sqrt{129}$. This is not anymore a problem since there is a three-loop order term preventing a $1/f_2$ -behavior, see the comparison of (23) and (25). The coefficient of the three-loop order can itself vanish. For (134), this happens for $N \equiv \bar{N} \approx 10.5324$. Since (24) and (25) imply a behavior like $1/f_3$, it is legitimate to wonder about poles. The answer is simple: if the coefficient of the three-loop term vanishes, then the problem is formally equivalent to evaluating the strong-coupling limit of a two-loop series. The coefficient of the linear and quadratic terms are however different from the two-loop result since the linear term has a factor $\rho(\rho + 1)/2$ instead of ρ and the quadratic one a coefficient $(2\rho - 1)$ instead of a factor 1. We conclude that when the three-loop term vanishes, the strong-coupling limit should be well-behaved, giving a smooth curve around \bar{N} . This discussion shows that the function r in (24) is not always zero here because r contains the coefficient f_2 of the two-loop term, and its zero govern the behavior of the solution (24). We note here the important following point: the positive square root $+r$ was chosen in (24) in order to match with a vanishing f_3 . We explained, and this was used when evaluating η , that the negative root might play a role as well. For η to three-loop order, we had a negative r . Let us see what happens for $f_2 = 0$. The expansion to be optimized is $f = f_0 + \bar{f}_1 \hat{u}_B + f_3 \hat{u}_B^3$, such that we have to solve $\bar{f}_1 + 3f_3(\hat{u}_B^*)^2 = 0$. For the critical exponents, the signs of \bar{f}_1 and f_3 are the same because the series are alternating. For this reason, this equation has no real solution, and we must solve the turning-point equation $6f_3 \hat{u}_B^* = 0$, which is $\hat{u}_B^* = 0$, leading to the optimized result $f^* = f_0$. For the amplitude functions, we have already seen that the alternating property is not necessarily true, so that the solution $\hat{u}_B^2 = -\bar{f}_1/(3f_3)$ is real. At the point where f_2 vanishes, we can see that the optimal value is

$$f^* = f_0 \pm \frac{2}{3} \bar{f}_1 \sqrt{-\frac{2\bar{f}_1}{3f_3}} \quad (135)$$

the positive or negative sign being chosen to get a continuity of the solution around f_2 .

For the difference function $u(F_- - F_+)$, it is possible to follow exactly the strong-coupling limit as a function of N . Depending on N , there are four different solutions: below $\bar{N}_1 \approx 2.48527$ and above $\bar{N}_3 \approx 16.6066$, the argument of the square root of r is negative, and one uses Eq. (25). For $N \in [\bar{N}_2, \bar{N}_3[$, one uses Eq. (24) with the positive root ($r = |r|$), where $\bar{N}_3 = -5 + \sqrt{129} \approx 6.35782$ is the value of N for which f_2 changes sign. The zero of f_3 lies within the same region. Finally, the last region is within the range $N \in]\bar{N}_1, \bar{N}_2]$, for which we use Eq. (24), but with the negative root $r = -|r|$. More precisely,

$$u^*(F_-^* - F_+^*) = \frac{1}{2} + \frac{2(N-4)(N^2 + 10N - 104)\rho(\rho+1)(2\rho-1)}{6 \left[F_-^{(3)} - F_+^{(3)} + 24(7N^2 - 14N - 440) \right]} \left(1 - \frac{2}{3}r \right)$$

$$- \frac{2[2(N^2 + 10N - 104)]^3(2\rho - 1)^3}{27 \left[F_-^{(3)} - F_+^{(3)} + 24(7N^2 - 14N - 440) \right]^2} (1 - r), \quad (136)$$

with $r = 0$ for $N \lesssim \bar{N}_1$ and $N \gtrsim \bar{N}_3$, and r the negative or positive square root of

$$r^2 = 1 - 3 \frac{(N - 4) \left[F_-^{(3)} - F_+^{(3)} + 24(7N^2 - 14N - 440) \right] \rho(\rho + 1)}{2[2(N^2 + 10N - 104)]^2(2\rho - 1)^2} \quad (137)$$

for $N \in]\bar{N}_1, \bar{N}_2]$ or $N \in [\bar{N}_2, \bar{N}_3[$, respectively.

Thus, the pole in N of the two-loop approximation to $u^*(F_-^* - F_+^*)$ was only an artifact of the low order. At the three-loop level, the singularity is avoided by the interplay between the different possible solutions of (24) arising from the different branches of r : $r = 0, \pm|r|$, with $|r|$ to be identify with the function r defined below Eq. (24). This possibility was not exploited in previous works [2,5] because of the alternating signs for the critical exponents. (See however η which required $r = -1$ for the strong-coupling limit of (74).)

In Figure 10, we show the strong-coupling limit of $u^*(F_-^* - F_+^*)$. For comparison, we also give the direct difference $u^*\Delta F_{\pm}^*$ between $u^*F_-^*$ and $u^*F_+^*$, as obtained from Eqs. (133) and (132), as well as its two-loop counterpart (121). The range for N has been increased to 30 in order to investigate the regions delimited by \bar{N}_1 , \bar{N}_2 and \bar{N}_3 .

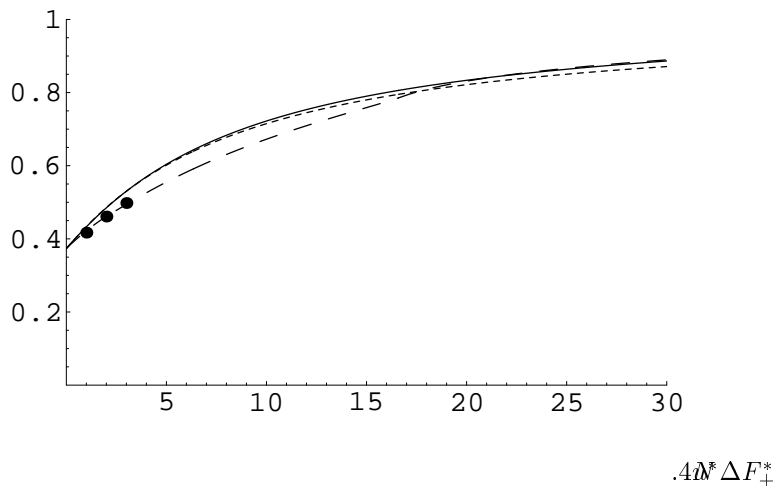


FIG. 10. Comparison between the strong-coupling limit of the two-loop (shot-dashed) and three-loop amplitude function (solid) $u^*\Delta F_{\pm}^* = u^*F_-^* - u^*F_+^*$. The three-loop (long-dashed) evaluation of $u^*(F_-^* - F_+^*)$ is in better agreement with the resummed values (dots) obtained in [42] using a five-loop ($N = 1$) or three-loop ($N \neq 0$) Borel resummation.

For the direct difference, the changes brought about by the three-loop is very small, as before in Figs. 8 and 9. The difference between $u^*(F_-^* - F_+^*)$ and $u^*\Delta F_{\pm}^*$ is however somewhat larger. To facilitate the comparison, the following table should be of help:

N	0	1	2	3	4
$u^* \Delta F_{\pm}^*$ (2 loops)	0.372596	0.433608	0.485714	0.530258	0.568519
$u^* \Delta F_{\pm}^*$ (3 loops)	0.374166	0.432926	0.484899	0.530224	0.569615
$u^*(F_-^* - F_+^*)$ (3 loops)	0.374166	0.421864	0.461436	0.489995	1/2
$u^*(F_-^* - F_+^*)$ (Ref. [41])		0.4179	0.461	0.498	

The simple value $1/2$ for the case $N = 4$ comes from Eqs. (136) and (137): for $N = 4$, r is vanishing, meaning the third term of (136) does not contribute. Since the second term is also proportional to $N - 4$, only the zero loop order survives for the Higgs case. Our results for $u^*(F_-^* - F_+^*)$ are in good agreement with the Borel results of Ref. [41]. This is probably not a coincidence since we now resum the same function as they did. We note however that, for the Ising model ($N = 1$), we are not within the error bars of [41]. We have already noted this for the strong-coupling limit of $u^*F_-^*$.

Before cloturing the investigation of $u(F_- - F_+)$, we recall the case of f_{Υ} , whose direct two-loop strong-coupling limit gave Eq. (111), exhibiting a pole. We know the strong-coupling limit should not have been far from $f_{\chi_T}^*/8$, see the discussion leading to Eq. (113). We have shown in this section how a pole in $u^*(F_-^* - F_+^*)$ at the two-loop level might disappear at the three-loop one. This is probably the case for f_{Υ} . It would be very useful to get its three-loop order.

We can now turn to the strong-coupling limit of the renormalization group constant of the vacuum $B(u)$. Its three-loop value has been given in Eq. (101).

Upon inserting the relation between the renormalized and the bare coupling constant (14), we obtain

$$uB(u) = \frac{N}{2}\bar{u}_B - 2N(N+8)\bar{u}_B^2 + N(8N^2 + 167N + 686)\bar{u}_B^3. \quad (138)$$

The series is alternating and behaves as for the critical exponents. No subtleties arises here as in the case of f_{ϕ} and $u^*(F_-^* - F_+^*)$. In particular, the argument of the square root of r in Eq. (24) is negative for all N and we have to work with (25). Using (25), the strong-coupling limit is

$$u^*B^* = \frac{N(N+8)\rho(\rho+1)(2\rho-1)}{6(8N^2 + 167N + 686)} - \frac{16N(N+8)^3(2\rho-1)^3}{27(8N^2 + 167N + 686)^2}. \quad (139)$$

This result is plotted in Figure 11 together with the two-loop result $u^*N/2$, see Eq. (122). We also indicate the approximate result $u^*B^* = u^*N/2$ with u^* from the three-loop expansion (84). There is no visible difference between the latter and (139).

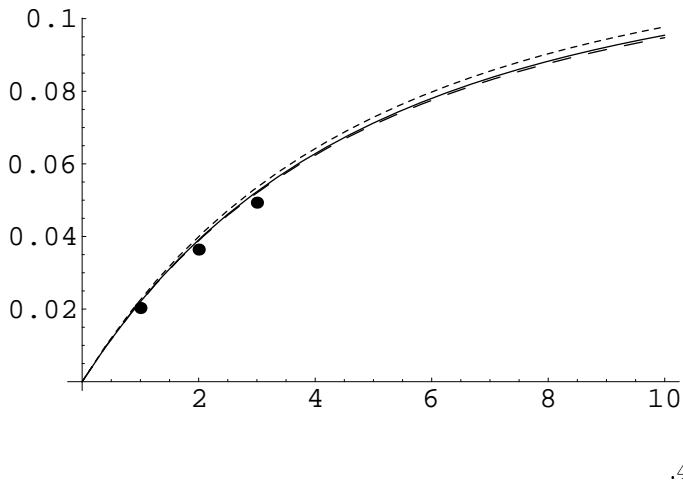


FIG. 11. Comparison between the strong-coupling limit of the two-loop (short-dashed) and three-loop (solid) renormalization group constant of the vacuum $u^* B^*$. For completeness, we also give the approximate three-loop (long-dashed) result $u_{(3)}^* N/2$. A five-loop calculation [41] using Borel resummation is included (dots) for values of N available.

To facilitate the comparison between the different approximations, we recapitulate the numerical results in the next table:

N	0	1	2	3	4
$u^* B^*$ (2 loops)	0	0.0226337	0.04	0.0535312	0.0642361
$u^* B^*$ (3 loops)	0	0.0221074	0.0391089	0.0523643	0.0628447
$u^* B^* = u_{(3)}^* N/2$	0	0.0219975	0.0388885	0.0520441	0.0624386
$u^* B^*$ (Ref. [41])	0	0.020297	0.0363919	0.049312	

For the comparison with [41], we have multiplied their five-loop results for B^* with their five-loop u^* . These five-loop results are within 7% from our three-loop strong-coupling calculation. This confirms that $B^* \approx N/2$ to all orders. We shall however see in the next section that this 7% difference leads to a non-negligible difference in the universal combination R_C .

The next quantity we shall resum to three loops is the polynomial P_+ . The three-loop bare expansion of the renormalized P_+ was given in Eq. (F4) and resummed to two loops in (123). The series in the bare coupling constant is alternating, and behaves as for the critical exponents. The strong-coupling limit of P_+^* to three loops is then given by

$$P_+^* = 1 + \frac{9}{2} \frac{(N+2)(2N+17)\rho(\rho+1)(2\rho-1)}{[-3(36N^2+837N+3920)+24(N+8)\pi^2+4(43N+182)\ln(3/4)+288(N+8)\text{Li}_2(-1/3)]} + 54 \frac{(N+2)(2N+17)^3(2\rho-1)^3}{[-3(36N^2+837N+3920)+24(N+8)\pi^2+4(43N+182)\ln(3/4)+288(N+8)\text{Li}_2(-1/3)]^2}, \quad (140)$$

with ρ from Eq. (64).

In Figure 12, we compare (140) with the two-loop result from Eq. (123). Almost no difference is found between our two- and three-loop expansions.

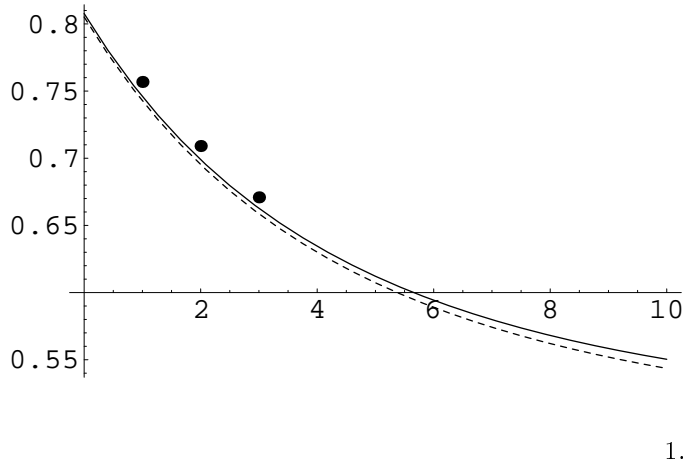


FIG. 12. Comparison between the strong-coupling limit of the two-loop (short-dashed) and three-loop (solid) polynomial P_+^* . The values [42] obtained using a five-loop Borel resummation are indicated by the dots for values of N available.

For a better comparison, we quote the numerical values for $N = 0, 1, 2, 3, 4$ in the next table:

N	0	1	2	3	4
P_+^* (2 loops)	0.805147	0.74269	0.695238	0.658642	0.63
P_+^* (3 loops)	0.807683	0.745874	0.698901	0.662717	0.63447
P_+^* (Ref. [41])		0.7568	0.7091	0.6709	

The two- and three-loop results agree within 1%. The results agree fairly well with the five-loop Borel resummation performed in [41]. We shall however see later that amplitude ratios depends crucially on the exact value of P_+^* . For this reason, our three-loop calculation is probably not precise enough. We shall present in the last section a numerical five-loop strong-coupling evaluation of P_+^* to more firmly settle this statement.

To conclude this section, we discuss the amplitude of the susceptibilities above and below T_c to three loops. We already know from the previous section that the two-loop amplitude above T_c is identical to the order zero: $f_{\chi_+}^* = 1$, see Eq. (108). As for the case of uf_ϕ , we then expect a very small deviation from the zero-order value as well as a very smooth N -dependence. The series to evaluate in the strong-coupling limit is given in Eq. (C9). It is alternating and behaves like the series of the critical exponents. Moreover, with a vanishing linear term, but with a negative coefficient of the quadratic term, the solution of the optimization problem is at variance with the case of the exponent η or the amplitude f_ϕ if, as for these quantities, we admit that the solution is a maximum. From Eq. (127), we determine

that the optimal value is $\hat{u}_B^* = 0$, so that

$$f_{\chi_+}^* = 1 \quad (141)$$

remains true at the three-loop level: The amplitude of the susceptibility above T_c at the three-loop level does not depend on N ! This is in contrast to [55], where a N -dependent fit, using Borel resummation, has been performed. Because our resummed value up to three loops is $f_{\chi_+} = 1$, it is tempting to conjecture that this is true for all orders. However, contrary to the case of η and f_ϕ , we have here no argument to tell that the maximum has to be chosen instead of the minimum. Only when going to higher orders, then having more expansion coefficients, can we decide which solution is the right one. For this reason, we also mention below this other solution, which differs from unity for at most 2.5%:

$$f_{\chi_+}^* = 1 - \frac{48668(N+2)^3(2\rho-1)^3}{[27(N+2)(N+8)]^2[-113+12\pi^2+128\ln(3/4)+144\text{Li}_2(-1/3)]^2}. \quad (142)$$

The comparison between the two curves is given in Figure 13, as well as a comparison with the fit

$$f_{\chi_+} = 1 - 92(N+2)u^2(1+b_{\chi_+}u)/27 \quad (143)$$

taken from Table 1 of [55], with $b_\chi = 9.68$ ($N = 1$), 11.3 ($N = 2$), 12.9 ($N = 3$), combined with the five-loop u^* of Ref. [41].

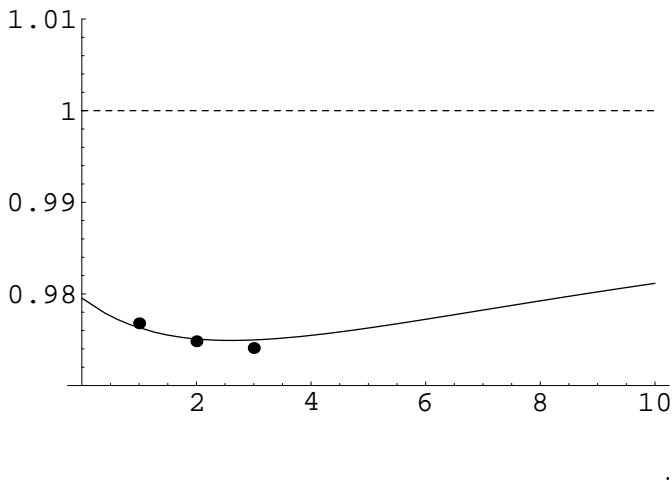


FIG. 13. Comparison between the two-loop strong-coupling limit (short-dashed) of the amplitude $f_{\chi_+}^*$ of the susceptibility above T_c and the second possible solution (142) at the three-loop level (solid). The values [55] obtained using a five-loop Borel resummation (dots) are given for values of N available.

More precisely, we have the table:

N	0	1	2	3	4
$f_{\chi_+}^*$ (2 loops)	1	1	1	1	1
$f_{\chi_+}^*$ (3 loops) from (141)	1	1	1	1	1
$f_{\chi_+}^*$ (3 loops) from (142)	0.979543	0.976298	0.975082	0.974977	0.975472
$f_{\chi_+}^*$ (Refs. [41,55])		0.976791	0.9748331	0.9740978	

The fact that our three-loop calculation (142) agrees very well with Refs. [41,55] might be an indication that (142) should be preferable to (141). However, from a variational perturbation theory point of view, nothing can be said. Only the determination of the next order might resolve the ambiguity.

Finally, we determine the strong-coupling limit of the amplitude of the susceptibility below T_c for $N = 1$. We have checked that the parameter r in (24) is zero, i.e., we have to work with the turning-point equation (25). Applying it to (D4), we have

$$f_{\chi_-}^* = 1 + \frac{4352\rho(\rho+1)(2\rho-1)}{3[19904 - 11664c_1 + 3\pi^2 + 10480\ln(4/3) + 36\text{Li}_2(-1/3)]} - \frac{164852924416(2\rho-1)^2}{19683[19904 - 11664c_1 + 3\pi^2 + 10480\ln(4/3) + 36\text{Li}_2(-1/3)]^2}, \quad (144)$$

with ρ from Eq. (64). Numerically, this is evaluated as $f_{\chi_-}^* \approx 2.09387$, to be compared with the two-loop (124) result $211/103 \approx 2.048544$. They agree within 3%.

For the ratio (105), the calculation of f_{χ_-}/f_{χ_+} is needed. Its strong-coupling limit can be determined using the individual strong-coupling limit of the numerator and the denominator. In that case, the ambiguity on $f_{\chi_-}^*$ at the three-loop level is relevant. According to the choice $f_{\chi_+}^* = 1$ from (141) or $f_{\chi_+}^* \approx 0.976298$ from (142) with $N = 1$, we have

$$\frac{f_{\chi_-}^*}{f_{\chi_+}^*} = 2.09387, \quad (145)$$

$$\frac{f_{\chi_-}^*}{f_{\chi_+}^*} = 2.1447, \quad (146)$$

respectively.

The strong-coupling limit of the ratio f_{χ_-}/f_{χ_+} can also be computed from its perturbative expansion. It has been derived in Appendix D, see Eq. (D6). The strong-coupling limit reads

$$\left(\frac{f_{\chi_-}}{f_{\chi_+}}\right)^* = 1 + \frac{1420\rho(\rho+1)(2\rho-1)}{3[19184 - 11664c_1 + 99\pi^2 + 9456\ln(4/3) + 1188\text{Li}_2(-1/3)]} - \frac{5726576000(2\rho-1)^2}{729[19184 - 11664c_1 + 99\pi^2 + 9456\ln(4/3) + 1188\text{Li}_2(-1/3)]^2}. \quad (147)$$

Its numerical value is 2.11227, to be compared with the two-loop (125) result $751/355 \approx 2.11549$. The three-loop level is in very good agreement with the two-loop result, within less than 0.2%. However, this by no means signify that the asymptotic limit has been reached, and the ratios (145) or (146) might be closer to the true ratio than (147). This is due to the fact the even and odd orders are on different converging lines because odd (even) terms come from an extremum (turning-point) condition or vice-versa. To see the speed of convergence, it would be necessary to compare the fourth-loop order with the two-loop order, and the fifth-loop order with the third-loop order.

E. Amplitude ratios from two- and three-loop expansions

We have now everything in hand in order to compute the ratio of the heat capacity A^+/A^- , the universal combination R_C and the ratio of the susceptibilities Γ_+/Γ_- . This section is restricted to a full two- and three-loop calculation. In order to improve the ratios, we shall break our rule of being selfconsistent in the next section and use there the maximum information available.

We start with the heat capacity A^+/A^- . Since we have a preference for (100) over (118), we shall work with the separate strong-coupling limit evaluation of $u^*F_-^*$ and $u^*F_+^*$. We have checked that the effect on the ratio A_+/A_- is negligible. The exponent α entering it is calculated from the two- or three-loop result for ν given in (52) and (81), respectively, using the hyperscaling relation $\alpha = 2 - D\nu$.

Combining the different results derived previously, we have

N	0	1	2	3	4
A_+/A_- (2 loops)	0	0.489106	0.843065	1.12691	1.37015
A_+/A_- (3 loops)	0	0.491088	0.862439	1.16719	1.43243

Regarding the fact that the critical exponent α is far away from its asymptotic limit (it is still positive for $N = 2$, while the shuttle experiment [28] shows clearly a negative value), the results of this table are promising: For $N = 2$, we obtain $A_+/A_- \approx 0.862439$ at the three-loop level, while the shuttle experiment [28] gives $A_+/A_- \approx 1.0442$, see Eq. (1). We shall see in the next section that working with asymptotic critical exponents leads to a better agreement with experiments.

The next ratio we examine is (104), the universal combination R_C . The results are best displayed in a table:

N	0	1	2	3	4
R_C (2 loops)	0	0.062474	0.124819	0.184355	0.239967
R_C (3 loops, $f_{\chi_+}^* = 1$)	0	0.05944	0.121628	0.182413	0.239691
R_C (3 loops, $f_{\chi_+}^*$ from (142))	0	0.060883	0.124736	0.187094	0.245718

We see an overall agreement between the two- and three-loop results. We have also checked that the ratio R_C calculated with the formula (103) used in [42] is within less than 1%. Moreover, our results are in agreement with the values $R_C(N = 2) = 0.123$ and $R_C(N = 3) = 0.189$ given in Ref. [42]. Since we expect that using the true critical exponent leads to a better ratio A_+/A_- , it is important to see how R_C evolves. Will the agreement with [42] be lost? This issue is investigated in the next section.

To end this section, we study the ratio of the susceptibilities Γ_+/Γ_- for the Ising model, see Eq. (105).

The two-loop result for the amplitude ratio (105) is, with $\nu = 56/90$ from (52), with $P_+^* = 127/171$ from (123) and with $f_{\chi_+}^* = 1$:

$$\frac{\Gamma_+}{\Gamma_-} = \frac{211}{103} \left(\frac{4\nu P_+^*}{3 - 4\nu P_+^*} \right)^{2\nu} = \frac{211}{103} \left(\frac{14224}{8861} \right)^{56/45} \approx 3.69171. \quad (148)$$

This is still far from the value ≈ 4.7 quoted in the literature [18,56]. A small improvement is obtained using the direct strong-coupling evaluation of f_{χ_-}/f_{χ_+} of Eq. (125):

$$\frac{\Gamma_+}{\Gamma_-} = \frac{751}{355} \left(\frac{4\nu P_+^*}{3 - 4\nu P_+^*} \right)^{2\nu} = \frac{751}{355} \left(\frac{14224}{8861} \right)^{56/45} \approx 3.81236. \quad (149)$$

This value is still far from the expected ratio 4.7. However, the ratio depends sensibly on the value of the critical exponent ν . For example, using $\nu = 0.63$, we increase (149) to $\Gamma_+/\Gamma_- = 4.002$. The sensibility is also seen when calculating the three-loop value of the ratio:

$$\frac{\Gamma_+}{\Gamma_-} \approx 3.88785, \quad (150)$$

where the ratio $(f_{\chi_-}/f_{\chi_+})^*$ has been obtained from (147).

F. Amplitude ratios using maximum information

Up to now, we have followed the strategy to make a fully consistent two- and three-loop calculation. The comparison between the two- and three-loop amplitude functions has made us believed that the resummed values are close to the extrapolated limit $L \rightarrow \infty$, although one has to take care that odd and even approximations are on different

converging lines. For the critical exponents, it is primordial going to the asymptotic limit. For example, we have $\alpha(N = 2)$ still positive at the three-loop level, while the shuttle experiment, see second reference of [28] and Eq. (1), shows a value of $\alpha(N = 2) = -0.01056$.

In this section, we shall relax our constrain of working only with two- and three-loop quantities and will take the maximum available information, i.e., our three-loop result for the amplitudes and extrapolated, or experimental, value for the critical exponents. We shall also see the effect of using uB to five loops.

Except for $\alpha(N = 2)$ that we took from the shuttle experiment [28], the exponents are taken from the $D = 3$ tables of [18], i.e., we are working with, for $N = 0, 1, 2, 3, 4$:

$$\begin{aligned} \nu &= 0.5882 & 0.6304 & 0.6703 & 0.7073 & 0.741 \\ \alpha &= 0.235 & 0.109 & -0.01056 & -0.122 & -0.223 \end{aligned} \quad . \quad (151)$$

Combining the three-loop strong-coupling limit of the amplitudes performed in section IV D with these exponents, we obtain, for A_+/A_- :

N	0	1	2	3	4
A_+/A_-	0	0.543406	1.04516	1.54386	2.0444
A_+/A_- (Ref. [41])	0	0.540	1.056	1.51	

We have checked that the increase from the three-loop value (for $N = 2$, this ratio was 0.862439) is mainly due to using the correct α . For example, with the correct α but still using the three-loop ν of (81), we would have obtained, for $N = 2$, a ratio 1.04711. It also does not depend too sensitively on using the five-loop strong-coupling limit of u^*B^* and P_+^* , neither on using $u^*(F_-^* - F_+^*)$ instead of the separate calculation of $u^*F_-^*$ and $u^*F_+^*$. For example, playing with all these quantities, the ratio, for $N = 2$, could be changed from $A_+/A_- = 1.04516$ to, at most, $A_+/A_- = 1.049$, depending which quantities are taken to five loops. A complete numerical study of this ratio, using variational perturbation theory up to five loops, will be presented elsewhere [58].

For $N = 2$, our result 1.04516 coincide remarkably well with the shuttle experiment, see second reference of [28]. For $N = 1$, we have 0.543406, which agrees reasonably well with Ref. [25] ($A_+/A_- \approx 0.541$) and with the values quoted in Table 5 of [18], values which are both experimental and theoretical. In Ref. [41], the authors obtained 1.056 for $N = 2$. Their Table 4 make a comparison between their result and other works and experiments, for $N = 1, 2, 3$. We see that the agreement is good. In our table, we have listed only the values calculated in the work [41] since the model is the same.

For the universal combination R_C , we obtain, using the five-loop critical exponents (151) and the three-loop amplitudes of section IV D

$$R_C = 0, 0.0616257, 0.130341, 0.201404, 0.270882 \quad (152)$$

for $N = 0, 1, 2, 3, 4$.

Here also, we have checked that the main effect is due to choosing the correct α . Working with ν at the three-loop level only modifies slightly the result. While working with the true exponents for the ratio A_+/A_- had considerably improved it, making it coincide with the experimental values, we see for R_C that the values of the previous section, with a wrong α were in better agreement with the quoted values in [42]: $R_C = 0.123, 0.189$ for $N = 2, 3$, respectively. We have checked that our result for $N = 2$ is not changed if we take the values of α and ν taken in [42]. Also, the result does not depend sensibly on u^*F_+ , although our value differs from their. We have traced back the difference between our result and [41] to uB at the critical point: limiting ourselves to $N = 2$, we have $u^*B^* \approx 0.0391089$ while [41] gives a value $u^*B^* \approx 0.0363919$. This difference is all is needed to explain the difference between our result and the result of Ref. [42], apart from a very small difference coming also from our use of Eq. (104) instead of (103). Since u^*B^* has been obtained in [41] using a five-loop Borel resummation, it is tempting to believe it is more accurate. For this reason, we have also determined numerically the five-loop strong-coupling limit of uB . We shall show a detailed numerical resummation in [58], showing here only the main steps. Starting from the five-loop expansion [41,43]

$$uB(u) = \frac{N}{2}u + \frac{N(N+2)}{48}u^3 + \frac{N(N+2)(N+8)[-25 + 12\zeta(3)]}{648}u^4 + N(N+2) \times \frac{[-319N^2 + 13968N + 64864 + 16(3N^2 - 382N - 1700)\zeta(3) + 96(4N^2 + 39N + 146)\zeta(4) - 1024(5N + 22)\zeta(5)]}{41472}u^5, \quad (153)$$

and using the algorithm given by Eq. (17), the corresponding strong-coupling limit is:

N	0	1	2	3	4
u^*B^* (5 loops)	0	0.0209552	0.0372717	0.0502225	0.0605918
u^*B^* (Ref. [41])	0	0.020297	0.0363919	0.049312	

Our five-loop result is now much nearer to the Borel resummed values of [41] than our three loop order of Section IV D. For this reason, we believe our five-loop result is near the infinite-loop limit extrapolation. More details will be given in [58], which also contains the effect of variations of P_+ , which is the second source, after u^*B^* , of error for R_C .

Finally, our best values for the ratio R_C are collected in the next table:

N	0	1	2	3	4
R_C	0	0.05803	0.12428	0.19402	0.26285
R_C (Ref. [41])	0		0.123	0.189	

To our knowledge no experimental value of this ratio is known for $N = 2$. The case $N = 3$ is presented in Table 7.6 of Ref. [56]. For $N = 1$, the value of the ratio has only slightly changed compared to the results based on three-loop α and ν . This is due to the fact that, for $N = 1$, α is positive and its effect on R_C is less sensitive. In the work [18], the theoretical and experimental values of R_C are also given for $N = 1$. The theoretical values seem to prefer a value around 0.057 while the experimental values are around 0.050. From Table 7.1 of [56], we however see that values close to 0.06 might as well be obtained. A result ($R_C \approx 0.0594$) close to this latter value was also obtained theoretically in [25].

Better experiments or other theoretical studies are needed in order to see if our predictions are correct or have to be ruled out.

Finally, we conclude this section with the ratio of the susceptibilities for the Ising model. Using the critical exponent ν to five loops (151), we obtain

$$\Gamma_+/\Gamma_- = 4.06419, \quad (154)$$

where we took the ratio $(f_{\chi_-}/f_{\chi_+})^*$ from (147). We might have slightly increased Γ_+/Γ_- using the value 2.1447 of Eq. (146). However, we would still be far from the value 4.77 of [18], value confirmed in the work [25]. The only possible quantity we may still vary in the ratio (105) is P_+^* . Our three-loop value is 0.745874, while the five-loop result given in [41] using Borel resummation is 0.7568. Using this value in our formula for the ratio, we find

$$\Gamma_+/\Gamma_- = 4.27154. \quad (155)$$

The ratio of the susceptibilities depends sensitively on P_+^* . We postpone to [58] the application of variational perturbation theory up to five loops for the resummation of P_+^* and Γ_+/Γ_- . We do not however expect a resummed P_+^* different from [41]. For this reason, the ratio (155) is probably the best we can obtain. A ratio of 4.77 obtained in [18] and references therein seems to be ruled out from our analysis.

V. CONCLUSION

In this paper, we have shown that variational strong-coupling theory [2,5] can be applied not only to critical exponents, but also to various amplitude ratios. We have focused on two- and three-loop results where analytical results for the amplitude functions are known [41,42,46] for all N both above and below T_c . Our results are analytical expressions, except in the last section where we used more information to find A_+/A_- , R_C and Γ_+/Γ_- . The results are quite sensitive to the precise value of the critical exponents. In addition, a five-loop evaluation of the renormalization constant B^* was necessary. The ratio R_C was so sensitive to it that a three-loop calculation was not sufficient. The same remark holds for P_+^* , which affects mainly Γ_+/Γ_- . A numerical study of the known five-loop amplitudes will be done in [58], which will contain refined results compared to Section IV F.

One interesting observation of our work is that we can evaluate series which have caused problems in previous Borel resummations when the expansion coefficients in terms of the renormalized coupling constant are not alternating to low orders. For these functions, strong-coupling theory turned out to work well.

Having obtained analytical expressions in N , we have shown that the coefficient of the series in the bare coupling constant may vanish and change sign. At the two-loop level, this led to diverging results near certain values of N . We have seen that the problem disappears at the three-loop level, because of the interplay of the different coefficients of the series. We could show precisely how it works because all our results were analytical and not restricted to integer values of N .

When using variational perturbation theory, nothing is known on the nature of the optimal variational parameter, which can be a minimum, a maximum, or a turning point. The analysis performed here should help to identify the correct (numerical) solution at higher-loop order. See for example the amplitude $f_{\chi_+}^*$, for which it is not yet clear which of the solutions (141) or (142) has to be chosen.

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APPENDIX A: EXPONENT ω FROM STRONG-COUPPLING THEORY

The power p/q of the leading power behavior $\bar{u}_B^{p/q}$ of a function W_L whose perturbative expansion has been given in (20) can be obtained taking the logarithmic derivative, giving (21). A subtlety arises for functions going to a constant in the strong-coupling limit. For such functions, p vanishes and the corresponding f^* in Eq. (17) vanishes. Care has to be taken: the limit $f^* \rightarrow 0$ is different from imposing $f^* = 0$. In the former case, we can identify q (or ω) by matching the series to achieve $f^* = 0$. Working directly with a series which has $f^* = 0$ implies a leading behavior $p'/q = -\omega/\epsilon$. The algorithm (17) serves then to identify the coefficient c_0 of the right-hand-side of Eq. (18). As an example how to use the series, let us derive the relation [2,5,13]

$$-\frac{\omega}{\epsilon} - 1 = \frac{d \log W'_L}{d \log \bar{u}_B}. \quad (\text{A1})$$

The left-hand-side is of the type (18), and the algorithm (17) can be applied. Formula (A1) follows directly from (18). Alternative derivation starts from (21): If p/q is vanishing, this means that its series has a leading exponent $p'/q = -\omega/\epsilon$, which we derive in the following manner. Start from formula (21) with its exponent p'/q which we know from the general behavior (18) with $f^* \equiv p/q = 0$, i.e.,

$$\frac{p'}{q} = -\frac{\omega}{\epsilon} = \frac{d \log(p/q)}{d \log \bar{u}_B} \quad (\text{A2})$$

where p/q is not yet taken at its asymptotic zero value, but is given as the right-hand-side of (21). It then follows

$$-\frac{\omega}{\epsilon} = 1 + \bar{u}_B \left(\frac{W''_L}{W'_L} - \frac{W'_L}{W_L} \right) = 1 + \bar{u}_B \frac{W''_L}{W'_L} - \frac{p}{q}. \quad (\text{A3})$$

Taking the limit $\bar{u}_B \rightarrow \infty$, the term p/q vanishes by hypothesis, and we end up once more with formula (A1).

Although the algorithm (17) cannot be applied directly for the right-hand-side of (21) if p/q is vanishing exactly but only in the limit $p/q \rightarrow 0$, we can nevertheless use a trick to circumvent this problem: If the series for W_L has a vanishing leading power p/q , then W_L/\bar{u}_B has a power $p'/q = -1$. This allows to deduce

$$\frac{p'}{q} \equiv -1 = \frac{d \log(W_L/\bar{u}_B)}{d \log \bar{u}_B} = \bar{u}_B \frac{W'_L}{W_L} - 1 = \frac{p}{q} - 1. \quad (\text{A4})$$

This shows that the right-hand-side of (21) can be used to reach the limit 0. Then, ω can be extracted either from (21) or from (A1). It is also clear from the expression (A3) that the right-hand-side has to be resummed blockwise: we have to use the intermediate result $p/q = 0$ before tempting to resum. Using a full resummation of the right-hand-side of the latter equation would lead to badly resummed results (although the underlying ϵ expansion would be the same): It is necessary to use $p/q = 0$ in Eq. (A3), and not its analytical form which would have been mixed up with the power series of W''_L/W'_L .

APPENDIX B: FREE ENERGY TO THREE LOOPS

From the model Hamiltonian (4), the analytical calculation of the Gibbs free energy $\Gamma_B(m_B'^2, u_B, M_B)$ near the coexistence curve below T_c and for $M_B^2 \equiv \langle \phi_B^2 \rangle = 0$ above T_c has been obtained at the two-loop order in [46] and at the three-loop order in [41], thus extending the $N = 1$ calculation of Rajantie [35]. We write directly the three-loop result:

$$\Gamma_B = \frac{1}{2} m_B'^2 M_B^2 + u_B M_B^2 + \sum_{b=1}^3 \sum_{l=0}^{b-1} \sum_{k=0}^1 (-1)^k 2^{-l-k} F_{blk}(\bar{\omega}, N) (24u_B)^{3-l} (M_B^2)^l \left[\frac{r_{0L}}{(24u_B)^2} \right]^{\frac{4-b-2l}{2}} \ln \left[\frac{r_{0L}}{(24u_B)^2} \right]^k, \quad (\text{B1})$$

where $r_{0L} = m_B'^2 + 12u_B M_B^2$ is the longitudinal bare mass, the transverse one $r_{0T} = m_B'^2 + 4u_B M_B^2$ being included in the parameter $\bar{\omega} = r_{0T}/r_{0L}$. The nonanalyticity in the coupling constant is seen in the last term.

The functions F_{blk} can be found in [41,46]. Since we need them later on in this Appendix, we shall write the nonzero components:

$$F_{100} = -\frac{1}{12\pi} \left[1 + (N-1)\bar{\omega}^{3/2} \right], \quad (\text{B2})$$

$$F_{200} = \frac{1}{384\pi^2} \left[3 + 2(N-1)\bar{\omega}^{1/2} + (N^2-1)\bar{\omega} \right], \quad (\text{B3})$$

$$F_{210} = \frac{1}{288\pi^2} (N-1) \ln \frac{1+2\bar{\omega}^{1/2}}{3}, \quad (\text{B4})$$

$$F_{211} = -\frac{1}{288\pi^2} (N+2). \quad (\text{B5})$$

$$F_{300} = \frac{1}{18432\pi^3} \left(15 + 24 \ln \frac{3}{4} - (N-1) \left\{ \bar{\omega}^{-1/2} + 2N - 6 + 8 \ln \frac{2+2\bar{\omega}^{1/2}}{3} + \bar{\omega}^{1/2} \left[N^2 - 6N - 9 + 4(N+1) \ln \frac{16\bar{\omega}}{9} + 8 \ln \frac{2+2\bar{\omega}^{1/2}}{3} \right] + \bar{\omega}(N-1) \right\} \right), \quad (\text{B6})$$

$$F_{301} = \frac{1}{2304\pi^3} \left\{ 3 + (N-1) \left[1 + (N+2)\bar{\omega}^{1/2} \right] \right\}, \quad (\text{B7})$$

$$F_{310} = \frac{1}{27648\pi^3} \left(9\pi^2 - 18 + 108 \text{Li}_2 \left(-\frac{1}{3} \right) - (N-1) \left\{ 4\bar{\omega}^{-1/2} + 4N + 2 - (N+2)\pi^2 - 12 \text{Li}_2 \left(\frac{1}{3} \right) - 32 \ln 2 - 6(\ln 3)^2 + \bar{\omega}^{1/2} [10N + 32 - 16(2N+3) \ln 2 + 48 \ln 3 - 8(N+1) \ln \bar{\omega}] + \frac{1}{3} \bar{\omega} (84N - 100 - 128 \ln 2) \right\} \right), \quad (\text{B8})$$

$$F_{320} = \frac{1}{165888\pi^3} \left[432 \ln \frac{4}{3} - 324 \text{Li}_2 \left(-\frac{1}{3} \right) - 432c_1 - 27\pi^2 - (N-1) \left(16\bar{\omega}^{-1/2} + \frac{3N+14}{3} \pi^2 + 18(\ln 3)^2 + 36 \text{Li}_2 \left(\frac{1}{3} \right) + 16 \left[c_2 + 4 \text{Li}_2(-2) - 2 \text{Li}_2 \left(-\frac{1}{2} \right) + \left(6 \ln 3 - \ln 2 - \frac{13}{3} \right) \ln 2 \right] - \frac{128}{3} + 16\bar{\omega}^{1/2} [7 - N + (N+1) \ln(16\bar{\omega}) + 2 \ln 2 - 6 \ln 3] \right) \right]$$

$$+ 4\bar{w} \left\{ 4c_2 - 12N - \frac{224}{5} + \pi^2 + 6 \left(6 \ln 3 - \ln 2 - \frac{16}{15} \right) \ln 2 + 12 \left[2 \text{Li}_2(-2) - \text{Li}_2 \left(-\frac{1}{2} \right) \right] \right\}. \quad (\text{B9})$$

where, in the coefficients F_{310} and F_{320} below T_c , terms of order $O(\bar{w}^{3/2}, \bar{w}^{3/2} \ln \bar{w})$ have been neglected (the coefficients are calculated in the vicinity of the coexistence curve where an expansion with respect to \bar{w} is justified). The constants c_1 and c_2 have been defined in the main text, see Eqs. (96) and (97).

Inverting the equation of state

$$h_B = \frac{\partial}{\partial M_B} \Gamma_B \quad (\text{B10})$$

gives, in the limit $h_B \rightarrow 0$, and to this order, the square of the magnetization:

$$\begin{aligned} M_B^2 = & \frac{1}{8u_B} (-2m_B'^2) + \frac{3}{4\pi} (-2m_B'^2)^{1/2} + \frac{u_B}{8\pi^2} \left[10 - N + 4(N-1) \ln 3 - 2(N+2) \ln \frac{-2m_B'^2}{(24u_B)^2} \right] \\ & + \frac{u_B^2 (-2m_B'^2)^{-1/2}}{1920\pi^3} \left\{ -2736N - 5904 - 6480c_1 + 240(N-1)c_2 - (75N^2 - 5N + 875)\pi^2 \right. \\ & - 1260 \left[(N-1) \text{Li}_2 \left(\frac{1}{3} \right) + 9 \text{Li}_2 \left(-\frac{1}{3} \right) \right] + 960(N-1) \left[2 \text{Li}_2(-2) - \text{Li}_2 \left(-\frac{1}{2} \right) \right] \\ & - 630(N-1)(\ln 3)^2 - 48 \ln 2 \left[10(N-1) \ln 2 - 60(N-1) \ln 3 + 111N - 561 \right] \\ & \left. + 240(12N - 57) \ln 3 - 1440(N+2) \ln \frac{-2m_B'^2}{(24u_B)^2} \right\}. \quad (\text{B11}) \end{aligned}$$

The logarithmic terms in u_B are nonanalyticities which can be removed using the length ξ_- instead of $m_B'^2 < 0$, see [38]. Up to the three-loop order, the relation between ξ_- and $m_B'^2 < 0$ is

$$\begin{aligned} -2m_B'^2 = & \xi_-^{-2} \left\{ 1 + \frac{N+2}{\pi} u_B \xi_- - \frac{N+2}{\pi^2} (u_B \xi_-)^2 \left[\frac{1385}{108} + 4 \ln(24u_B \xi_-) \right] \right. \\ & \left. + \frac{N+2}{108\pi^3} (u_B \xi_-)^3 \left[3(438N + 4349) + 576(N+8) \text{Li}_2 \left(-\frac{1}{3} \right) + 48(N+8)\pi^2 + 8(43N + 182) \ln \frac{3}{4} \right] \right\}. \quad (\text{B12}) \end{aligned}$$

Using (B12) in (B11) one obtains an analytic function of u_B , from which one extract the amplitude f_ϕ of Eqs. (85) and (93), after proper normalization with the help of Z_ϕ . This has been done in [42,46] and will not be repeated here. The equations we have quoted here are mentioned because we shall need them below for obtaining the amplitude functions f_{χ_+} and f_{χ_-} which, to our knowledge, have not been determined analytically within this model.

APPENDIX C: THREE-LOOP AMPLITUDE FUNCTION OF THE ISOTROPIC SUSCEPTIBILITY

ABOVE T_c

By definition, the amplitude of the susceptibility above T_c is obtained from the susceptibility at zero momentum $f_{\chi_+}^B = \xi_+^2 \chi_{+,B}^{-1}$, where the inverse susceptibility is given by the two-point function $\Gamma_B^{(2)}$ at zero momentum. The

correlation length above the critical temperature ξ_+ is defined as in Refs. [41,42,46]:

$$\xi_+^2 = \chi_{+,B}(q) \partial \chi_{+,B}^{-1}(q) / \partial q^2 \Big|_{q^2=0}. \quad (\text{C1})$$

Combining with the definition of $f_{\chi_+}^B$, we have

$$f_{\chi_+}^B = \frac{\partial \chi_{+,B}^{-1}}{\partial q^2} \Big|_{q^2=0} = \frac{\partial \Gamma_B^{(2)}}{\partial q^2} \Big|_{q^2=0}. \quad (\text{C2})$$

The derivative of $\Gamma_B^{(2)}$ with respect to q^2 is needed. This is in contrast to Refs. [41,42,46] where only the combination (C1) was needed. For this reason, the intermediate result leading to Eq. (C2) was not published. Being needed to determine the ratio R_C in (104) and the ratio of the susceptibilities (105), we derive it in the following. The two-point function can be written as $\Gamma_B^{(2)} = r_0 + q^2 - \Sigma_B(q, r_0, \bar{u}_B)$, where the self-energy has the expansion $\Sigma_B(q, r_0, \bar{u}_B) = \sum_{m=1}^{\infty} (-\bar{u}_B)^m \Sigma_B^{(m)}(q, r_0)$. The two-loop results have first been given in Appendix A of Ref. [46], with the result, up to order q^2 :

$$\Gamma^{(2)} = q^2 + r_0 - 4(N+2)A_D u_B r_0^{1/2} + 8(N+2)^2 A_D^2 u_B^2 - 32 \frac{(N+2)}{(4\pi^3)} \left[\frac{2\pi}{D-3} - \frac{2\pi q^2}{27 r_0} \right] u_B^2. \quad (\text{C3})$$

The pole at $D = 3$ can be eliminated by subtraction, leading to the masses m_B^2 and m_B^2 . This is however of no concern here since we are interested in taking the derivative with respect to q^2 :

$$\frac{\partial \Gamma_B^{(2)}}{\partial q^2} \Big|_{q^2=0} = 1 + \frac{N+2}{27\pi^2} \frac{u_B^2}{r_0}. \quad (\text{C4})$$

For the three-loop expansion, one must calculate the diagrams in Appendix B of [42]. Again, we concentrate on the derivative of the susceptibility at zero momentum, focusing on the diagrammatic Eq. (B5) of [42]. The corresponding vacuum diagrams have been given by Rajantie in [35], see in particular its Eqs. (15) and (25) and, taking the appropriate derivative with respect to the mass, we obtain the contribution of the three-loop diagrams:

$$\frac{\partial \Sigma_B^{(3)}}{\partial q^2} \Big|_{q^2=0} = \left\{ \frac{(N+2)^2}{27\pi^3} - \frac{(N+2)(N+8)}{54\pi^3} \left[-8 + 3\pi^2 + 32 \ln \frac{3}{4} + 36 \text{Li}_2 \left(-\frac{1}{3} \right) \right] \right\} r_0^{-3/2}. \quad (\text{C5})$$

This has to be combined with (C4) to yield the expansion

$$\frac{\partial \Gamma_B^{(2)}}{\partial q^2} \Big|_{q^2=0} = 1 + \frac{N+2}{27\pi^2} \frac{u_B^2}{r_0} + \left\{ \frac{(N+2)^2}{27\pi^3} - \frac{(N+2)(N+8)}{54\pi^3} \left[-8 + 3\pi^2 + 32 \ln \frac{3}{4} + 36 \text{Li}_2 \left(-\frac{1}{3} \right) \right] \right\} \frac{u_B^3}{r_0^{3/2}}. \quad (\text{C6})$$

Since there is no linear term u_B , the amplitude of the susceptibility above T_c to three loops requires only the one-loop order of the correlation-length ξ_+ , i.e., the one-loop order of the susceptibility. To three loops, the following expression

$$m_B'^2 = \xi_+^{-2} \left\{ 1 + \frac{N+2}{\pi} u_B \xi_+ + \frac{N+2}{\pi^2} (u_B \xi_+)^2 \left[\frac{1}{27} + 2 \ln(24 u_B \xi_+) \right] + \frac{N+2}{\pi^3} (u_B \xi_+)^3 \left[3(3N+22) - 144(N+8) \text{Li}_2 \left(-\frac{1}{3} \right) - 12(N+8)\pi^2 - 2(43N+182) \ln \frac{3}{4} \right] \right\}. \quad (\text{C7})$$

is found in the literature, see [42]. This is the analogue of (B12) above T_c . At the one-loop level, there is no distinction between r_0 and m_B^2 , and we identify $r_0 = \xi_+^{-2} [1 + (N+2)u_B \xi_+ / \pi]$. Together with Eq. (C6), we arrive at

$$f_{\chi_+}^B \equiv \left. \frac{\partial \Gamma_B^{(2)}}{\partial q^2} \right|_{q^2=0} = 1 + \frac{N+2}{27\pi^2} (u_B \xi_+)^2 - \frac{(N+2)(N+8)}{54\pi^3} \left[-8 + 3\pi^2 + 32 \ln \frac{3}{4} + 36 \text{Li}_2 \left(-\frac{1}{3} \right) \right] (u_B \xi_+)^3. \quad (\text{C8})$$

This has to be compared with the numerical coefficients $a_m^{(2)}$ of Table 2 in [55].

Having calculated the bare amplitude function $f_{\chi_+}^B$, we can now turn to the normalized one f_{χ_+} . Since the latter is related to a two-point function, the normalization factor is equal to the wave function renormalization constant Z_ϕ : $f_{\chi_+} = Z_\phi f_{\chi_+}^B$, with Z_ϕ being supplied by (16). Using the relation (13) between u_B and $\bar{u}_B = u_B \xi_+ / (4\pi)$, we arrive at the normalized amplitude function of the susceptibility above T_c , expressed in terms of the reduced bare coupling constant \bar{u}_B :

$$f_{\chi_+} = 1 - \frac{92}{27} (N+2) \bar{u}_B^2 - \frac{8}{27} (N+2)(N+8) \left[-113 + 12\pi^2 + 128 \ln \frac{3}{4} + 144 \text{Li}_2 \left(-\frac{1}{3} \right) \right] \bar{u}_B^3. \quad (\text{C9})$$

The corresponding expansion in terms of the renormalized coupling constant gives

$$f_{\chi_+} = 1 - \frac{92}{27} (N+2) u^2 - \frac{8}{27} (N+2)(N+8) \left[-21 + 12\pi^2 + 128 \ln \frac{3}{4} + 144 \text{Li}_2 \left(-\frac{1}{3} \right) \right] u^3, \quad (\text{C10})$$

where we used (14). Contrary to (C9) which is well-behaved regarding strong-coupling theory, Eq. (C10), which coincide with the numerical coefficients $c_m^{(2)}$ of Table 4 of [55], is problematic when considering the Borel resummation scheme: All its coefficients are negative. For this reason, we have not been able to reproduce the Borel resummation made in Ref. [55]. We shall however make, in the main text, a comparison between the strong-coupling limit of (C9) and the resummation performed in [55].

APPENDIX D: THREE-LOOP AMPLITUDE FUNCTION OF THE $N = 1$ -SUSCEPTIBILITY BELOW T_c

In Ref. [40], the amplitude function of the susceptibility below T_c for $N = 1$ has been calculated numerically to five loops. We have quoted in (91) the corresponding two-loop part. This amplitude function enters the ratio of the susceptibilities (105). Since f_{χ_+} has been obtained to three loops in the previous section, it is also interesting to obtain f_{χ_-} analytically: The ratio (105) will thus be analytical.

In Ref. [42], the free energy Γ_B has been given analytically up to three loops. We shall use this knowledge to determine f_{χ_-} . We have recalled the relevant equations in the first part of this Appendix, which have to be evaluated for $N = 1$ and $\bar{\omega} = 0$. The derivative of the free energy with respect to the magnetization leads to the equation of state (B10) which can be inverted to obtain the magnetization [42]. We have recalled its expression in (B11). The equation of state can itself be derived with respect to the magnetization, defining the inverse susceptibility below T_c : $\chi_{-,B}^{-1} = \partial h_B / \partial M_B$. Only at this stage is the external field h_B taken to be vanishing. The length ξ_- [42], which we recalled in (B12), is then used to remove the nonanalyticity coming from logarithms of the coupling constant. Doing so, and using the magnetization given by the equation of state, we have been able to obtain the inverse bare susceptibility below T_c : $\chi_{-,B}^{-1} = \xi_-^{-2} f_{\chi_-}^B$, with

$$f_{\chi_-}^B = 1 + \frac{9}{2\pi}(u_B \xi_-) - \frac{1061}{36\pi^2}(u_B \xi_-)^2 + \left[19472 - 11664c_1 + 3\pi^2 + 10480 \ln \frac{4}{3} + 36 \text{Li}_2 \left(-\frac{1}{3} \right) \right] \frac{(u_B \xi_-)^3}{64\pi^3}. \quad (\text{D1})$$

Numerically, this expansion reads

$$1 + 1.43239(u_B \xi_-) - 2.98616(u_B \xi_-)^2 + 11.2134(u_B \xi_-)^3. \quad (\text{D2})$$

This result agrees perfectly with the numerical expansion given in the last column of Table 2 in [40]. Using the relation (14) between u_B and \bar{u}_B , we obtain

$$f_{\chi_-}^B = 1 + 18\bar{u}_B - \frac{4244}{9}\bar{u}_B^2 + \left[19472 - 11664c_1 + 3\pi^2 + 10480 \ln \frac{4}{3} + 36 \text{Li}_2 \left(-\frac{1}{3} \right) \right] \bar{u}_B^3. \quad (\text{D3})$$

The renormalized version of (D3) is found by multiplying it with Z_ϕ from (16):

$$f_{\chi_-} = 1 + 18\bar{u}_B - \frac{4352}{9}\bar{u}_B^2 + \left[19904 - 11664c_1 + 3\pi^2 + 10480 \ln \frac{4}{3} + 36 \text{Li}_2 \left(-\frac{1}{3} \right) \right] \bar{u}_B^3. \quad (\text{D4})$$

This is the amplitude to be evaluated in the strong-coupling limit and entering the ratio (105).

The bare amplitudes (C8) and (D3) might as well be chosen to enter the amplitude ratio (105) since the renormalization constant Z_ϕ drops out, being the same above and below T_c . We have however chosen to work with the renormalized quantities (C9) and (D4).

For completeness, we also state the expansion of f_{χ_-} in terms of the renormalized coupling constant. Using (14) and (D4), we obtain

$$f_{\chi_-} = 1 + 18u + \frac{1480}{9}u^2 + \left[1072 - 11664c_1 + 3\pi^2 + 10480 \ln \frac{4}{3} + 36 \text{Li}_2 \left(-\frac{1}{3} \right) \right] u^3. \quad (\text{D5})$$

Taking the inverse of this equation, we recover the coefficients of the second column of Table 3 of Ref. [40].

For an application to the evaluation of the amplitude ratio of the susceptibilities, we also give the perturbative expansion of the ratio f_{χ_-}/f_{χ_+} at $N = 1$. Combining (C9) with (D4), we obtain

$$\frac{f_{\chi_-}}{f_{\chi_+}} = 1 + 18\bar{u}_B - \frac{1420}{3}\bar{u}_B^2 + \left[19184 - 11664c_1 + 99\pi^2 + 9456 \ln \frac{4}{3} + 1188 \text{Li}_2 \left(-\frac{1}{3} \right) \right] \bar{u}_B^3. \quad (\text{D6})$$

APPENDIX E: AMPLITUDE RATIOS

To get the different amplitude ratios of Section IV B, we make use of the relations

$$\chi_+ = Z_\phi \frac{\xi_+^2}{f_{\chi_+}} \exp \left[- \int_{u(l_+)}^u \frac{\gamma_\phi}{\beta_u} du' \right], \quad (\text{E1})$$

$$\chi_- = Z_\phi \frac{\xi_-^2}{f_{\chi_-}} \exp \left[- \int_{u(l_-)}^u \frac{\gamma_\phi}{\beta_u} du' \right], \quad (\text{E2})$$

$$A^\pm = \frac{(b^\pm)^2}{(\xi_\pm^0)^D} \frac{A_D}{4} (4\nu B^* + \alpha F_\pm^*), \quad (\text{E3})$$

$$\langle \phi_B \rangle^2 = Z_\phi \frac{f_\phi}{\xi_-^{D-2}} \exp \left[- \int_{u(l_-)}^u \frac{\gamma_\phi}{\beta_u} du' \right], \quad (\text{E4})$$

which were derived in [37] for (E1), [40] for (E2) and (E4), and [38] for (E3). All the quantities have been defined in the main text, except for

$$l_\pm = \exp \left(\int_u^{u(l_\pm)} \frac{du'}{\beta_u} \right), \quad (\text{E5})$$

with $u(1) = l_\pm$ and the flow parameter chosen as $l_\pm \mu \xi_\pm = 1$, and with $\xi_\pm = \xi_\pm^0 |t|^{-\nu}$.

The amplitude ratio of the heat capacity (100) follows trivially from (E3), while the amplitude ratio for the susceptibilities (105) is a direct consequence of (E1) and (E2). The only missing information is the ratio ξ_+^0/ξ_-^0 , given explicitly in [38] as

$$\frac{\xi_+^0}{\xi_-^0} = \left(\frac{b^+}{b^-} \right)^\nu. \quad (\text{E6})$$

Because our derivation (104) of the universal combination R_C does not coincide with (103) derived by the authors of [42], we reproduce below our calculation. We need the amplitude A_M , related to (E4) by [56] $\langle \phi_B \rangle \equiv M_B \approx A_M |t|^\beta$.

We deduce

$$A_M^2 = Z_\phi \frac{f_\phi}{(\xi_-^0)^{(D-2)}} |t|^{\nu(D-2)-2\beta} \exp \left[- \int_{u(l_-)}^u \frac{\gamma_\phi}{\beta_u} \right] \Big|_{l_- \rightarrow 0}, \quad (\text{E7})$$

where we have specified that the right-hand-side is evaluated at the critical point.

In the same way, the amplitude of the susceptibility is obtained from (E1) using [56] $\chi_+ \approx \Gamma^+ |t|^{-\gamma}$. We deduce

$$\Gamma^+ = Z_\phi \frac{(\xi_+^0)^2}{f_{\chi_+}} |t|^{\gamma-2\nu} \exp \left[- \int_{u(l_+)}^u \frac{\gamma_\phi}{\beta_u} \right] \Big|_{l_+ \rightarrow 0}. \quad (\text{E8})$$

Taking the ratio Γ^+/A_M^2 , we have directly

$$\frac{\Gamma^+}{A_M^2} = \frac{(\xi_+^0)^2}{f_{\chi_+}^* f_\phi^*} (\xi_-^0)^{(D-2)}. \quad (\text{E9})$$

The dependence in $|t|$ has disappeared, as it should, due to the identity $\gamma - 2\nu + 2\beta - \nu(D-2) = 0$.

Combining with the definition of A^+ in (E3), we get

$$R_C \equiv \frac{\Gamma^+ A^+}{A_M^2} = \frac{(b^+)^2}{f_{\chi_+}^* f_\phi^*} \left(\frac{\xi_-^0}{\xi_+^0} \right)^{(D-2)} \frac{A_D}{4} (4\nu B^* + \alpha F_\pm^*) = \frac{(b^+)^{2-\nu(D-2)}}{(b^-)^{-\nu(D-2)}} \frac{A_D}{4} (4\nu B^* + \alpha F_\pm^*) \frac{1}{f_{\chi_+}^* f_\phi^*}. \quad (\text{E10})$$

where we used (E6) to obtain the last equality. Using $b^+ = 2\nu P_+$ and $b^- = 3/2 - 2\nu P_+$ [38], as well as $A_3 = 1/(4\pi)$ from (8), we arrive to the amplitude ratio R_C given in (104).

APPENDIX F: DETERMINATION OF THE POLYNOMIAL P_+ TO THREE LOOPS

In this section, we want to derive the analytical expression for the polynomial P_+ up to three loops. It has been given numerically, and resummed, for $N = 1, 2, 3$ up to five loops in [55], so that our analytical result will have to match this reference. Above T_c , the relation between $m_B'^2$ and the correlation length has been given in (C7) at the three-loop level. A polynomial P_+^B in powers of u_B is defined through the relation

$$P_+^B = \partial m_B'^2 / \partial \xi_+^{-2}, \quad (\text{F1})$$

leading to

$$P_+^B = 1 + \frac{N+2}{2\pi} (u_B \xi_+) - \frac{N+2}{\pi^2} (u_B \xi_+)^2 + \frac{N+2}{108\pi^3} \left[-3(3N+22) + 12(N+8)\pi^2 + 2(43N+182) \ln \frac{3}{4} + 144(N+8) \text{Li}_2 \left(-\frac{1}{3} \right) \right] (u_B \xi_+)^3. \quad (\text{F2})$$

The numeric coefficient b_m of Table 2 of [55] coincide perfectly, up to three loops, with our analytical expression, which has the advantage of being valid for all N . Its renormalized counter part is defined by

$$P_+ = Z_r^{-1} P_+^B, \quad (\text{F3})$$

where the renormalization constant Z_r^{-1} has been given to three loops in Eq. (15). The corresponding power series in \bar{u}_B follows readily:

$$P_+ = 1 - 2(N+2)\bar{u}_B + 4(N+2)(2N+17)\bar{u}_B^2 + \frac{8}{27}(N+2) \left[-3(36N^2 + 837N + 3920) + 24(N+8)\pi^2 + 4(43N+182)\ln\frac{3}{4} + 288(N+8)\text{Li}_2\left(-\frac{1}{3}\right) \right] \bar{u}_B^3, \quad (\text{F4})$$

where we have used the relation between u_B and \bar{u}_B given in Eq. (13), the scale μ being identified with the inverse of the correlation length: $\bar{u}_B = u_B \xi_+ A_3$. Eq. (F4) is the polynomial whose strong-coupling expansion has to be calculated. The corresponding power series in the renormalized coupling constant u follows from Eq. (14):

$$P_+ = 1 - 2(N+2)u + 4(N+2)u^2 + \frac{8}{27}(N+2) \left[-3(63N+572) + 24(N+8)\pi^2 + 4(43N+182)\ln\frac{3}{4} + 288(N+8)\text{Li}_2\left(-\frac{1}{3}\right) \right] u^3. \quad (\text{F5})$$

The reader can verify that the analytical result coincide, for $N = 1, 2, 3$, with the numerical values in Table 4 of Ref. [55]. It differs only in the fifth decimal place of the cubic term c_{P_3} of this table.

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