

# Five-Loop Critical Temperature Shift in Weakly Interacting Homogeneous Bose-Einstein Condensate

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Using variational perturbation theory, we calculate the shift in the critical temperature up to five loops to lowest order in the scattering length  $\Delta T_c/T_c \approx 0.93 \pm 0.13 an^{1/3}$ .

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The effect of a small repulsive interaction upon the critical temperature  $T_c$  of a Bose-Einstein condensate (BEC) has been a matter of controversy for many years, and the various results are converging only very slowly towards a common answer. Here we want to contribute the hopefully most accurate result so far.

The interacting Bose-Einstein condensate is described by the euclidean action

$$\mathcal{A}_E = \int_0^\beta d\tau \int d^3x \left\{ \psi^*(\mathbf{x}, \tau) \left( \partial_\tau - \frac{1}{2M} \nabla^2 \right) \psi(\mathbf{x}, \tau) - \mu \psi^*(\mathbf{x}, \tau) \psi(\mathbf{x}, \tau) + \frac{2\pi a}{M} [\psi(\mathbf{x}, \tau) \psi^*(\mathbf{x}, \tau)]^2 \right\}, \quad (1)$$

where  $M$  is the mass of the bosons,  $\beta$  the inverse temperature in natural units with  $\hbar = k_B = 1$ ,  $a$  is the  $s$ -wave scattering length, and  $\mu$  the chemical potential. The free system has a transition temperature

$$T_c^{(0)} = \frac{2\pi}{M} \left[ \frac{n}{\zeta(3/2)} \right]^{\frac{2}{3}}, \quad (2)$$

where  $n$  is the particle density. A small relative shift of  $T_c$  with respect to  $T_c^{(0)}$  can be calculated from the general formula

$$\frac{\Delta T_c}{T_c^{(0)}} = -\frac{2}{3} \frac{\Delta n}{n^{(0)}}, \quad (3)$$

where  $n^{(0)}$  is the particle density in the free condensate and  $\Delta n$  its change at  $T_c$  caused by the small interaction. For small  $a$ , this behaves like [1, 2]

$$\frac{\Delta T_c}{T_c^{(0)}} = c_1 an^{1/3} + [c'_2 \ln(an^{1/3}) + c_2] a^2 n^{2/3} + \mathcal{O}(a^3 n). \quad (4)$$

where  $c'_2 = -64\pi\zeta(1/2)/3\zeta(3/2)^{5/3} \simeq 19.7518$  can be calculated perturbatively, whereas  $c_1$  and  $c_2$  require nonperturbative techniques since infrared divergences at  $T_c$  make them basically strong-coupling results. The standard technique to reach this regime is based on a resummation of perturbation expansions using the renormalization group [3, 4], first applied in this context by Ref. [6]. Recently, however, it has been shown by calculating the best known critical exponent  $\alpha$  of superfluid helium from Satellite experiments [7] that the accuracy of strong-coupling results can be surpassed by much simpler variational perturbation theory [4, 8, 9].

Up to now,  $c_2$  has been inferred only from Monte Carlo data to be  $c_2 \approx 75.7 \pm 0.4$ . In order to find the leading coefficient  $c_1$ , one may take advantage of an important simplification due to the fact that  $\Delta n$  can be calculated from the classical limit of the field theory, which is governed by the three-dimensional action

$$\mathcal{A}_{3d} = \beta \int d^3x \left\{ \psi_0^*(\mathbf{x}) \left( -\frac{1}{2M} \nabla^2 - \mu \right) \psi_0(\mathbf{x}) + \frac{2\pi a}{M} [\psi_0^*(\mathbf{x}) \psi_0(\mathbf{x})]^2 \right\}. \quad (5)$$

This is a special case  $N = 2$  of the more general  $O(N)$ -invariant  $\phi^4$  field theory

$$\mathcal{A}_\phi = \int d^3x \left[ \frac{1}{2} |\nabla\phi|^2 + \frac{1}{2} m^2 \phi^2 + \frac{u}{4!} (\phi^2)^2 \right], \quad (6)$$

where the  $N$ -component field  $\phi = (\phi_1, \phi_2, \dots, \phi_N)$  is related to the original field  $\psi$  for  $N = 2$  by  $\psi(\mathbf{x}) = \sqrt{MT}[\phi_1(\mathbf{x}) + i\phi_2(\mathbf{x})]$ . The square mass is  $m^2 = -2M\mu$ , and the quartic coupling is  $u = 48\pi aMT$ . Using this relation, the shift of the critical temperature (3) can be found from the formula

$$\frac{\Delta T_c}{T_c^{(0)}} \approx -\frac{2}{3} \frac{MT_c^{(0)}}{n} \langle \Delta \phi^2 \rangle = -\frac{4\pi}{3} \frac{(MT_c^{(0)})^2}{n} 4! \left\langle \frac{\Delta \phi^2}{u} \right\rangle a = -\frac{4\pi}{3} (2\pi)^2 \frac{1}{[\zeta(3/2)]^{4/3}} 4! \left\langle \frac{\Delta \phi^2}{u} \right\rangle a n^{1/3}, \quad (7)$$

corresponding in Eq. (4) to

$$c_1 \approx -1103.09 \left\langle \frac{\Delta \phi^2}{u} \right\rangle. \quad (8)$$

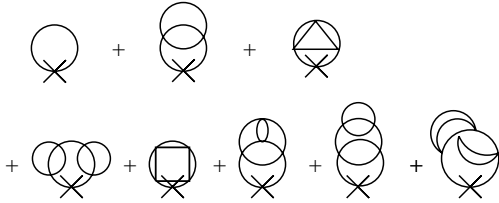


FIG. 1: Diagrams contributing to the expectation value  $\langle \phi^2 \rangle$ .

The three-dimensional theory is superrenormalizable and requires only mass counterterms which shift the original bare mass  $m$  to the renormalized mass  $m_r$ . A calculation of the Feynman diagrams in Fig. 1 yields the following five-loop perturbation expansion for the expectation value  $\langle \phi^2/u \rangle$  [10, 11]

$$\begin{aligned} \left\langle \frac{\phi^2}{u} \right\rangle = F(u) \equiv & -\frac{N}{4\pi} \frac{m_r}{u} - a_2 \frac{N(2+N)}{18(4\pi)^3} \frac{u}{m_r} + a_3 \frac{N(16+10N+N^2)}{108(4\pi)^5} \left( \frac{u}{m_r} \right)^2 \\ & - \left[ a_{41} \frac{N(2+N)^2}{324(4\pi)^7} + a_{42} \frac{N(40+32N+8N^2+N^3)}{648(4\pi)^7} + a_{43} \frac{N(44+32N+5N^2)}{324(4\pi)^7} \right. \\ & \left. + a_{44} \frac{N(2+N)^2}{324(4\pi)^7} + a_{45} \frac{N(44+32N+5N^2)u^4}{324m_r^3(4\pi)^7} \right] \left( \frac{u}{m_r} \right)^3 + \dots \end{aligned} \quad (9)$$

where  $a_2 \equiv \log(4/3)/2 \approx 0.143841$  and the other constants are only known numerically [12]:

$$a_3 = 0.642144, \quad a_{41} = -0.115069, \quad a_{42} = 3.128107, \quad a_{43} = 1.63, \quad a_{44} = -0.624638, \quad a_{45} = 2.39. \quad (10)$$

Writing the above expansion up to the  $L$ th term as  $F_L(u) = \sum_{l=-1}^L f_l (u/4\pi m_r)^l$ , the expansion coefficients for the relevant number of components  $N = 2$  are [12]:

$$f_{-1} = -126.651 \times 10^{-4}, \quad f_0 = 0, \quad f_1 = -4.04837 \times 10^{-4}, \quad f_2 = 2.39701 \times 10^{-4}, \quad f_3 = -1.80 \times 10^{-4}. \quad (11)$$

We need the value of the series  $F_L(u)$  in the critical limit  $m_r \rightarrow 0$ , which is obviously equivalent to the strong-coupling limit of  $F_L(u)$ . As mentioned above, this limit should be most accurately found with the help of variational perturbation theory [4, 8, 9].

If the series were of quantum mechanical origin, we could find this limit by applying the rules of *naive* variational perturbation theory [13]. We form the sequence of truncated expansions  $F_L(u)$  for 1, 2, 3 and replace each term  $(u/m_r)^l$  by  $K^l [1-1]_{L-l}^{-l/2}$  where the symbol  $[1-1]_k^r$  is defined as the binomial expansion of  $(1-1)^r$  truncated after the  $k$ th term

$$[1-1]_k^r \equiv \sum_{i=0}^k \binom{r}{i} (-1)^i = (-1)^k \binom{r-1}{k}. \quad (12)$$

TABLE I: Trial functions for the naive quantum-mechanical variational perturbation expansion

$$\begin{aligned}
W_1^{\text{QM}} &= -0.0596831K^{-1} - 0.0000322159K, \\
W_2^{\text{QM}} &= -0.0497359K^{-1} - 0.0000483239K + 1.51792 \cdot 10^{-6} K^2, \\
W_3^{\text{QM}} &= -0.0435189K^{-1} - 0.0000604049K + 3.03584 \cdot 10^{-6} K^2 - .908 \cdot 10^{-7} K^3.
\end{aligned}$$

The resulting expressions must be optimized in the variational parameter  $K$ . They are listed in Table I. The approximants  $W_{1,2,3}^{\text{QM}}$  have extrema  $W_{1,2,3}^{\text{QMext}} \approx -0.00277, +0.00405, -0.0029$ , corresponding, via (8), to  $c_1 \approx 3.059, -4.46, 3.01$ . These values have previously been obtained in Ref. [10] in a much more complicated way via a so-called  $\delta$ -expansion. Note the negative sign of the second approximation arising from the fact that an extremum exists only at negative  $K$ . According to our rules of variational perturbation theory one should, in this case, use the saddle point at positive  $K$  which would yield  $W_2^{\text{QM}} = -0.00153$  corresponding to  $c_1 \approx 1.69$  rather than  $-4.46$ , leading to the more reasonable approximation sequence  $c_1 \approx 3.059, 1.69, 3.01$ , which shows no sign of convergence. In  $W_3^{\text{QM}}$ , there is also a pair of complex extrema from which the authors of Ref. [10] extract the real part  $\text{Re } \tilde{W}_{3\text{complex}}^{\text{QM}} \approx -0.00134$  corresponding to  $c_1 \approx 1.48$ , which they state as their final result. There is, however, no acceptable theoretical justification for such a choice [14].

This lack of convergence is not astonishing since naive quantum mechanical variational perturbation theory is inapplicable to field theory, contrary to ubiquitous statements in the literature [15]. A simple but essential modification is necessary to allow for the well-known fact that there are *anomalous dimensions* in the critical regime of fluctuating fields. This modification was discovered in Ref. [8] and tested by the fact it reproduces in  $D = 4 - \epsilon$  dimensions *exactly* the known  $\epsilon$  expansions of renormalization group theory [9]. In  $D = 3$  dimensions, it leads to the most accurate critical exponents so far (see in particular Chapters 19 and 20 in the textbook [4]).

The correct procedure goes as follows: We form the logarithmic derivative of the expansion (9):

$$\beta(u) \equiv \frac{\partial \log F(u)}{\partial \log u} = -1 + 2 \frac{f_1}{f_{-1}} \left( \frac{u}{m_r} \right)^2 + 3 \frac{f_2}{f_{-1}} \left( \frac{u}{m_r} \right)^3 + \left( 4 \frac{f_3}{f_{-1}} - 2 \frac{f_1^2}{f_{-1}^2} \right) \left( \frac{u}{m_r} \right)^4 + \dots \quad (13)$$

In order for  $F(u)$  to go to a constant in the critical limit  $m_r \rightarrow 0$ , this function must go to zero in the strong-coupling limit  $u \rightarrow \infty$ . Writing the expansion as  $\beta_L(u) = -1 + \sum_{l=2}^L b_l (u/4\pi m_r)^l$ , the coefficients are

$$b_2 = 0.0639293, \quad b_3 = -0.056778, \quad b_4 = 0.0548799. \quad (14)$$

The sums  $\beta_L(u)$  have to be evaluated for  $u \rightarrow \infty$  allowing for the universal anomalous dimension  $\omega$  by which the physical observables of  $\phi^4$ -theories approach the scaling limit [3, 4]. This  $\omega$  coincides with the famous Wegner exponent [5] of renormalization group theory. Here it appears in the variational expression for the strong-coupling limit which is found [8, 9] by replacing  $(u/m_r)^l$  by  $K^l [1 - 1]_{L-l}^{-ql/2}$ , where  $q \equiv 2/\omega$ . Thus we obtain the variational expressions

$$W_3^\beta = -1 + \left( \frac{2f_1}{f_{-1}} + \frac{2f_1 q}{f_{-1}} \right) K^2 + \frac{3f_2}{f_{-1}} K^3 \quad (15)$$

$$W_4^\beta = -1 + \left( \frac{2f_1}{f_{-1}} + \frac{3f_1 q}{f_{-1}} + \frac{f_1 q^2}{f_{-1}} \right) K^2 + \left( \frac{3f_2}{f_{-1}} + \frac{9f_2 q}{2f_{-1}} \right) K^3 + \left( \frac{-2f_1^2}{f_{-1}^2} + \frac{4f_3}{f_{-1}} \right) K^4 \quad (16)$$

The first has a vanishing extremum at  $\omega_3 = 0.592$ , the second has neither an extremum nor a saddle point. However, a complex pair of extrema lies reasonably close to the real axis at  $\omega_4 = 0.635 \pm 0.116$ , whose real part is not far from the true exponent of approach  $\omega_\infty \approx 0.80$  [3, 4], to which  $\omega_L$  will converge for order  $L \rightarrow \infty$  [8]. Given these  $\omega$ -values, we now form the variational expressions  $W_L$  from  $F_L$  by the replacement  $(u/m_r)^l \rightarrow K^l [1 - 1]_{L-l}^{-ql/2}$ , which are

$$W_2 = f_{-1} \left( 1 - \frac{3}{4}q + \frac{1}{8}q^2 \right) K^{-1} + f_1 K, \quad (17)$$

$$W_3 = f_{-1} \left( 1 - \frac{11}{13}q + \frac{1}{4}q^2 - \frac{1}{48}q^3 \right) K^{-1} + f_1 \left( 1 + \frac{q}{2} \right) K + f_2 K^2, \quad (18)$$

$$W_4 = f_{-1} \left( 1 - \frac{25}{24}q + \frac{35}{96}q^2 - \frac{5}{96}q^3 + \frac{1}{384}q^4 \right) K^{-1} + f_1 \left( 1 + \frac{3}{4}q + \frac{1}{8}q^2 \right) K + f_2(1+q)K^2 + f_2 K^3. \quad (19)$$

The lowest function  $W_2$  is optimized with the naive growth parameter  $q = 1$  since to this order no anomalous value can be determined from the zero of the beta function (13). The optimal result is  $W_2^{\text{opt}} = -\sqrt{\log[4/3]}/6/8\pi^2 \approx -0.00277$  corresponding to  $c_1 \equiv 3.06$ . The next function  $W_3$  is optimized with the above determined  $q_3 = 2/\omega_3$  and yields  $W_3^{\text{opt}} \approx -0.00096$  corresponding to  $c_1 \equiv 1.06$ . Although  $\omega_4$  is not real we shall insert its real part into  $W_4$  and find  $W_4^{\text{opt}} \equiv -0.00096$  corresponding to  $c_1 \equiv 1.06$ . The three values of  $c_1$  for  $\bar{L} \equiv L - 1 = 1, 2, 3$  can well be fitted by a function  $c_1 \approx 1.053 + 2/\bar{L}^6$  (see Fig. 2). Such a fit is suggested by the general large- $L$  behavior  $a + be^{-c\bar{L}^{1-\omega}}$  which was derived in Refs. [13]. Due to the smallness of  $1 - \omega \approx 0.2$ , this can be replaced by  $\approx a' + b'/\bar{L}^s$ .

Alternatively, we may optimize the functions  $W_{1,2,3}$  using the known precise value of  $q_\infty = 2/\omega_\infty \approx 2/0.8$ . Then  $W_2$  turns out to have no optimum, whereas the others yield  $W_{3,4}^{\text{opt}} \approx -0.000529, -0.000706$ , corresponding via Eq. (8) to  $c_1 = 0.584, 0.78$ . If these two values are fitted by the same inverse power of  $\bar{L}$ , we find  $c_1 \approx 0.798 - 13.7/\bar{L}^6$ .

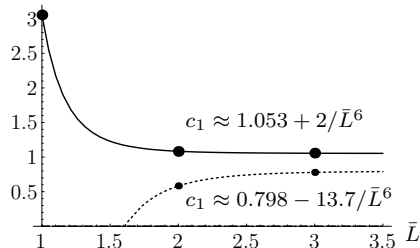


FIG. 2: The three approximants for  $c_1$  plotted against the order of variational approximation  $\bar{L} \equiv L - 1 = 1, 2, 3$ , and extrapolation to the infinite-order limit.

From the extrapolations to infinite order we estimate  $c_{1,\infty} \approx 0.93 \pm 0.13$ , such that the critical temperature shift is

$$\frac{\Delta T_c}{T_c^{(0)}} \approx (0.93 \pm 0.13) an^{1/3}. \quad (20)$$

Our result (20) is to be compared with latest Monte Carlo data which estimate  $c_1 \approx 1.32 \pm 0.02$  [16, 17]. Other analytic estimates are  $c_1 \approx 2.90$  [18], 2.33 from a  $1/N$ -expansion [19]), 1.71 from a next-to-leading order in a  $1/N$ -expansion [20], and 3.059 from an inapplicable  $\delta$ -expansion [21] to three loops, and 1.48 from the same  $\delta$ -expansion to five loops, with a questionable evaluation at a complex extremum [10], and some wrong expansion coefficients (see [12]). Remarkably, our result lies close to the average between the latest and the first Monte Carlo result  $c_1 \approx 0.34 \pm 0.03$  in Ref. [22].

As a cross check of the reliability of our theory consider the result in the limit  $N \rightarrow \infty$ . Here we drop the first term in the expansion (9) which vanishes at the critical point (but would diverge for  $N \rightarrow \infty$  at finite  $m_r$ ). The remaining expansion coefficients of  $\langle \phi^2/u \rangle / N$  in powers of  $Nu/4\pi m_r$  are

$$f_1 = -6.35917 \cdot 10^{-4}, \quad f_2 = 4.7315 \cdot 10^{-4}, \quad f_3 = -3.84146 \cdot 10^{-4}. \quad (21)$$

Using the  $N \rightarrow \infty$  limit of  $\omega$  which is equal to 1 implying  $q = 2$  in Eqs. (18) and (19), we obtain the two variational approximations

$$W_2^\infty = -0.00127183K + 0.00047315K^2, \quad W_3^\infty = -0.00190775K + 0.00141945K^2 - 0.000384146K^3, \quad (22)$$

whose optima yield the approximations  $c_1 \approx 1.886$  and 2.017, converging rapidly towards the exact large- $N$  result 2.33 of Ref. [19], with a 10% error as in the  $N = 2$ -result (20).

Numerically, the first two  $1/N$ -corrections found from a fit to large- $N$  results obtained by using the known large- $N$  expression for  $\omega = 1 - 8(8/3\pi^2 N) + 2(104/3 - 9\pi^2/2)(8/3\pi^2 N)^2$  [23] produce a finite- $N$  correction factor  $(1 - 3.1/N + 30.3/N^2 + \dots)$ , to be compared with  $(1 - 0.527/N + \dots)$  obtained in Ref. [20].

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integrals and confirming the correctness of the coefficients (10) of Ref. [11] rather than those of Ref. [10] listed in [12].

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